Kinematics of Particles: Plane Curvilinear Motion

Rectangular Coordinates (*x*-*y*)

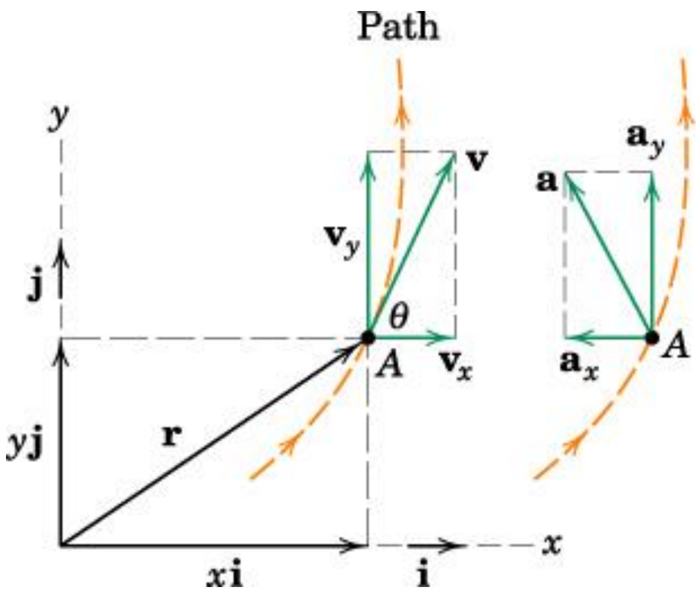
If all motion components are directly expressible in terms of horizontal and vertical coordinates

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j}$$

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j}$$

$$\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j}$$

$$v_x = \dot{x},\ v_y = \dot{y} \text{ and } a_x = \dot{v}_x = \ddot{x},\ a_y = \dot{v}_y = \ddot{y}$$



$$v^2 = v_x^2 + v_y^2 \qquad v = \sqrt{v_x^2 + v_y^2} \qquad \tan\theta = \frac{v_y}{v_x}$$

$$a^2 = a_x^2 + a_y^2 \qquad a = \sqrt{a_x^2 + a_y^2}$$

Time derivatives of the unit vectors are zero because their magnitude and direction remains constant.

Also, $dy/dx = \tan\theta = v_y/v_x$

Kinematics of Particles: Plane Curvilinear Motion

Normal and Tangential Coordinates (*n*-*t*)

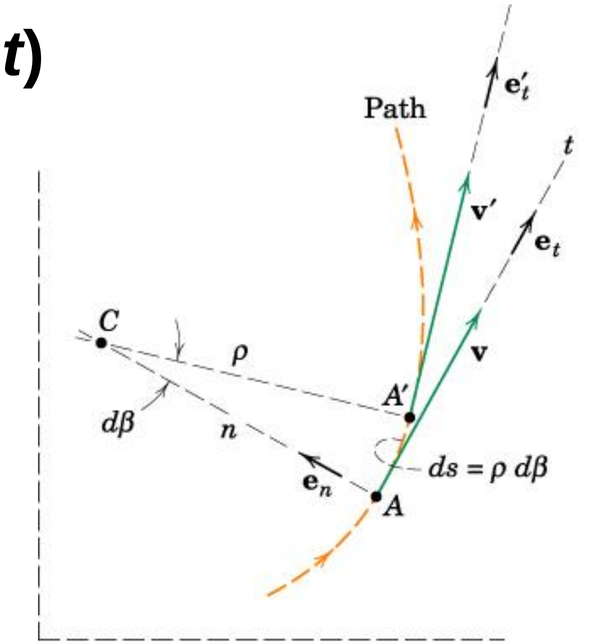
Determination of $\dot{\mathbf{e}}_t$:

→ change in $\mathbf{e}_t$ during motion from *A* to *A'*

→ The unit vector changes to $\mathbf{e}'_t$

The vector difference $d\mathbf{e}_t$ is shown in the bottom figure.

- In the limit $d\mathbf{e}_t$ has magnitude equal to length of the arc $|\mathbf{e}_t|\ d\beta = d\beta$
- Direction of $d\mathbf{e}_t$ is given by $\mathbf{e}_n$

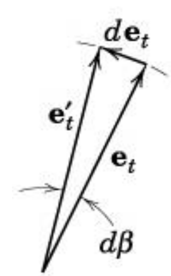


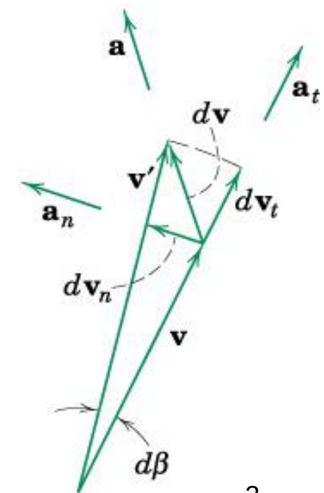
→ We can write: $d\mathbf{e}_t = \mathbf{e}_n\, d\beta$ → $\dfrac{d\mathbf{e}_t}{d\beta} = \mathbf{e}_n$

Dividing by *dt*: $d\mathbf{e}_t/dt = \mathbf{e}_n (d\beta/dt)\ \mathbf{e}_n$ → $\dot{\mathbf{e}}_t = \dot{\beta}\mathbf{e}_n$

Substituting this and $v = \rho\, d\beta/dt$ = $v = \rho\dot{\beta}$ in equation for acceleration:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d(v\mathbf{e}_t)}{dt} = v\dot{\mathbf{e}}_t + \dot{v}\mathbf{e}_t \quad \rightarrow \quad \mathbf{a} = \frac{v^2}{\rho}\mathbf{e}_n + \dot{v}\mathbf{e}_t$$





Here:

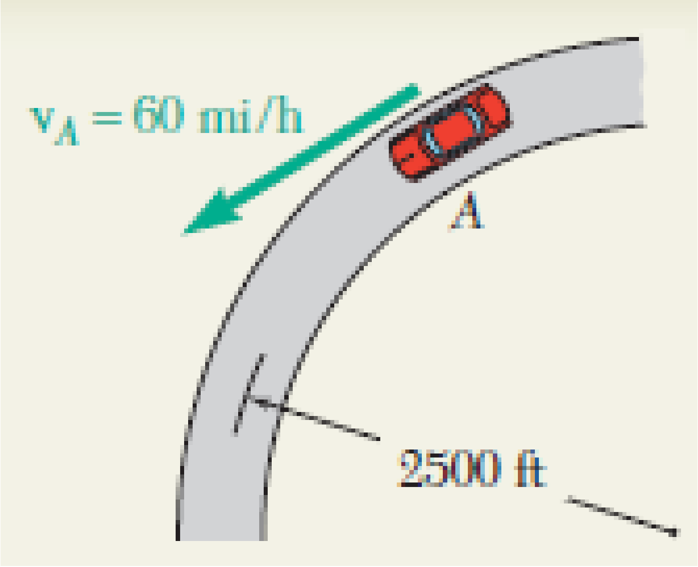
$$a_n = \frac{v^2}{\rho} = \rho\dot{\beta}^2 = v\dot{\beta}$$

$$a_t = \dot{v} = \ddot{s}$$

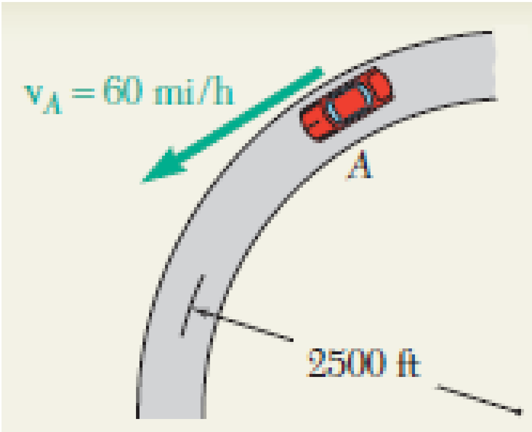
$$a = \sqrt{a_n^2 + a_t^2}$$

Example (1) on normal and tangential coordinates

A motorist is traveling on a curved section of highway of radius 2500 ft at the speed of 60 mi/h. The motorist suddenly applies the brakes, causing the automobile to slow down at a constant rate. Knowing that after 8 s the speed has been reduced to 45 mi/h, determine the acceleration of the automobile immediately after the brakes have been applied.



Example (1) on normal and tangential coordinates



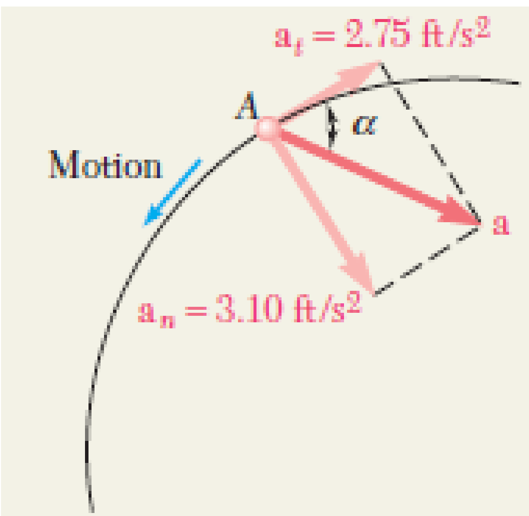
Tangential Component of Acceleration. First the speeds are expressed in ft/s.

$$60 \text{ mi/h} = \left(60 \frac{\text{mi}}{\text{h}}\right)\left(\frac{5280 \text{ ft}}{1 \text{ mi}}\right)\left(\frac{1 \text{ h}}{3600 \text{ s}}\right) = 88 \text{ ft/s}$$

$$45 \text{ mi/h} = 66 \text{ ft/s}$$

Since the automobile slows down at a constant rate, we have

$$a_t = \text{average } a_t = \frac{\Delta v}{\Delta t} = \frac{66 \text{ ft/s} - 88 \text{ ft/s}}{8 \text{ s}} = -2.75 \text{ ft/s}^2$$



Normal Component of Acceleration. Immediately after the brakes have been applied, the speed is still 88 ft/s, and we have

$$a_n = \frac{v^2}{\rho} = \frac{(88 \text{ ft/s})^2}{2500 \text{ ft}} = 3.10 \text{ ft/s}^2$$

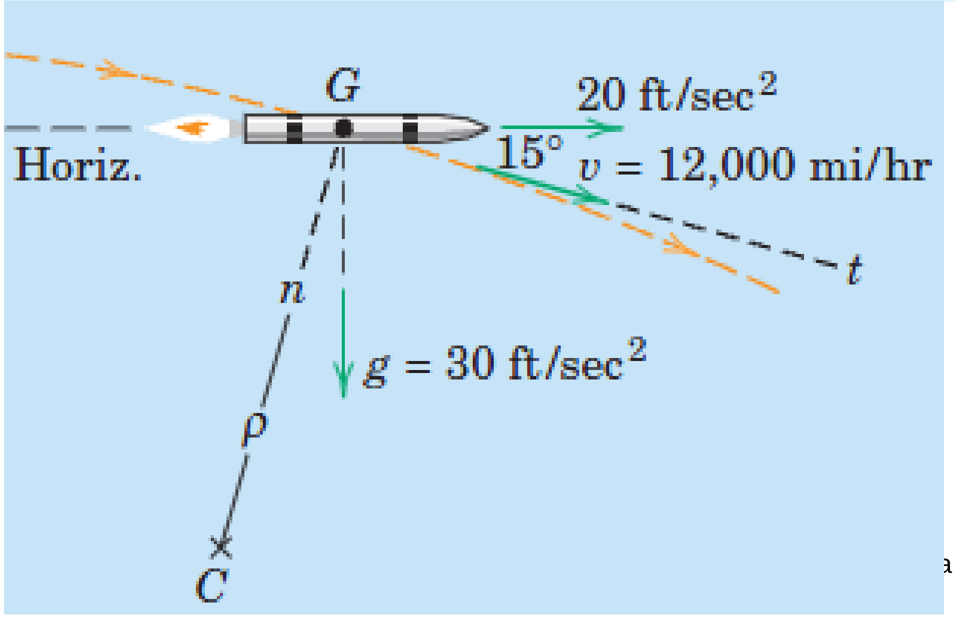
Magnitude and Direction of Acceleration. The magnitude and direction of the resultant $\mathbf{a}$ of the components $\mathbf{a}_n$ and $\mathbf{a}_t$ are

$$\tan \alpha = \frac{a_n}{a_t} = \frac{3.10 \text{ ft/s}^2}{2.75 \text{ ft/s}^2} \qquad \alpha = 48.4° \quad ◀$$

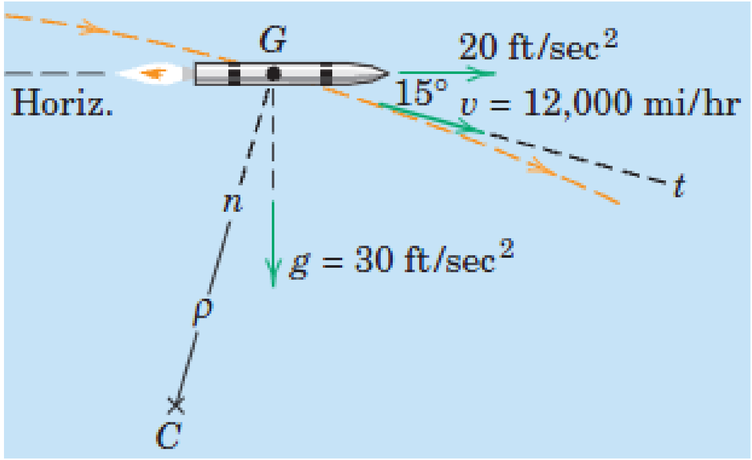
$$a = \frac{a_n}{\sin \alpha} = \frac{3.10 \text{ ft/s}^2}{\sin 48.4°} \qquad a = 4.14 \text{ ft/s}^2 \quad ◀$$

Example (2) on normal and tangential coordinates

A certain rocket maintains a horizontal attitude of its axis during the powered phase of its flight at high altitude. The thrust imparts a horizontal component of acceleration of 20 ft/sec^2, and the downward acceleration component is the acceleration due to gravity at that altitude, which is $g = 30$ ft/sec^2. At the instant represented, the velocity of the mass center G of the rocket along the 15° direction of its trajectory is 12,000 mi/hr. For this position determine (*a*) the radius of curvature of the flight trajectory, (*b*) the rate at which the speed v is increasing, (*c*) the angular rate $\dot{\beta}$ of the radial line from G to the center of curvature C, and (*d*) the vector expression for the total acceleration **a** of the rocket.



Example (2) on normal and tangential coordinates



$$a_n = 30 \cos 15^\circ - 20 \sin 15^\circ = 23.8 \text{ ft/sec}^2$$

$$a_t = 30 \sin 15^\circ + 20 \cos 15^\circ = 27.1 \text{ ft/sec}^2$$

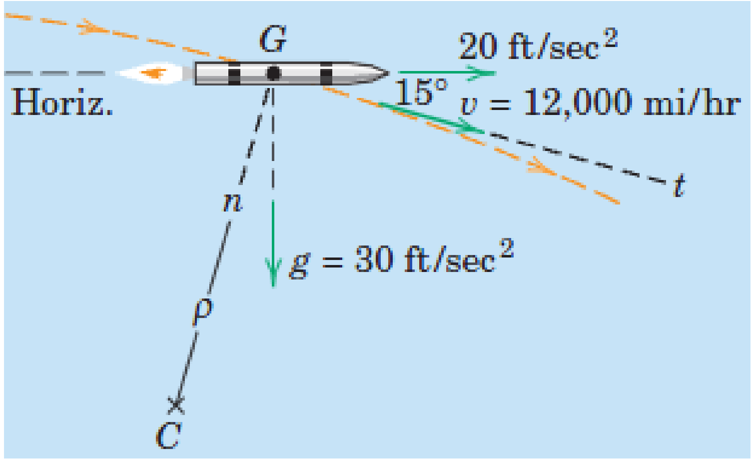
(a) We may now compute the radius of curvature from

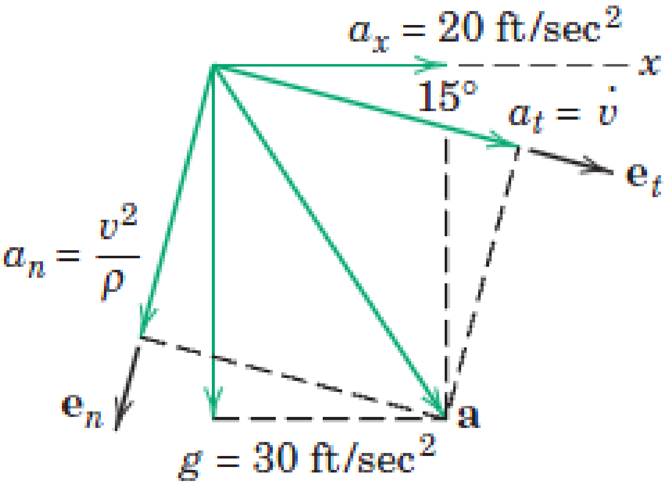
$[a_n = v^2/\rho]$ $$\rho = \frac{v^2}{a_n} = \frac{[(12{,}000)(44/30)]^2}{23.8} = 13.01(10^6) \text{ ft}$$ *Ans.*

(b) The rate at which v is increasing is simply the t-component of acceleration.

$[\dot{v} = a_t]$ $$\dot{v} = 27.1 \text{ ft/sec}^2$$ *Ans.*

Example (2) on normal and tangential coordinates





(c) The angular rate $\dot{\beta}$ of line GC depends on v and ρ and is given by

$[v = \rho\dot{\beta}]$ $$\dot{\beta} = v/\rho = \frac{12{,}000(44/30)}{13.01(10^6)} = 13.53(10^{-4}) \text{ rad/sec}$$ *Ans.*

(d) With unit vectors $\mathbf{e}_n$ and $\mathbf{e}_t$ for the n- and t-directions, respectively, the total acceleration becomes

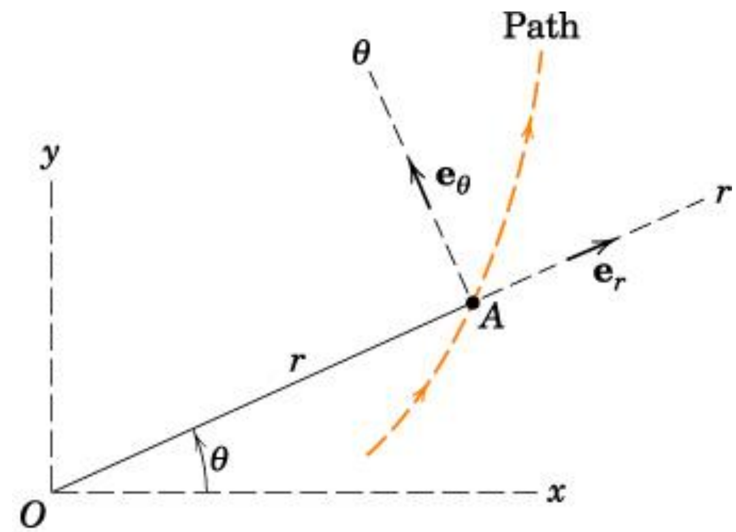
$$\mathbf{a} = 23.8\mathbf{e}_n + 27.1\mathbf{e}_t \text{ ft/sec}^2$$ *Ans.*

Kinematics of Particles: Plane Curvilinear Motion

Polar Coordinates (r - θ)

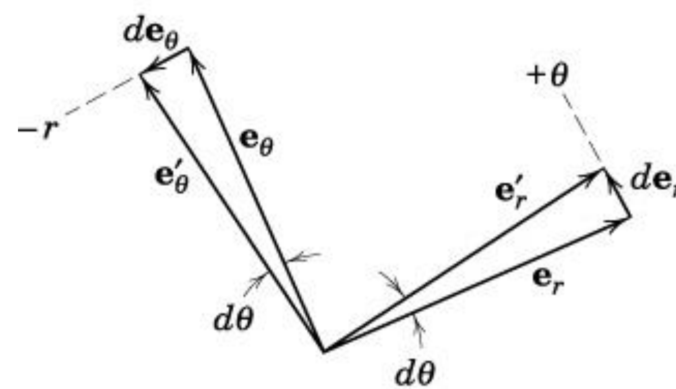
The particle is located by the radial distance r from a fixed point and by an angular measurement θ to the radial line.

- θ is measured from an arbitrary reference axis
- $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are unit vectors along +r & +θ dirns.



Location of particle at A: $\mathbf{r} = r\,\mathbf{e}_r$
By definition: $\mathbf{v} = d\mathbf{r}/dt$ and $\mathbf{a} = d^2\mathbf{r}/dt^2$
Therefore we need: $\dot{\mathbf{e}}_r$ *and* $\dot{\mathbf{e}}_\theta$

During time dt, the coordinate directions rotate through an angle $d\theta$: $\mathbf{e}_r \rightarrow \mathbf{e}'_r$ and $\mathbf{e}_\theta \rightarrow \mathbf{e}'_\theta$
Vector change $d\mathbf{e}_r$ is in the +ve θ direction
Vector change $d\mathbf{e}_\theta$ is in the -ve r direction

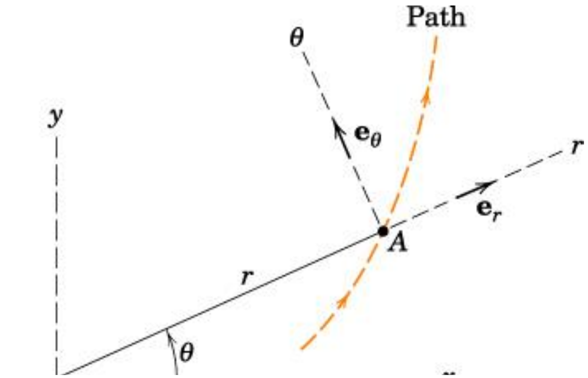


As already seen in the previous section: magnitudes of $d\mathbf{e}_r$ and $d\mathbf{e}_\theta$ in the limit are equal to the unit vector (radius) times $d\theta$ →
$d\mathbf{e}_r = \mathbf{e}_\theta\, d\theta$ and $d\mathbf{e}_\theta = -\,\mathbf{e}_r\, d\theta$

Kinematics of Particles: Plane Curvilinear Motion

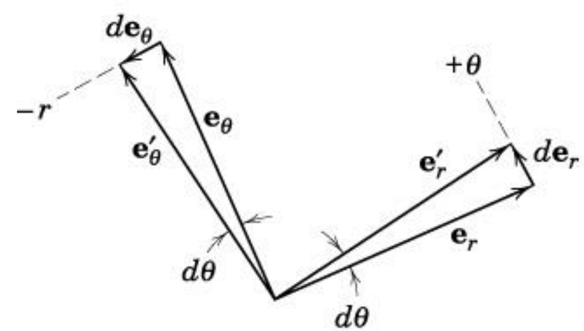
Polar Coordinates (*r* - *θ*)

$d\mathbf{e}_r = \mathbf{e}_\theta d\theta$ and $d\mathbf{e}_\theta = -\mathbf{e}_r d\theta$



- Dividing by $d\theta$ → $\frac{d\mathbf{e}_r}{d\theta} = \mathbf{e}_\theta \qquad \frac{d\mathbf{e}_\theta}{d\theta} = -\mathbf{e}_r$
- Dividing by dt → $\frac{d\mathbf{e}_r}{dt} = \mathbf{e}_\theta \frac{d\theta}{dt} \qquad \frac{d\mathbf{e}_\theta}{dt} = -\mathbf{e}_r \frac{d\theta}{dt}$

→ $\dot{\mathbf{e}}_r = \dot{\theta}\mathbf{e}_\theta \qquad \dot{\mathbf{e}}_\theta = -\dot{\theta}\mathbf{e}_r$



Relations for Velocity:

Differentiating $\mathbf{r} = r\,\mathbf{e}_r$ wrt time

Vector expression for velocity →

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\mathbf{e}}_r \quad \rightarrow \quad \mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$$

Magnitudes can be calculated as:

r-component of *v* is the rate at which the vector *r* stretches. *θ* component of *v* is due to the rotation of *r* along the circumference of a circle having radius *r*.

$$v_r = \dot{r}$$

$$v_\theta = r\dot{\theta}$$

$$v = \sqrt{v_r^2 + v_\theta^2}$$

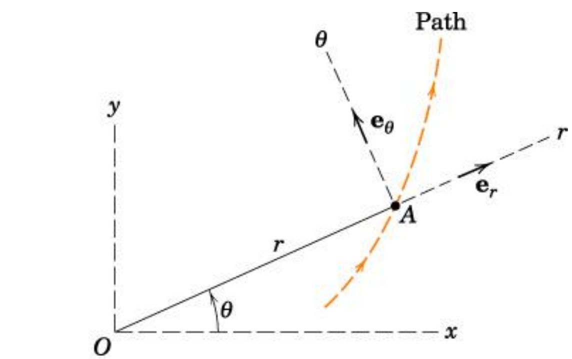
The term d*θ/dt* is called Angular Velocity (rad/s) since it represents time rate of change of angle *θ*

Kinematics of Particles: Plane Curvilinear Motion

Polar Coordinates (r - θ)

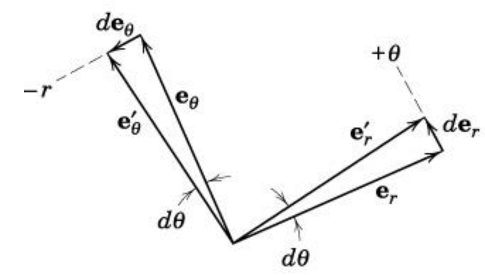
Relations for Acceleration:

Differentiating the expression $\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$ wrt time
The derivative of the second term will produce three terms since all three factors are variable.



$$\mathbf{a} = \dot{\mathbf{v}} = \left(\ddot{r}\mathbf{e}_r + \dot{r}\dot{\mathbf{e}}_r\right) + \left(\dot{r}\dot{\theta}\mathbf{e}_\theta + r\ddot{\theta}\mathbf{e}_\theta + r\dot{\theta}\dot{\mathbf{e}}_\theta\right)$$

We know: $\dot{\mathbf{e}}_r = \dot{\theta}\mathbf{e}_\theta \qquad \dot{\mathbf{e}}_\theta = -\dot{\theta}\mathbf{e}_r$



Vector expression for acceleration →

$$\mathbf{a} = \ddot{r}\mathbf{e}_r + \dot{r}\dot{\theta}\mathbf{e}_\theta + \dot{r}\dot{\theta}\mathbf{e}_\theta + r\ddot{\theta}\mathbf{e}_\theta - r\dot{\theta}^2\mathbf{e}_r$$

→ $$\mathbf{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\mathbf{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\mathbf{e}_\theta$$

Magnitudes can be calculated as:

$$a_r = \ddot{r} - r\dot{\theta}^2$$
$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$
$$a = \sqrt{a_r^2 + a_\theta^2}$$

The term $d^2\theta/dt^2$ is called Angular Accln since it represents change made in angular vel during an instant of time (rad/s^2)

θ-component can be alternatively written as: $a_\theta = \dfrac{1}{r}\dfrac{d}{dt}\left(r^2\dot{\theta}\right)$

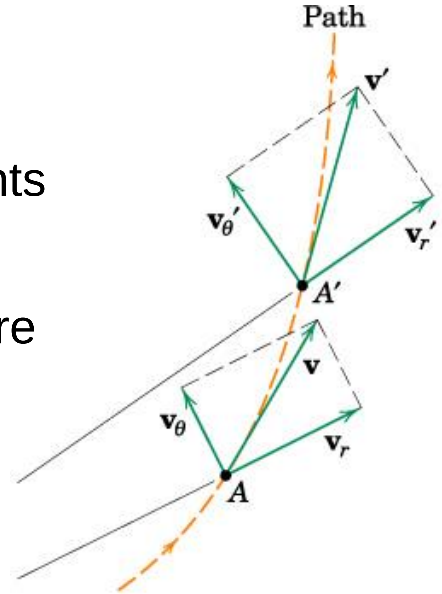
Kinematics of Particles: Plane Curvilinear Motion

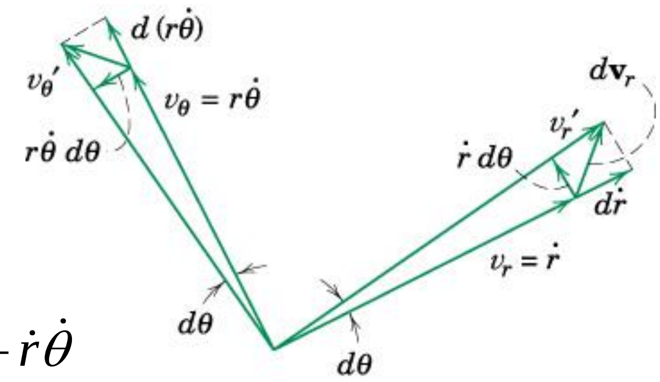
Polar Coordinates (r - θ)

Geometric Interpretations of the equations

Top figure shows velocity vectors and their r- and θ- components at positions A and A' after an infinitesimal movement.

Changes in magnitudes and directions of these components are shown in the bottom figure. Following are the changes:





(a) Magnitude change of $\mathbf{v}_r$:

= increase in length of v_r or $dv_r = d\dot{r}$

→ Accn term (in the + r-dirn): $d\dot{r}/dt = \ddot{r}$

(b) Direction change of $\mathbf{v}_r$:

Magnitude of this change = $v_r d\theta = \dot{r} d\theta$

→ Accn term (in the + θ-dirn): $\dot{r} d\theta / dt = \dot{r}\dot{\theta}$

(c) Magnitude change of $\mathbf{v}_\theta$:

= change in length of v_θ or $d(r\dot{\theta})$

→ Accn term (in the + θ-dirn): $d(r\dot{\theta})/dt = r\ddot{\theta} + \dot{r}\dot{\theta}$

(d) Direction change of $\mathbf{v}_\theta$:

Magnitude of this change = $v_\theta d\theta = r\dot{\theta}\, d\theta$

→ Accn term (in the - r-dirn): $r\dot{\theta}(d\theta/dt) = r\dot{\theta}^2$

Kinematics of Particles: Plane Curvilinear Motion

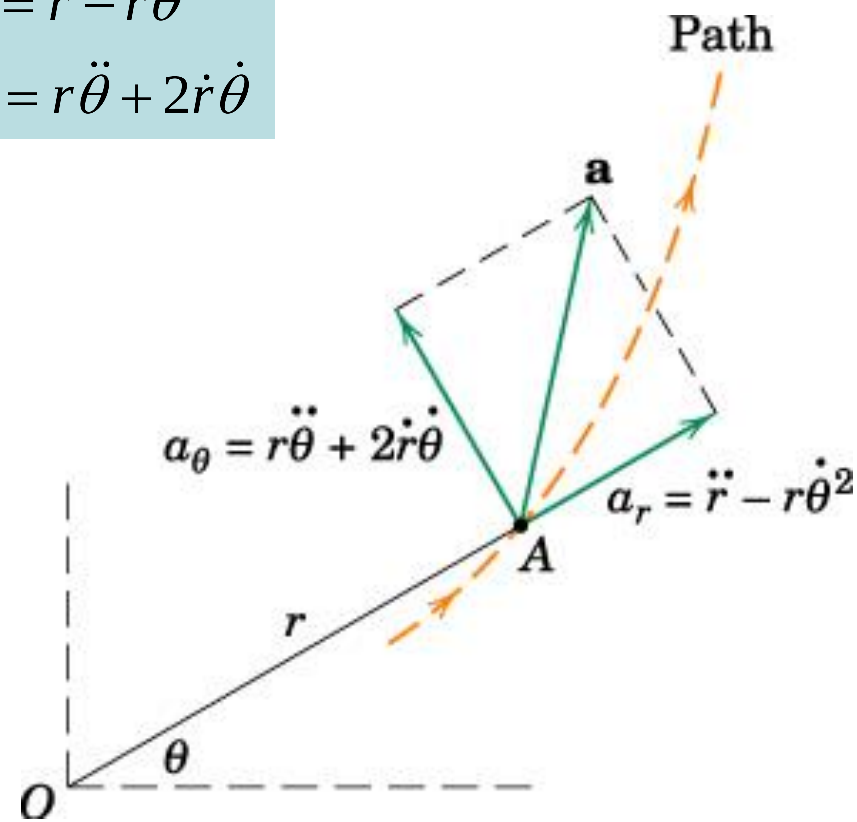
Polar Coordinates (r - θ)

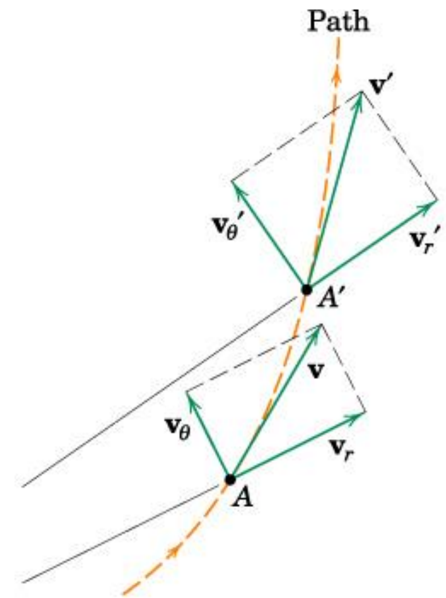
Geometric Interpretations of the equations

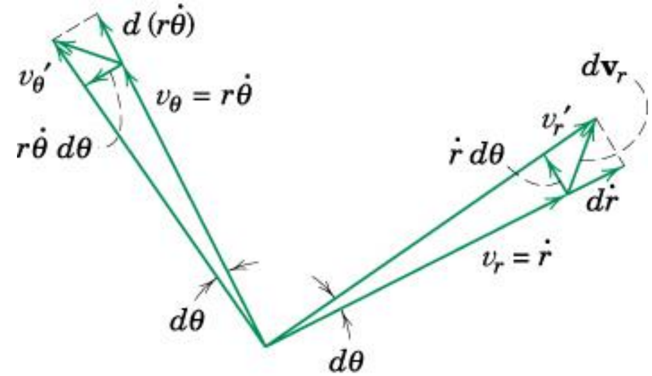
Collecting terms gives same relations as obtained previously

$$a_r = \ddot{r} - r\dot{\theta}^2$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$







Kinematics of Particles: Plane Curvilinear Motion

Polar Coordinates (*r* - θ)

Circular Motion: For motion in a circular path, *r* is constant

→ The components of velocity and acceleration become:

$$v_r = \dot{r} \quad \rightarrow \quad v_r = 0$$

$$v_\theta = r\dot{\theta} \qquad\qquad v_\theta = r\dot{\theta}$$

$$a_r = \ddot{r} - r\dot{\theta}^2 \quad \rightarrow \quad a_r = -r\dot{\theta}^2$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} \qquad\qquad a_\theta = r\ddot{\theta}$$

→ Same as that obtained with *n*- and *t*-components, where the θ and *t*-directions coincide but the +ve *r*-direction is along the –ve *n*-direction

→ $a_r = -a_n$ for circular motion centered at the origin of the polar coordinates.

Further the expressions for a_r and a_θ can also be obtained using rectangular coordinates $x = r\cos\theta$ and $y = r\sin\theta$

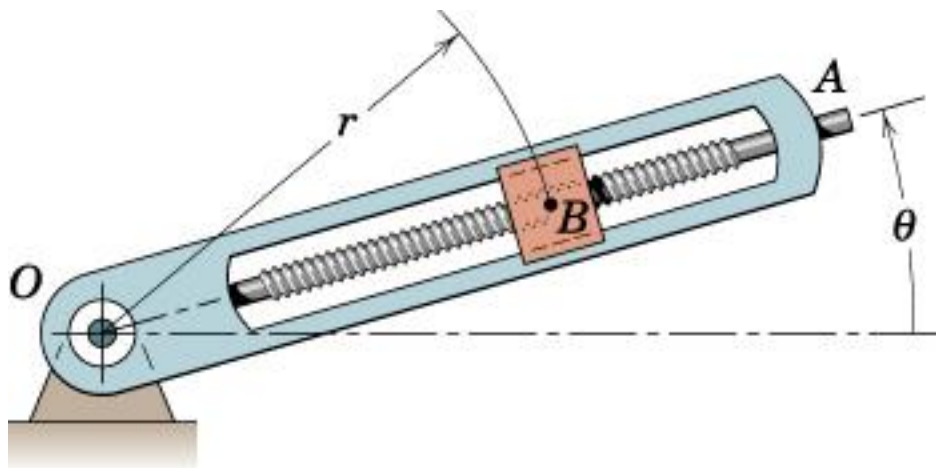
→ $a_x = \ddot{x}$ and $a_y = \ddot{y}$

these rectangular components can be resolved into *r*- and θ-components to get the same expressions as obtained above.

Example (1) on polar coordinates

Rotation of the radially slotted arm is governed by $\theta = 0.2t + 0.02t^3$. Simultaneously, the power screw in the arm engages the slider B and controls its distance from O according to $r = 0.2 + 0.04t^2$. Calculate the magnitudes of the velocity and acceleration of the slider for the instance when $t = 3$ s.

θ is in radians, r is in meters, and t is in seconds.



Example (1) on polar coordinates

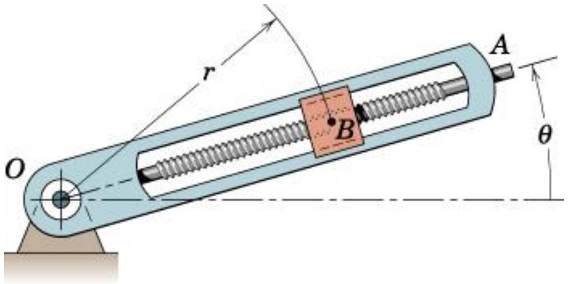
Solution:

Using the Polar Coordinates.

Available Equations:

$$v_r = \dot{r} \qquad v_\theta = r\dot{\theta} \qquad v = \sqrt{v_r^2 + v_\theta^2}$$

$$a_r = \ddot{r} - r\dot{\theta}^2 \qquad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} \qquad a = \sqrt{a_r^2 + a_\theta^2}$$



Obtaining the derivatives of *r* and θ at *t* = 3 s.

$$r = 0.2 + 0.04t^2 \qquad r_3 = 0.2 + 0.04(3^2) = 0.56 \text{ m}$$

$$\dot{r} = 0.08t \qquad \dot{r}_3 = 0.08(3) = 0.24 \text{ m/s}$$

$$\ddot{r} = 0.08 \qquad \ddot{r}_3 = 0.08 \text{ m/s}^2$$

$$\theta = 0.2t + 0.02t^3 \qquad \theta_3 = 0.2(3) + 0.02(3^3) = 1.14 \text{ rad}$$

$$\text{or } \theta_3 = 1.14(180/\pi) = 65.3°$$

$$\dot{\theta} = 0.2 + 0.06t^2 \qquad \dot{\theta}_3 = 0.2 + 0.06(3^2) = 0.74 \text{ rad/s}$$

$$\ddot{\theta} = 0.12t \qquad \ddot{\theta}_3 = 0.12(3) = 0.36 \text{ rad/s}^2$$

Example (1) on polar coordinates

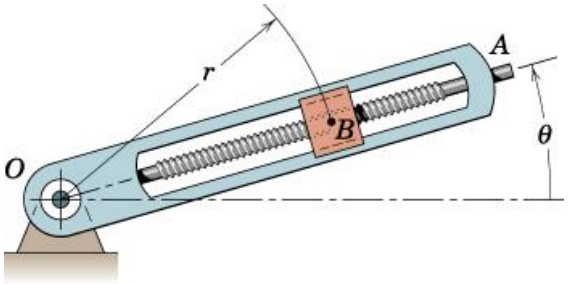
Solution:

Substituting in these eqns:

$$v_r = \dot{r} \qquad a_r = \ddot{r} - r\dot{\theta}^2$$

$$v_\theta = r\dot{\theta} \qquad a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

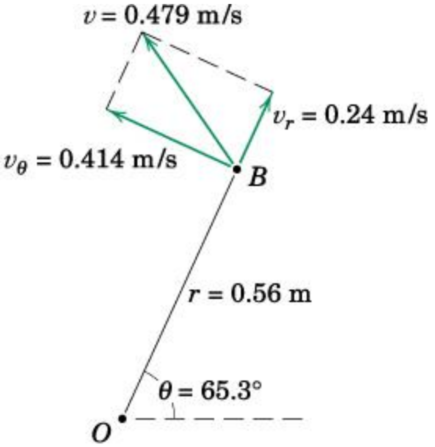
$$v = \sqrt{v_r^2 + v_\theta^2} \qquad a = \sqrt{a_r^2 + a_\theta^2}$$



$$v_r = 0.24 \text{ m/s}$$

$$v_\theta = 0.56(0.74) = 0.414 \text{ m/s}$$

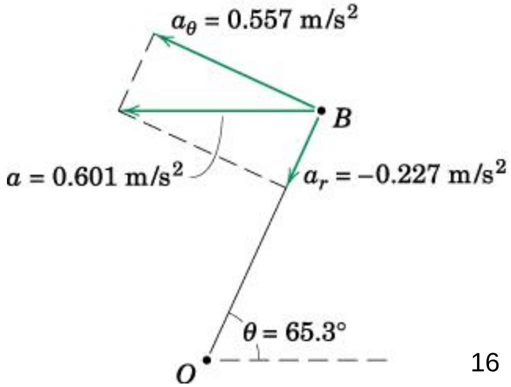
$$v = \sqrt{(0.24)^2 + (0.414)^2} = 0.479 \text{ m/s}$$



$$a_r = 0.08 - 0.56(0.74)^2 = -0.227 \text{ m/s}^2$$

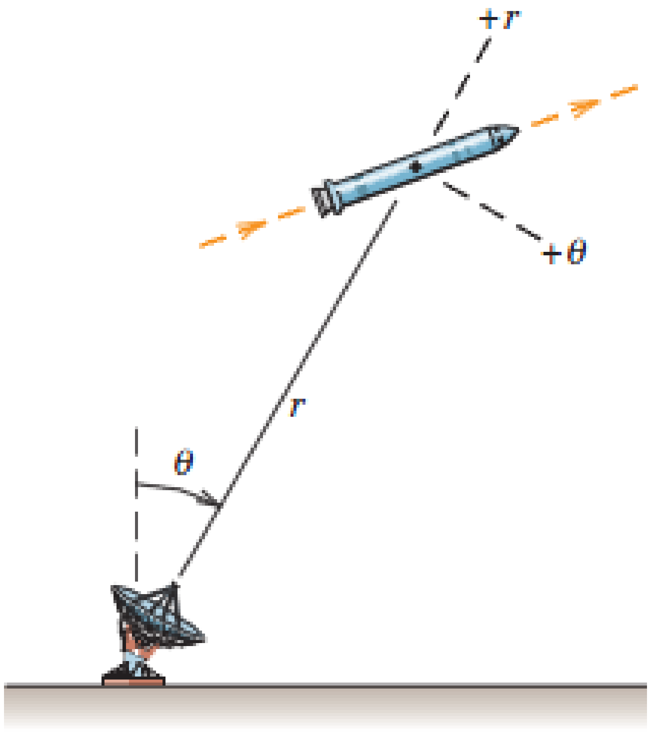
$$a_\theta = 0.56(0.36) + 2(0.24)(0.74) = 0.557 \text{ m/s}^2$$

$$a = \sqrt{(-0.227)^2 + (0.557)^2} = 0.601 \text{ m/s}^2$$



Example (2) on polar coordinates

A tracking radar lies in the vertical plane of the path of a rocket which is coasting in unpowered flight above the atmosphere. For the instant when $\theta = 30°$, the tracking data give $r = 25(10^4)$ ft, $\dot{r} = 4000$ ft/sec, and $\dot{\theta} = 0.80$ deg/sec. The acceleration of the rocket is due only to gravitational attraction and for its particular altitude is 31.4 ft/sec^2 vertically down. For these conditions determine the velocity v of the rocket and the values of $\ddot{r}$ and $\ddot{\theta}$.



Example (2) on polar coordinates

$[v_r = \dot{r}]$ $\quad v_r = 4000 \text{ ft/sec}$

$[v_\theta = r\dot{\theta}]$ $\quad v_\theta = 25(10^4)(0.80)\left(\dfrac{\pi}{180}\right) = 3490 \text{ ft/sec}$

$[v = \sqrt{v_r{}^2 + v_\theta{}^2}]$ $\quad v = \sqrt{(4000)^2 + (3490)^2} = 5310 \text{ ft/sec}$ *Ans.*

$$a_r = -31.4 \cos 30^\circ = -27.2 \text{ ft/sec}^2$$

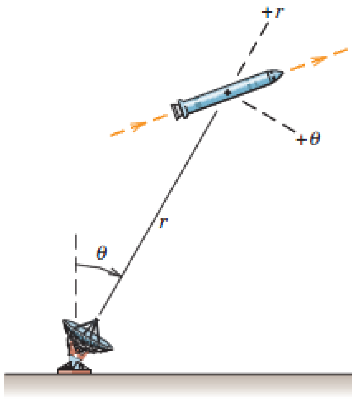
$$a_\theta = 31.4 \sin 30^\circ = 15.70 \text{ ft/sec}^2$$

$[a_r = \ddot{r} - r\dot{\theta}^2]$ $\quad -27.2 = \ddot{r} - 25(10^4)\left(0.80\,\dfrac{\pi}{180}\right)^2$

$\ddot{r} = 21.5 \text{ ft/sec}^2$ *Ans.*

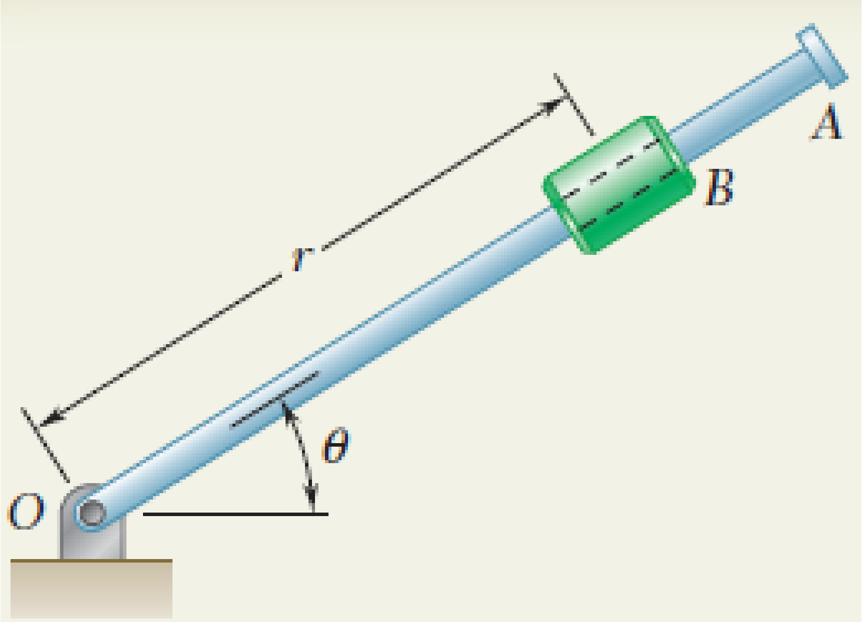
$[a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}]$ $\quad 15.70 = 25(10^4)\ddot{\theta} + 2(4000)\left(0.80\,\dfrac{\pi}{180}\right)$

$\ddot{\theta} = -3.84(10^{-4}) \text{ rad/sec}^2$ *Ans.*

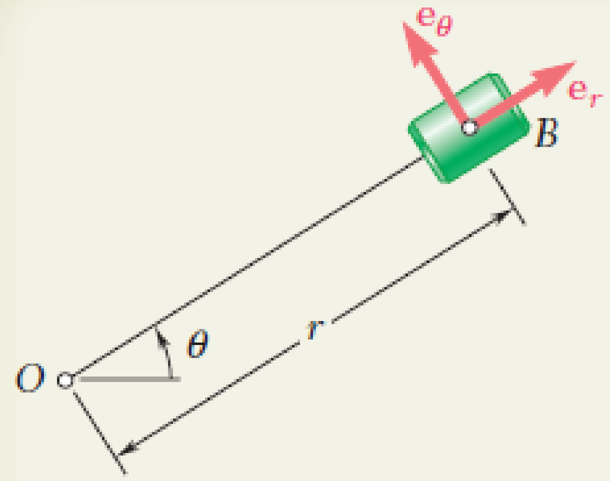


Example (3) on polar coordinates

The rotation of the 0.9-m arm OA about O is defined by the relation $\theta = 0.15t^2$, where θ is expressed in radians and t in seconds. Collar B slides along the arm in such a way that its distance from O is $r = 0.9 - 0.12t^2$, where r is expressed in meters and t in seconds. After the arm OA has rotated through 30°, determine (a) the total velocity of the collar, (b) the total acceleration of the collar, (c) the relative acceleration of the collar with respect to the arm.



Example (3) on polar coordinates



$\mathbf{v} = v_r \mathbf{e}_r + v_\theta \mathbf{e}_\theta$

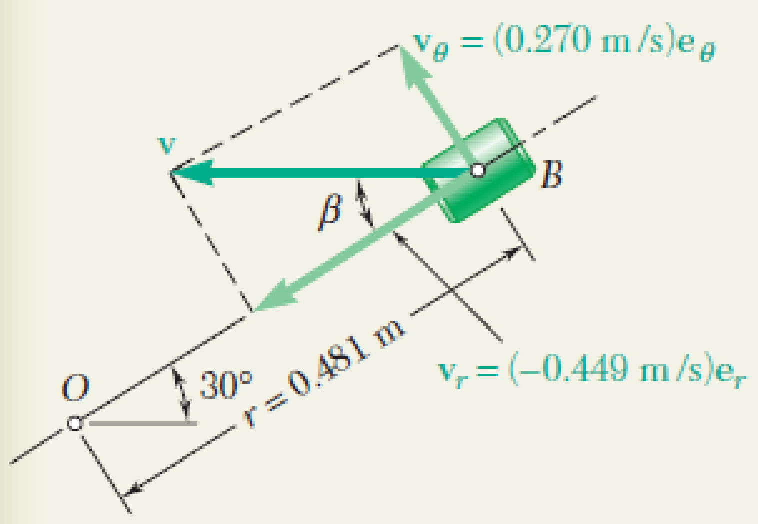
$\mathbf{a} = a_r \mathbf{e}_r + a_\theta \mathbf{e}_\theta$

Time t at which $\theta = 30°$. Substituting $\theta = 30° = 0.524$ rad into the expression for θ, we obtain

$$\theta = 0.15t^2 \qquad 0.524 = 0.15t^2 \qquad t = 1.869 \text{ s}$$

Equations of Motion. Substituting $t = 1.869$ s in the expressions for r, θ, and their first and second derivatives, we have

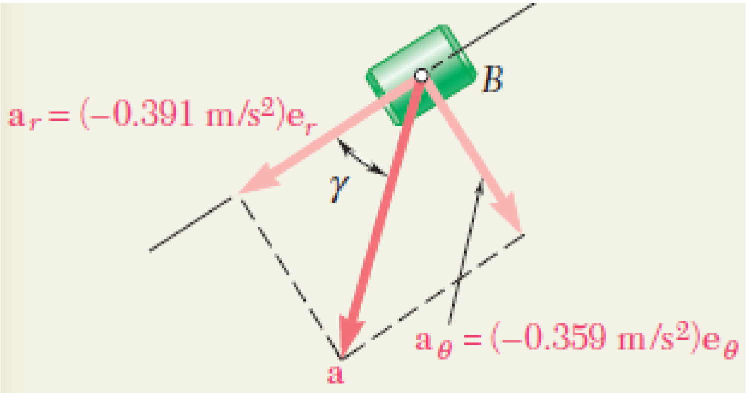
$$r = 0.9 - 0.12t^2 = 0.481 \text{ m} \qquad \theta = 0.15t^2 = 0.524 \text{ rad}$$
$$\dot{r} = -0.24t = -0.449 \text{ m/s} \qquad \dot{\theta} = 0.30t = 0.561 \text{ rad/s}$$
$$\ddot{r} = -0.24 = -0.240 \text{ m/s}^2 \qquad \ddot{\theta} = 0.30 = 0.300 \text{ rad/s}^2$$



$$v_r = \dot{r} = -0.449 \text{ m/s}$$
$$v_\theta = r\dot{\theta} = 0.481(0.561) = 0.270 \text{ m/s}$$

$$v = 0.524 \text{ m/s} \qquad \beta = 31.0°$$

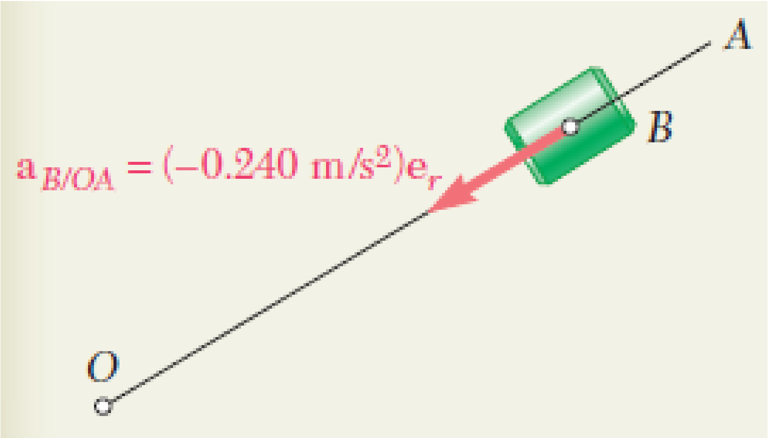
Example (3) on polar coordinates



Acceleration of B. Using Eqs. (11.46), we obtain

$$a_r = \ddot{r} - r\dot{\theta}^2$$
$$= -0.240 - 0.481(0.561)^2 = -0.391 \text{ m/s}^2$$
$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$
$$= 0.481(0.300) + 2(-0.449)(0.561) = -0.359 \text{ m/s}^2$$

$a = 0.531 \text{ m/s}^2 \qquad \gamma = 42.6°$



c. Acceleration of B with Respect to Arm OA. We note that the motion of the collar with respect to the arm is rectilinear and defined by the coordinate r. We write

$$a_{B/OA} = \ddot{r} = -0.240 \text{ m/s}^2$$

$a_{B/OA} = 0.240 \text{ m/s}^2$ toward O. ◀