On Semipositive Cones and their Extremals

Aritra N. Hisabia, and Manideepa Saha

Department of Mathematics, NIT Meghalaya, Shillong, India

An $m \times n$ real matrix A is called a semipositive (SP) matrix if it maps a positive vector to a positive vector. Non-negative matrices, M-matrices, P-matrices, and positive definite matrices are some important classes of semipositive matrices. The concept of semipositivity plays important role in variety of theoretical and practical problems, mainly in linear complementarity problems.

Let the sign '$\geq$ $(>)$' represent the entry wise inequality of matrices or vectors. For a given non-singular $n \times n$ matrix A, the aim of the paper is to study the cone $S_A = \{x : Ax \geq 0\}$, and its subcone K_A lies on the positive orthant, called as semipositive cone. In [2, 1], authors explained the geometric mapping properties of the semipositive matrix A in terms of its semipositive cone K_A, and also established that K_A is a proper polyhedral cone. In this paper, we prove that S_A is a simplicial cone and analysis the properties of its extremals.

An $m \times n$ matrix A is called a *semipositive* matrix if there exists a vector $x \geq 0$ such that $Ax > 0$, and the vector x is referred to as a *semipositivity vector* of A. For a semipositive matrix A if none of the column deleted submatrix of A is semipositive, then it is known as *minimally semipositive* matrix.

For any subset S of $\mathbb{R}^n$, the set *generated by* S consists of all finite non-negative linear combinations of elements of S, and we denote the set by S^G. A set $K \in \mathbb{R}^n$ is said to be a *cone* if $K = S^G$. In case S is finite, the cone K is called a *polyhedral cone*

A set is called *convex*, if it contains the line segment joining any two of its points. A convex cone K in $\mathbb{R}^n$ is called

(i) *pointed* if $K \cap (-K) = \{0\}$.

(ii) *solid* if K has non-empty interior.

A closed, pointed and solid convex cone is called a *proper cone.* A proper cone K always generates a partial order in $\mathbb{R}^n$ via $y \overset{K}{\leq} x$ if and only if $x - y \in K$. A vector x in $\mathbb{R}^n$ is an *extremal* of K $\mathbb{R}^n$ if $0 \overset{K}{\leq} y \overset{K}{\leq} x$ implies that $y = \alpha x$, for some $\alpha \geq 0$. If a cone K has exactly n extremals, then the cone K is called a *simplicial cone.*

Main results are stated below:

Theorem 1. For an invertible matrix $A \in \mathbb{R}^{n \times n}$, S_A is a proper polyhedral cone.

Theorem 2. For an invertible matrix $A \in \mathbb{R}^{n \times n}$, S_A is a simplicial cone.

Theorem 3. For an invertible semipositive matrix $A \in \mathbb{R}^{n \times n}$, x is an extremal of K_A if and only if Ax is an extremal of $K_{A^{-1}}$.

Corollary 1. For an invertible semipositive matrix A, the number of extremals of K_A and $K_{A^{-1}}$ are same.

Theorem 4. Any matrix $A \in \mathbb{R}^{n \times n}$ is minimally semipositive if and only if $K_A = S_A$ and K_A is a simplicial cone.

Theorem 5. For a square minimally semipositive matrix A, extremals of K_A are the columns of A^{-1}.

Reference

1. K. C. Sivakumar and M. J. Tsatsomeros. Semipositive matrices and their semipositive cones. *Positivity*, 22:379–398, 2018.
2. M. J. Tsatsomeros. Geometric mapping properties of semipositive Matrices. *Linear Algebra Appl.*, 498:349–359, 2016.