Non-chiral Bosonization of Fermions in One Dimension

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An alternative to the conventional approach to bosonization in one dimension that invokes the Dirac equation in 1+1 dimension with chiral 'right-movers' and 'left-movers' is proposed that works directly with the bounded parabolic energy bands relevant to Condensed Matter problems. This technique allows us to use a basis different from the plane wave basis that makes this non-chiral approach ideally suited to study Luttinger liquids that have boundary or impurities that break translational symmetry. We provide a simple solution to the electron Green function for the problem of Luttinger liquid (LL) with a boundary and also for a LL with a single impurity. The present method is significantly easier than the g-ology based standard bosonization and other methods that require a combination of RG along with bosonization/refermionization techniques but our results are broadly consistent with the ones obtained using those methods.

I. A CRITIQUE OF CONVENTIONAL 'CHIRAL' BOSONIZATION

The subject of what is currently referred to as 'bosonization' started with the works of Coleman and also independently by Luther. While Coleman showed that the fermion Green function massive of the Thirring model has a independent description in terms of bosonic variables of an 'equivalent' model involving commuting variables namely the so-called Sine-Gordon theory, other authors such as Luther and Mandelstam took this to mean that the Fermi field operator itself has an expression in terms of bosonic variables. This latter assertion is much stronger and is stated without proof in those articles making them subject to criticism. The stronger assertion has been used by later researchers in Condensed Matter Physics to generate hamiltonians of fermions in one dimension that go under the name 'g-ology'. Here we argue that the stronger assertion is in fact false making the g-ology program in Condensed Matter Physics of questionable validity. This approach has been used in highly cited works of Kane and Fisher on impurities in Luttinger liquids and nearly everywhere in the textbook by Giamarchi. An alternative is proposed, mainly for Condensed Matter problems, involving an action for fermions in terms of hydrodynamic variables and a prescription for generating the N-point functions of fermions that allows one to go beyond the linear dispersion approximation and also the random phase approximation. The particle-hole excitations of the Fermi system that make the kinetic energy diagonal in these operators are not bosons even 'approximately' making the term 'bosonization' a misnomer (The belief in conventional bosonization that the kinetic energy $K = \Sigma_k \epsilon_k c_k^{\dagger} c_k = \Sigma_{p>0} v_F p \; b_{p,R}^{\dagger} b_{p,R} + \Sigma_{p>0} v_F p \; b_{p,L}^{\dagger} b_{p,L} + const$ is an operator identity is shown to be false, it is merely a mnemonic for generating the correlation functions. Coleman's assertion that $\bar{\psi}\gamma^{\mu}\psi \equiv -\frac{\beta}{2\pi}\epsilon^{\mu\nu}\partial_{\nu}\varphi$ is a metaphor only, for the left hand side is a Grassmann number and the right hand side is a real number - a Grassmann number can never be equal to a real number though it can be equivalent to a real number).

We have some serious issues with the conventional approach to bosonization. The purpose of this section is to motivate the rest of the article by these criticisms. In his excellent textbook, based on pioneering works of Haldane, Giamarchi trumpets the formula for the field operator (Luther's construction),

$$\psi_r^{\dagger}(x) = Lim_{\epsilon\rightarrow 0} \; \psi_r^{\dagger}(x;\epsilon) = Lim_{\epsilon\rightarrow 0}\frac{1}{\sqrt{2\epsilon L}} e^{-ir(k_F - \frac{\pi}{L})x} e^{i\phi_r^{\dagger}(x,\epsilon) + i\phi_r(x,\epsilon)} \tag{1}$$

where $r = \pm$ corresponds to right and left movers and,

$$\phi_r(x,\epsilon) = -\frac{\pi r x}{L} N_r + i \sum_{p\neq 0} \left(\frac{2\pi}{L|p|}\right)^{\frac{1}{2}} e^{-\frac{L\epsilon|p|}{2\pi}} Y(rp) b_p e^{ipx} \tag{2}$$

The main point we are making is that Eq.(1) is mathematically meaningless as the limit $Lim_{\epsilon\rightarrow 0}$ almost certainly does not exist. In particular, it is not true that matrix elements of the right hand side of Eq.(1) are the matrix elements of the field operator (at least no such proof is forthcoming). Some authors recommend that we postpone taking the $\epsilon \rightarrow 0$ limit until the end of the 'calculation'. Until the end of what calculation ? What if I don't want to do any calculation ? What if I just want to stare at this operator itself ? If one insists on a calculation, how about calculating the matrix elements of the field operator ? All these rhetorical questions lead to one conclusion - that is, all the ϵ's have be the same for all the field operators in the N-point function calculation and the fermion commutation rules are recovered only at the level of correlation functions and not at the level of operators. This is in fact clear if one consults Coleman's pioneering paper on the equivalence of the massive Thirring model and Sine Gordon equation, he only shows that the Green functions come out right in both the languages, it is never shown that the matrix elements of the field operator come out right. Thus the g-ology program which involves a literal interpretation of the Luther construction is on shaky ground.

Therefore, the following set of judgemental characterizations are in order.

(a) **Preposterous** : The Fermi field operator has an expression in terms of bosons constructed out of Fermi bilinears and other objects like Klein factors. No such claim has ever been proven in the literature. In other words no proof exists that all matrix elements, or indeed any matrix element of the nonlocal combination of bosons is equal to the corresponding matrix elements of the field operator.

(b) **Plausible but still untrue** : Equal-time number conserving products of Fermi fields are expressible in terms of Fermi bilinears that are bosonic in character. This has also never been proven – just showing that the propagator comes out right is not enough. All matrix elements have to come out right, showing the finite temperature case is also not enough – that is just the diagonal matrix elements.

(c) **Possible Fact** : N-point functions have a non local integral representation involving commuting variables that may be simply related to Fermi bilinears such as current and densities.

In the literature, the operator description is sometimes replaced by a path integral version based on Hubbard Stratanovich transformation making these ideas appear more legitimate. However, both these approaches are flawed for the same reason - they brazenly manipulate infinities under the euphemism known as 'normal ordering'. Our approach differs from all these in several respects. The n-point Fermi functions are reduced to a closed form that is shown to be exact in the RPA limit $k_F, m \rightarrow \infty, v_F = k_F/m < \infty$. It is shown to reproduce the expected results when the impurity is turned off and the two-body potential is on and vice versa. These features are unlike the other approaches where some sort of renormalization group analysis is also needed. Besides, no mathematically questionable manipulations such as 'normal ordering' are made. Now we go on to discuss our approach.

II. IMPURITY IN A LUTTINGER LIQUID

For free electrons in an impurity potential $V(x) = V_0\delta(x)$, we list the following expressions for the propagator, which is nothing but the sum of these four pieces (here $\theta(z)$ is the Heaviside unit step function).

$$< T\ \Psi_R(x,t)\Psi^{\dagger}_R(x^{'},t^{'}) >= e^{ik_F(x-x^{'})}\frac{1}{[(x-x^{\prime})-v_F(t-t^{\prime})]}$$

$$[\frac{i}{2\pi} - \frac{V_0}{2\pi v_F}(\frac{\theta(x^{'})\theta(-x)}{\left(1-V_0\frac{i}{v_F}\right)}$$

$$-\frac{\theta(x)\theta(-x^{'})}{\left(1+V_0\frac{i}{v_F}\right)})]$$

$$< T\ \Psi_L(x,t)\Psi^{\dagger}_L(x^{'},t^{'}) >= e^{-ik_F(x-x^{'})}\frac{1}{[-(x-x^{\prime})-v_F(t-t^{\prime})]}$$

$$[\frac{i}{2\pi} - \frac{V_0}{2\pi v_F}(\frac{\theta(-x^{'})\theta(x)}{\left(1-V_0\frac{i}{v_F}\right)} - \frac{\theta(-x)\theta(x^{'})}{\left(1+V_0\frac{i}{v_F}\right)})]$$

$$< T\ \Psi_R(x,t)\Psi^{\dagger}_L(x^{'},t^{'}) >= e^{ik_F(x+x^{'})}\frac{V_0}{2\pi v_F}(\frac{\theta(-x^{'})\theta(-x)}{\left(1-V_0\frac{i}{v_F}\right)}$$

$$-\frac{\theta(x)\theta(x^{'})}{\left(1+V_0\frac{i}{v_F}\right)})\frac{1}{(-x^{\prime}-x)-v_F(t^{\prime}-t)}$$

$$< T\ \Psi_L(x,t)\Psi^{\dagger}_R(x^{'},t^{'}) >= e^{ik_F(-x-x^{'})}\frac{V_0}{2\pi v_F}(\frac{\theta(x^{'})\theta(x)}{\left(1-V_0\frac{i}{v_F}\right)}$$

$$-\frac{\theta(-x)\theta(-x^{'})}{\left(1+V_0\frac{i}{v_F}\right)})\frac{1}{(x^{\prime}+x)-v_F(t^{\prime}-t)} \tag{3}$$

The average density and density correlation may be written as follows.

$$< \rho(x,t) > -\rho_0 = -\frac{V_0}{\pi^2}\ \frac{m v_F^2}{(V_0^2+v_F^2)}\int_0^{\infty}\frac{dq}{q}\ cos(qx)\ Log\left[\frac{2k_F+q}{|2k_F-q|}\right] \tag{4}$$

The density-density correlation function is,

$$< T\ \rho(x,t)\rho(x^{'},t^{'}) > - < T\ \rho(x,t) >< T\ \rho(x^{'},t^{'}) >$$

$$= - < T\ \Psi(x,t)\Psi^{\dagger}(x^{'},t^{'}) >< T\ \Psi(x^{'},t^{'})\Psi^{\dagger}(x,t^{+}) >$$

If $\rho_s(x,t)$ is the slowly varying part of the density,

$$< T\ \rho_s(x,t)\rho_s(x^{'},t^{'}) > - < T\ \rho_s(x,t) >< T\ \rho_s(x^{'},t^{'}) >=$$

$$-\frac{V_0^2}{(2\pi)^2}\frac{\theta(xx^{'})}{(v_F^2+V_0^2)}[\frac{1}{[(x'+x)+v_F(t'-t)]^2}+\frac{1}{[(x+x')+v_F(t-t')]^2}]$$

$$+\frac{1}{[(x-x')+v_F(t-t')]^2}[-\frac{\theta(xx^{'})}{(2\pi)^2}-\frac{\theta(-xx^{'})}{(2\pi)^2}\frac{v_F^2}{(v_F^2+V_0^2)}]$$

$$+\frac{1}{[(x-x')-v_F(t-t')]^2}[-\frac{\theta(xx^{'})}{(2\pi)^2}-\frac{\theta(-xx^{'})}{(2\pi)^2}\frac{v_F^2}{(v_F^2+V_0^2)}] \quad (5)$$

The following expression for the field operator (also a mnemonic only)

$$\psi(x,t) =$$
$$C_{R,1}(x)\ e^{ik_F x}$$
$$e^{i\pi \int_{sgn(x)\infty}^{x} dy\ \tilde{\rho}_s(y,t) + i\int_{\infty\ sgn(x)}^{x} dy\ v(y,t)}$$
$$+C_{R,2}(x)\ e^{ik_F x}$$
$$e^{i\pi \int_{sgn(x)\infty}^{x} dy\ \tilde{\rho}_s(y,t) + 2\pi i\int_{sgn(x)\infty}^{x} dy\ \tilde{\rho}_s(-y,t) + i\int_{\infty\ sgn(x)}^{x} dy\ v(y,t)}$$
$$+C_{L,1}(x)\ e^{-ik_F x}$$
$$e^{-i\pi \int_{sgn(x)\infty}^{x} dy\ \tilde{\rho}_s(y,t) + i\int_{\infty\ sgn(x)}^{x} dy\ v(y,t)}$$
$$+C_{L,2}(x)\ e^{-ik_F x}$$
$$e^{-i\pi \int_{sgn(x)\infty}^{x} dy\ \tilde{\rho}_s(y,t) - 2\pi i\int_{sgn(x)\infty}^{x} dy\ \tilde{\rho}_s(-y,t) + i\int_{\infty\ sgn(x)}^{x} dy\ v(y,t)} \tag{6}$$

This should be thought of as short hand for the n-point functions it generates. These functions will have the products of several of the C-functions. These have to be independently fixed by making contact with the corresponding expressions obtained from Fermi algebra. This has been done and the following expressions for the Green function of the interacting system has been obtained. One last point before we do this, fermion commutation rules inferred from Eq.(6) do come out as expected namely $\psi(x,t)\psi(x^{'},t) = -\psi(x^{'},t)\psi(x,t)$ and $\psi(x,t)\psi^{\dagger}(x^{'},t) = -\psi^{\dagger}(x^{'},t)\psi(x,t)$ for $x \neq x^{'}$ as can be easily verified.

Particle Propagator : If $xx^{'} > 0$:

$$< T\ \Psi_R(x,t)\Psi^{\dagger}_R(x^{'},t^{'}) > = e^{ik_F(x-x^{'})}\ [\frac{i}{2\pi}]$$

$$< e^{(i\pi \int_{sgn(x)\infty}^{x} dy\ \tilde{\rho}_s(y,t) + i \int_{\infty\ sgn(x)}^{x} dy\ v(y,t))}$$

$$e^{(-i\pi \int_{sgn(x^{'})\infty}^{x^{'}} dy^{'}\ \tilde{\rho}_s(y^{'},t^{'}) - i \int_{\infty\ sgn(x^{'})}^{x^{'}} dy^{'}\ v(y^{'},t^{'}))} >$$

$$< T\ \Psi_L(x,t)\Psi^{\dagger}_L(x^{'},t^{'}) > = e^{-ik_F(x-x^{'})}[\frac{i}{2\pi}]$$

$$< e^{(-i\pi \int_{sgn(x)\infty}^{x} dy\ \tilde{\rho}_s(y,t) + i \int_{\infty\ sgn(x)}^{x} dy\ v(y,t))}$$

$$e^{(i\pi \int_{sgn(x^{'})\infty}^{x^{'}} dy^{'}\ \tilde{\rho}_s(y^{'},t^{'}) - i \int_{\infty\ sgn(x^{'})}^{x^{'}} dy^{'}\ v(y^{'},t^{'}))} >$$

$$< T\ \Psi_R(x,t)\Psi_L^{\dagger}(x^{'},t^{'}) > =$$

$$e^{ik_F(x+x^{'})}\frac{V_0}{4\pi v_F}(\frac{\theta(-x^{'})\theta(-x)}{\left(1-V_0\frac{i}{v_F}\right)}$$

$$-\frac{\theta(x)\theta(x^{'})}{\left(1+V_0\frac{i}{v_F}\right)})$$

$$e^{\frac{2V_0^2+v_F^2}{2(V_0^2+v_F^2)}Log(2x)}$$

$$< e^{(i\pi\int_{sgn(x)\infty}^{x}dy\ \ \tilde{\rho}_s(y,t)+2\pi i\int_{sgn(x)\infty}^{x}dy\ \ \tilde{\rho}_s(-y,t)+i\int_{\infty\ \ sgn(x)}^{x}dy\ \ v(y,t))}$$

$$e^{(i\pi\int_{sgn(x^{'})\infty}^{x^{'}}dy^{'}\ \ \tilde{\rho}_s(y^{'},t^{'})-i\int_{\infty\ \ sgn(x^{'})}^{x^{'}}dy^{'}\ \ v(y^{'},t^{'}))} >$$

$$+e^{ik_F(x+x^{'})}\frac{V_0}{4\pi v_F}\left(\frac{\theta(x^{'})\theta(x)}{\left(1+V_0\frac{i}{v_F}\right)}-\frac{\theta(-x)\theta(-x^{'})}{\left(1-V_0\frac{i}{v_F}\right)}\right)$$

$$e^{\frac{2V_0^2+v_F^2}{2(V_0^2+v_F^2)}Log(2x^{'})}$$

$$< e^{(i\pi\int_{sgn(x)\infty}^{x}dz\ \ \tilde{\rho}_s(z,t)+i\int_{\infty\ \ sgn(x)}^{x}dz\ \ v(z,t))}$$

$$e^{(i\pi\int_{sgn(x^{'})\infty}^{x^{'}}dz^{'}\ \ \tilde{\rho}_s(z^{'},t^{'})+2\pi i\int_{sgn(x^{'})\infty}^{x^{'}}dz^{'}\ \ \tilde{\rho}_s(-z^{'},t^{'})-i\int_{\infty\ \ sgn(x^{'})}^{x^{'}}dz^{'}\ \ v(z^{'},t^{'}))} >$$

$$< T\ \Psi_L(x,t)\Psi_R^{\dagger}(x^{'},t^{'}) > = e^{ik_F(-x-x^{'})}\frac{V_0}{4\pi v_F}$$

$$(\frac{\theta(x^{'})\theta(x)}{\left(1-V_0\frac{i}{v_F}\right)}$$

$$-\frac{\theta(-x)\theta(-x^{'})}{\left(1+V_0\frac{i}{v_F}\right)})$$

$$e^{\frac{2V_0^2+v_F^2}{2(V_0^2+v_F^2)}Log(2x)}$$

$$< e^{(-i\pi\int_{sgn(x)\infty}^{x}dy\ \ \tilde{\rho}_s(y,t)-2\pi i\int_{sgn(x)\infty}^{x}dy\ \ \tilde{\rho}_s(-y,t)+i\int_{\infty\ \ sgn(x)}^{x}dy\ \ v(y,t))}$$

$$e^{(-i\pi\int_{sgn(x^{'})\infty}^{x^{'}}dy^{'}\ \ \tilde{\rho}_s(y^{'},t^{'})-i\int_{\infty\ \ sgn(x^{'})}^{x^{'}}dy^{'}\ \ v(y^{'},t^{'}))} >$$

$$+e^{-ik_F(x+x^{'})}\frac{V_0}{4\pi v_F}\left(\frac{\theta(-x^{'})\theta(-x)}{\left(1+V_0\frac{i}{v_F}\right)}-\frac{\theta(x)\theta(x^{'})}{\left(1-V_0\frac{i}{v_F}\right)}\right)$$

$$e^{\frac{2V_0^2+v_F^2}{2(V_0^2+v_F^2)}Log(2x^{'})}$$

$$< e^{(-i\pi\int_{sgn(x)\infty}^{x}dz\ \ \tilde{\rho}_s(z,t)+i\int_{\infty\ \ sgn(x)}^{x}dz\ \ v(z,t))}$$

$$e^{(-i\pi\int_{sgn(x^{'})\infty}^{x^{'}}dz^{'}\ \ \tilde{\rho}_s(z^{'},t^{'})-2\pi i\int_{sgn(x^{'})\infty}^{x^{'}}dz^{'}\ \ \tilde{\rho}_s(-z^{'},t^{'})-i\int_{\infty\ \ sgn(x^{'})}^{x^{'}}dz^{'}\ \ v(z^{'},t^{'}))} > \tag{7}$$

For obtaining $< \psi^{\dagger}(x^{'},t^{'})\psi(x,t) >$ we simply place all the $x^{'}$ terms to the left of all the x terms. The long wavelength part of the density density correlation in the RPA limit with mutual interaction v_q (forward scattering only) and the impurity potential V_0 is given by,

$$< \rho(x,t)\rho(x^{'},t^{'}) >=$$

$$\frac{\theta(xx^{'})}{2\pi}\,(\frac{1}{((x+x')+v'(t-t'))^2}\frac{v_F^3}{2\pi v'(V_0^2+v_F^2)}$$

$$-\frac{1}{((x+x')+v(t-t'))^2}\frac{v_F}{2\pi v})$$

$$+\frac{1}{2\pi}\,(-\theta(-xx^{'})\,\frac{1}{((x-x')+v'(t-t'))^2}\frac{v_F^3}{2\pi v'(V_0^2+v_F^2)}$$

$$-\theta(xx^{'})\,\frac{1}{((x-x')+v(t-t'))^2}\frac{v_F}{2\pi v})$$

$$+\frac{\theta(xx^{'})}{2\pi}(\frac{1}{((x+x')-v'(t-t'))^2}\frac{v_F^3}{2\pi v'(V_0^2+v_F^2)}$$

$$-\frac{1}{((x+x')-v(t-t'))^2}\frac{v_F}{2\pi v})$$

$$+\frac{1}{2\pi}(-\theta(-xx^{'})\,\frac{1}{((x-x')-v'(t-t'))^2}\frac{v_F^3}{2\pi v'(V_0^2+v_F^2)}$$

$$-\theta(xx^{'})\,\frac{1}{((x-x')-v(t-t'))^2}\frac{v_F}{2\pi v}) \tag{8}$$

where,

$$v^2 = v_F^2 + \frac{v_F v_q}{\pi}$$

$$v'^2 = v_F^2 \frac{(v^2 + V_0^2)}{(V_0^2 + v_F^2)}$$

We can see that the equation Eq.(8) is consistent with Eq.(5) since the former reduces to the latter when mutual interaction between fermions is absent ($v = v' = v_F$, $V_0 \neq 0$). Conversely, when the impurity is absent but mutual interactions are present ($V_0 = 0, v' = v \neq v_F$) then,

$$< \rho(x,t)\rho(x',t') >_{V_0=0} \;=\; -\frac{v_F}{v}\frac{1}{(2\pi)^2}\Big(\frac{1}{((x-x')+v(t-t'))^2} + \frac{1}{((x-x')-v(t-t'))^2}\Big) \qquad (9)$$

as it should be. Lastly, when $V_0 = \infty$ we expect the results to coincide with those of a Luttinger liquid with a boundary obtained in an earlier section. The velocity and density are related in the RPA limit as follows.

$$v(x,t) = -\pi\, \partial_{v_F t} \int_{sgn(x)\infty}^{x} \tilde{\rho}(y',t)\, dy' \qquad (10)$$

Using the form of the density-density correlation in the long-wavelength limit, and the relation between velocity and density and the Baker Hausdorff theorem we may evaluate the single particle propagator.

The particle propagator for right movers may be evaluated to yield,

(a) For $xx^{'} > 0$

$$< \psi_R(x,t)\psi^{\dagger}_R(x^{'},t^{'}) >=$$

$$e^{ik_F(x-x^{'})} \; e^{-[\frac{v_F^2}{(V_0^2+v_F^2)}(-\frac{v_F}{4v^{'}}+\frac{v^{'}}{4v_F})+(\frac{v_F}{4v}-\frac{v}{4v_F})]Log(4xx^{'})}$$

$$e^{\frac{v_F^2}{(V_0^2+v_F^2)}(-\frac{v_F}{4v^{'}}+\frac{v^{'}}{4v_F})Log((x+x^{'})^2-v^{'2}(t-t^{'})^2)}$$

$$e^{(\frac{v_F}{4v}-\frac{v}{4v_F})Log((x+x^{'})^2-v^2(t-t^{'})^2)}$$

$$e^{(\frac{1}{2}-\frac{v_F}{4v}-\frac{v}{4v_F})Log((x-x^{'})+v(t-t^{'}))}$$

$$e^{(-\frac{v_F}{4v}-\frac{v}{4v_F}-\frac{1}{2})Log((x-x^{'})-v(t-t^{'}))} \tag{11}$$

It is easy to see that this also has all the right limits. For instance, when the impurity is absent $V_0 = 0$, $v = v^{'} \neq v_F$, Eq.(11) reduces to,

$$< \psi_R(x,t)\psi^{\dagger}_R(x^{'},t^{'}) >_{V_0=0}= e^{ik_F(x-x^{'})}$$

$$e^{(\frac{1}{2}-\frac{v_F}{4v}-\frac{v}{4v_F})Log((x-x^{'})+v(t-t^{'}))}$$

$$e^{(-\frac{v_F}{4v}-\frac{v}{4v_F}-\frac{1}{2})Log((x-x^{'})-v(t-t^{'}))} \tag{12}$$

as it should. Conversely, if mutual interactions between fermions are absent but the impurity is present then, $V_0 \neq 0$

but $v = v^{'} = v_F$. In this case,

$$< \psi_R(x,t)\psi_R^{\dagger}(x^{'},t^{'}) >_{v=v^{'}=v_F} = e^{-Log((x-x^{'})-v_F(t-t^{'}))} \qquad (13)$$

again as it should be. Lastly, if $V_0 = \infty$ (this also means $v = v^{'}$) we expect the results to coincide with those of a LL with a boundary. In this case Eq.(11) becomes

$$< \psi_R(x,t)\psi_R^{\dagger}(x^{'},t^{'}) >_{V_0=\infty} =$$

$$e^{ik_F(x-x^{'})}\ e^{-(\frac{v_F}{4v}-\frac{v}{4v_F})\ Log(4xx^{'})}$$

$$e^{(\frac{v_F}{4v}-\frac{v}{4v_F})Log((x+x^{'})^2-v^2(t-t^{'})^2)}$$

$$e^{(\frac{1}{2}-\frac{v_F}{4v}-\frac{v}{4v_F})Log((x-x^{'})+v(t-t^{'}))}$$

$$e^{(-\frac{v_F}{4v}-\frac{v}{4v_F}-\frac{1}{2})Log((x-x^{'})-v(t-t^{'}))} \qquad (14)$$

Now we may extract the dynamical density of states for right movers.

For this we examine the equal space and unequal time part of the Green function. First we verify that far away from the impurity we get what we expect.

$$< \psi_R(x \rightarrow \infty, t)\psi_R^{\dagger}(x \rightarrow \infty, t^{'}) > \;\; \sim \;\; e^{(-\frac{v_F}{2v} - \frac{v}{2v_F})Log(v(t-t^{'}))} \tag{15}$$

So that far away from the impurity $D(\omega) \sim |\omega|^{\delta}$ where $\delta = \frac{v}{2v_F} + \frac{v_F}{2v} - 1$. At the impurity we expect the exponent to be very different.

$$< \psi_R(x = 0, t)\psi_R^{\dagger}(x = 0, t^{'}) >=$$

$$e^{[\frac{v_F^2}{(V_0^2+v_F^2)}(-\frac{v_F}{2v^{'}} + \frac{v^{'}}{2v_F}) - \frac{v}{v_F}] \; Log(t-t^{'})} \tag{16}$$

From this we may conclude that at the impurity, the density of states is $D(\omega) = |\omega|^{\delta^{'}}$ where $\delta^{'} = \frac{v_F^2}{(V_0^2+v_F^2)}(\frac{v_F}{2v^{'}} - \frac{v^{'}}{2v_F}) + \frac{v}{v_F} - 1$. A plot of $\delta^{'}$ and δ versus v and V_0 indicates that $\delta^{'} > \delta$ for repulsive interactions $(v > v_F)$ and $\delta^{'} < \delta$ for attractive interactions $(v < v_F)$. We may surmise that the exponent associated with a.c. conductivity is the difference between these two δ's. Thus we may suspect $\sigma(\omega) \sim |\omega|^{(\delta^{'} - \delta)}$. In other words, there is breaking of the chain when repulsive interactions and the impurity are both present and a healing of the defect when attractive interactions and impurity are present. When any one of these is absent, the conductivity becomes nonsingular (unremarkable).

III. CLOSED FORM FOR PROPAGATOR OF LUTTINGER LIQUID IN ONE DIMENSION WITH A SINGLE IMPURITY

If,

$$v^2 = v_F^2 + \frac{v_F v_q}{\pi}$$

$v_q = const$ mutual interaction forward scattering only.

$$v'^2 = v_F^2 \frac{(v^2 + V_0^2)}{(V_0^2 + v_F^2)}$$

V_0 is impurity strength.

In general, we may write for the full particle propagator,

$$< T\ \Psi_R(x,t)\Psi^{\dagger}_R(x',t') > =$$

$$e^{ik_F(x-x')}\ [\frac{i}{2\pi} - \frac{V_0}{2\pi v_F}\left(\frac{\theta(x')\theta(-x)}{\left(1-V_0\frac{i}{v_F}\right)} - \frac{\theta(x)\theta(-x')}{\left(1+V_0\frac{i}{v_F}\right)}\right)]$$

$$e^{\frac{v_F^3}{2v'(V_0^2+v_F^2)}(-\frac{1}{2}+\frac{1}{2}\frac{v'^2}{v_F^2})\ Log((x+x')^2-v'^2(t-t')^2)}$$

$$e^{(\frac{v_F}{4v}-\frac{v}{4v_F})\ Log((x+x')^2-v^2(t-t')^2)}$$

$$e^{(-\frac{v_F}{4v}+\frac{1}{2}-\frac{v}{4v_F})\ Log((x-x')+v(t-t'))}$$

$$e^{(-\frac{v_F}{4v}-\frac{1}{2}-\frac{v}{4v_F})\ Log((x-x')-v(t-t'))}$$

$$e^{-\frac{v_F^3}{2v'(V_0^2+v_F^2)}(-\frac{1}{2}+\frac{1}{2}\frac{v'^2}{v_F^2})\ Log(4xx')}$$

$$e^{-(\frac{v_F}{4v}-\frac{v}{4v_F})\ Log(4xx')}$$

$$< T\ \Psi_L(x,t)\Psi^{\dagger}_L(x',t') > =$$

$$e^{-ik_F(x-x')}[\frac{i}{2\pi}-\frac{V_0}{2\pi v_F}\left(\frac{\theta(-x')\theta(x)}{\left(1-V_0\frac{i}{v_F}\right)}-\frac{\theta(-x)\theta(x')}{\left(1+V_0\frac{i}{v_F}\right)}\right)]$$

$$e^{\frac{v_F^3}{2v'(V_0^2+v_F^2)}(-\frac{1}{2}+\frac{1}{2}\frac{v'^2}{v_F^2})\ Log((x+x')^2-v'^2(t-t')^2)}$$

$$e^{(\frac{v_F}{4v}-\frac{v}{4v_F})\ Log((x+x')^2-v^2(t-t')^2)}$$

$$e^{(-\frac{v_F}{4v}-\frac{v}{4v_F}-\frac{1}{2})\ Log((x-x')+v(t-t'))}$$

$$e^{(-\frac{v_F}{4v}-\frac{v}{4v_F}+\frac{1}{2})\ Log((x-x')-v(t-t'))}$$

$$e^{-\frac{v_F^3}{2v'(V_0^2+v_F^2)}(-\frac{1}{2}+\frac{1}{2}\frac{v'^2}{v_F^2})\ Log(4xx')}\ e^{-(\frac{v_F}{4v}-\frac{v}{4v_F})\ Log(4xx')}$$

$$< T\ \Psi_R(x,t)\Psi_L^{\dagger}(x^{'},t^{'}) > = e^{ik_F(x+x^{'})}\frac{V_0}{4\pi v_F}$$

$$\left(\frac{\theta(x)\theta(x^{'})}{\left(1+V_0\frac{i}{v_F}\right)} - \frac{\theta(-x^{'})\theta(-x)}{\left(1-V_0\frac{i}{v_F}\right)}\right)$$

$$e^{\frac{v_F^3}{2v^{'}(V_0^2+v_F^2)}(-\frac{1}{2}+\frac{1}{2}\frac{v^{'2}}{v_F^2})\ Log((x+x^{'})^2-v^{'2}(t-t^{'})^2)}$$

$$e^{(-\frac{v_F}{4v}-\frac{v}{4v_F}+\frac{1}{2})\ Log((x+x^{'})+v(t-t^{'}))}$$

$$e^{(-\frac{v_F}{4v}-\frac{v}{4v_F}-\frac{1}{2})\ Log((x+x^{'})-v(t-t^{'}))}$$

$$e^{(\frac{v_F}{4v}-\frac{v}{4v_F})\ Log((x-x^{'})^2-v^2(t-t^{'})^2)}$$

$$(e^{\gamma_1\ Log(2x)}e^{\gamma_3\ Log(2x^{'})} + e^{\gamma_2 Log(2x^{'})}e^{\gamma_3\ Log(2x)})$$

$$< T \; \Psi_L(x,t)\Psi^{\dagger}_R(x',t') > = e^{-ik_F(x+x')}\frac{V_0}{4\pi v_F}$$

$$\left(\frac{\theta(x')\theta(x)}{\left(1 - V_0\frac{i}{v_F}\right)} - \frac{\theta(-x)\theta(-x')}{\left(1 + V_0\frac{i}{v_F}\right)}\right)$$

$$e^{\frac{v_F^3}{2v'(V_0^2+v_F^2)}(-\frac{1}{2}+\frac{1}{2}\frac{v'^2}{v_F^2})\; Log((x+x')^2-v'^2(t-t')^2)}$$

$$e^{(-\frac{v_F}{4v}-\frac{v}{4v_F}-\frac{1}{2})\; Log((x+x')+v(t-t'))}$$

$$e^{(-\frac{v_F}{4v}-\frac{v}{4v_F}+\frac{1}{2})\; Log((x+x')-v(t-t'))}$$

$$e^{(\frac{v_F}{4v}-\frac{v}{4v_F})\; Log((x-x')^2-v^2(t-t')^2)}$$

$$(e^{\gamma_2 Log(2x)}e^{\gamma_3 Log(2x')} + e^{\gamma_1 Log(2x')}e^{\gamma_3 \; Log(2x)})$$

where,

$$\gamma_1 = -\frac{v_F^3}{2v'(V_0^2+v_F^2)}\left(\frac{1}{2}-2+\frac{1}{2}\frac{v'^2}{v_F^2}\right)-\left(\frac{v_F}{4v}-\frac{v}{4v_F}+\frac{v_F}{v}\right)+\frac{2V_0^2+v_F^2}{2(V_0^2+v_F^2)}$$

$$\gamma_3 = -\frac{v_F^3}{2v'(V_0^2+v_F^2)}\left(-\frac{1}{2}+\frac{1}{2}\ \frac{v'^2}{v_F^2}\right)-\left(\frac{v_F}{4v}-\frac{v}{4v_F}\right)$$

$$\gamma_2 = -\frac{v_F^3}{2v'(V_0^2+v_F^2)}\left(-\frac{1}{2}-1+\frac{1}{2}\frac{v'^2}{v_F^2}\right)-\left(\frac{v_F}{4v}-\frac{v}{4v_F}+\frac{v_F}{v}\right)+\frac{2V_0^2+v_F^2}{2(V_0^2+v_F^2)}$$

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Thank You