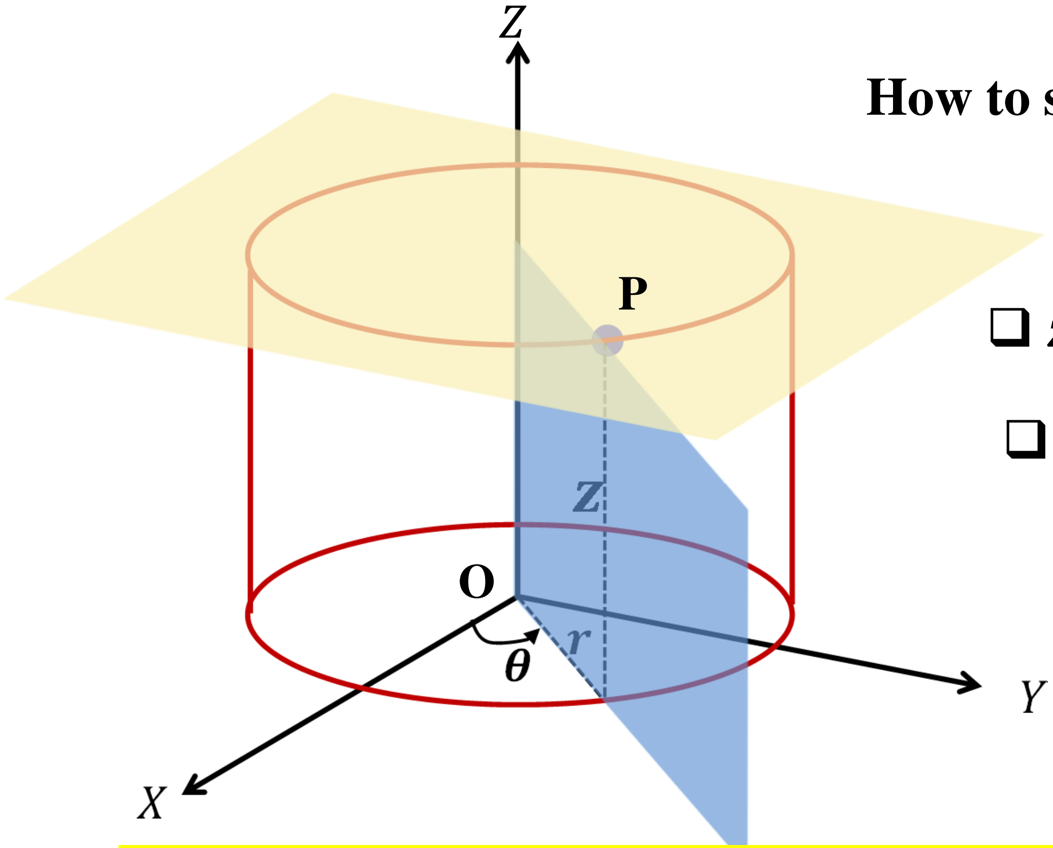
PH101: PHYSICS1

Lecture 3

II. Cylindrical coordinate system (r, θ, z)



How to specify a point 'P' in space ?

$$(\boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{z})$$

- ❑ $\boldsymbol{z}$-Height from the XY plane
- ❑ (r, θ) Coordinate of the foot of the point in XY plane

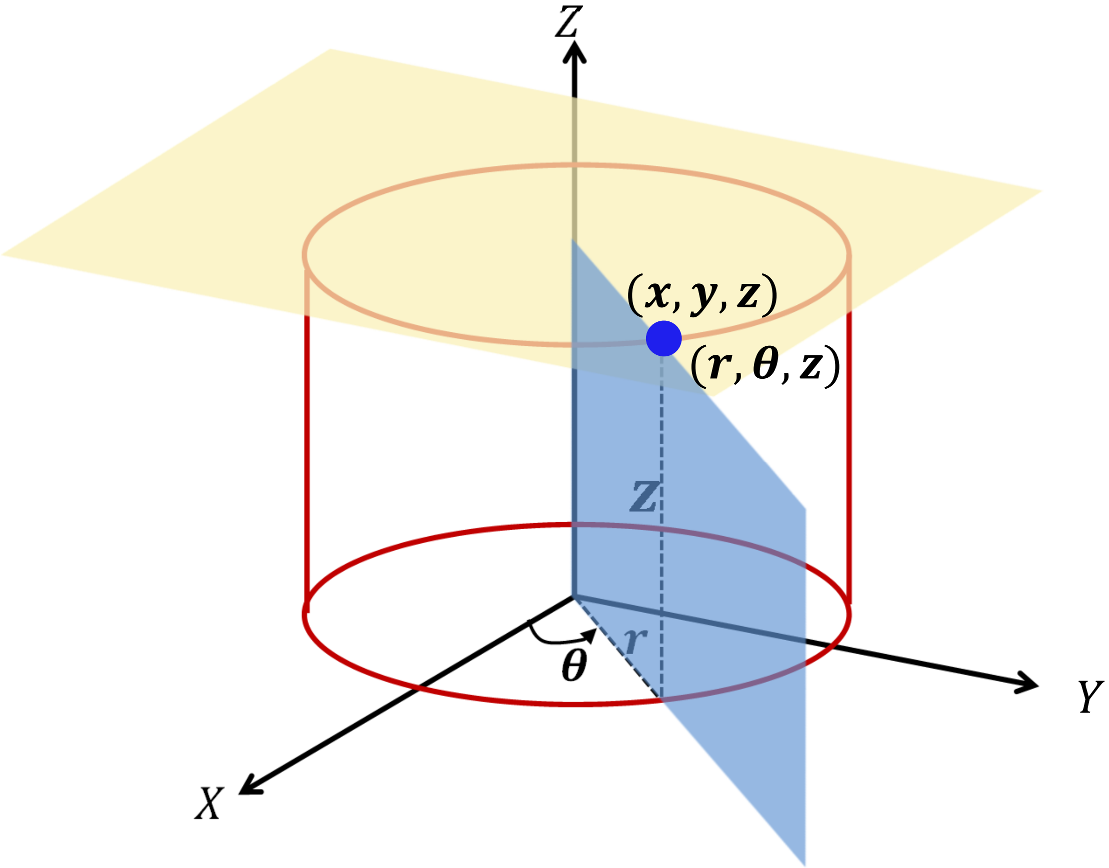
❑ $(\boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{z})$ coordinates system is known as cylindrical coordinate system

Why the name cylindrical?

❑ Point 'P' is the intersection of three surfaces: A cylindrical surface $\boldsymbol{r} = \boldsymbol{constant}$; A half plane containing z-axis with **$\boldsymbol{\theta}$=constant** and a plane **$\boldsymbol{z}$=constant**.

Coordinate transformation: Cartesian to cylindrical

Transformation equation is very similar to polar coordinate with additional *z*-coordinate.



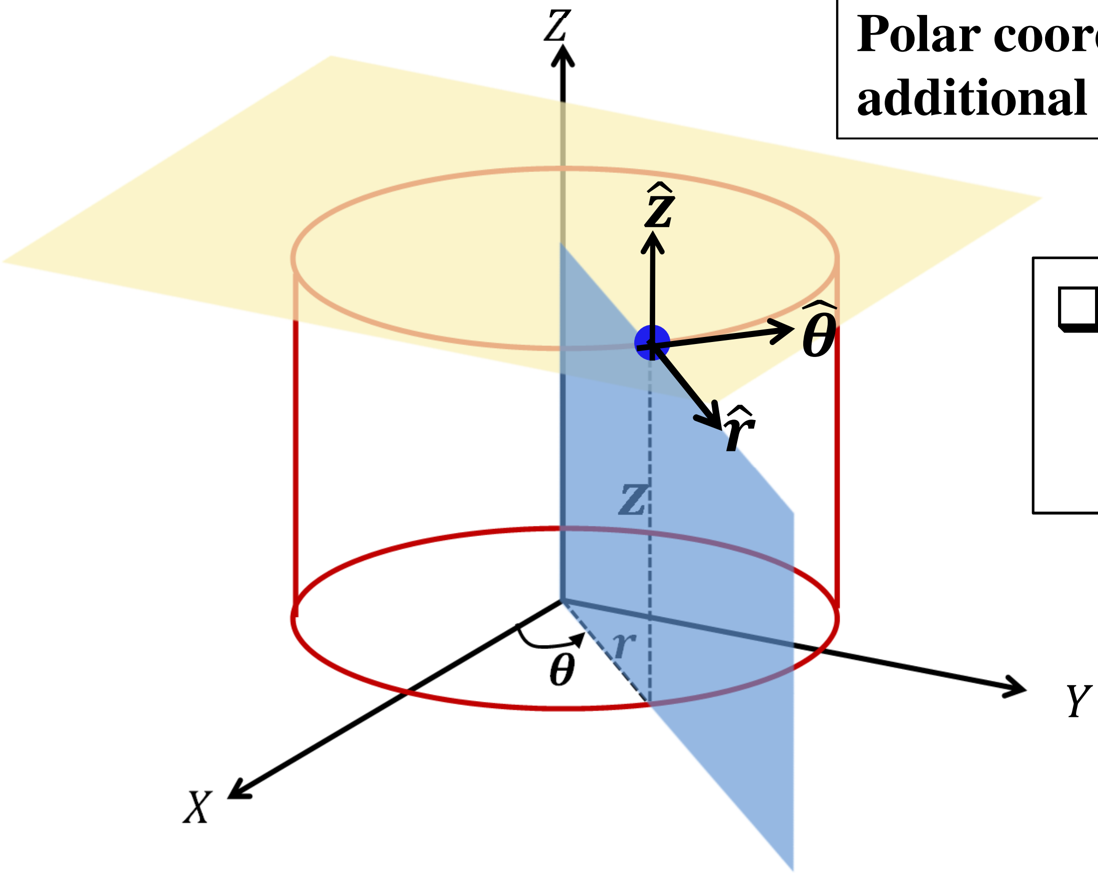
$$\boldsymbol{x = r\cos\theta}$$
$$\boldsymbol{y = r\sin\theta}$$
$$\boldsymbol{z = z}$$

Reverse transformation

$$\boldsymbol{r = \left(x^2 + y^2\right)^{1/2}}$$
$$\boldsymbol{\theta = \tan^{-1}\frac{y}{x}}$$
$$\boldsymbol{z = z}$$

Note: Instead of (r, θ) many books use notation (ρ, φ).

Unit vectors in cylindrical coordinate system



Polar coordinate unit vectors ($\hat{r}, \hat{\theta}$) + additional unit vector in the z −direction.

- $\hat{r}, \hat{\theta}$ **and** $\hat{z}$ are unit vectors along increasing direction of coordinates r, θ **and** z.

$$\hat{r} = \cos\theta\ \hat{x} + \sin\theta\ \hat{y}$$
$$\hat{\theta} = -\sin\theta\ \hat{x} + \cos\theta\ \hat{y}$$
$$\hat{z} = \hat{z}$$

$\hat{r}$ and $\hat{\theta}$ are **orthogonal** but their directions depend on location.

Position, Velocity, Acceleration, Newton's law in cylindrical coordinate system

Vector components are very similar to polar coordinate+ z −component

Position vector → $\overrightarrow{OP} = \vec{R} = r\hat{r} + z\hat{z}$

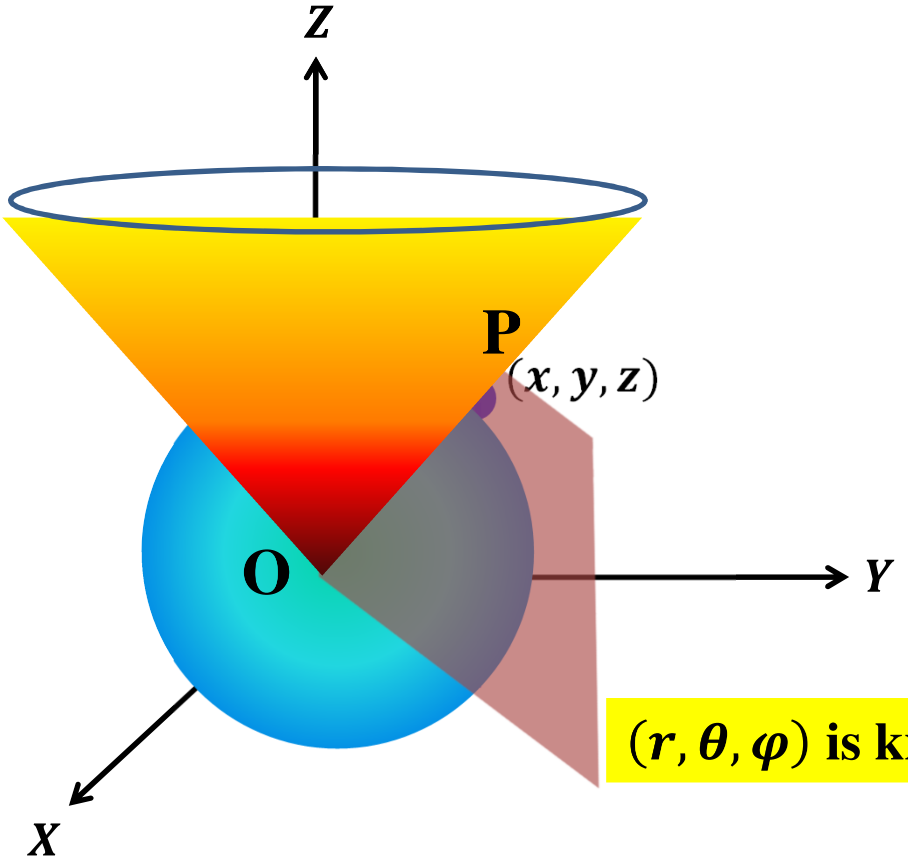
Velocity → $\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + \dot{z}\hat{z}$

Acceleration → $\vec{a}=(\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} + \ddot{z}\hat{z}$

Newton's law →

$$\vec{F} = F_r\hat{r} + F_\theta\hat{\theta} + F_z\hat{z}$$
$$= m[(\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} + \ddot{z}\hat{z}]$$

III. Spherical polar coordinate system



$\boldsymbol{r}$ → Radial distance from origin

$\boldsymbol{\theta}$ →Angle of radial vector with z-axis.

$\boldsymbol{\varphi}$ → Angle between X-axis and the projection of radial vector in XY plane

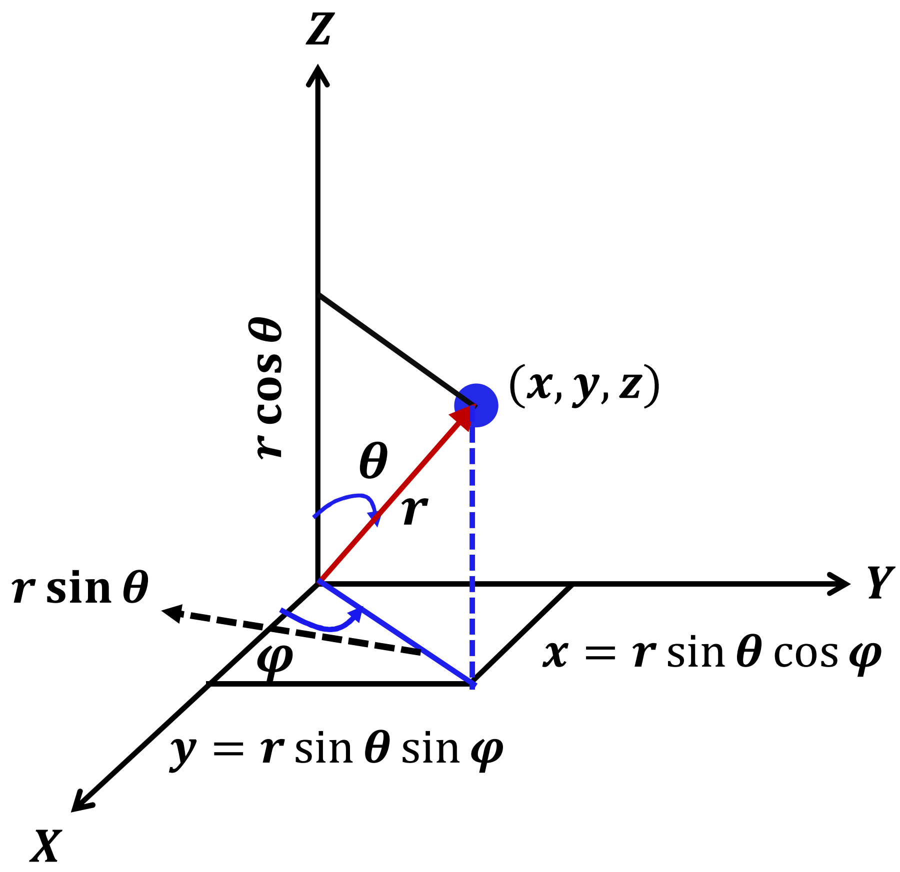
$(\boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{\varphi})$ is known as spherical polar coordinate

Note that point (r, θ, φ) is at the intersection of three surfaces

- ❑ A sphere where $\boldsymbol{r}$ =**Constant**
- ❑ A cone about z=axis with **$\boldsymbol{\theta}$=constant**.
- ❑ A half plane containing z-axis and **$\boldsymbol{\varphi}$= constant**

Be careful, notations are different.
$\boldsymbol{r}$ and $\boldsymbol{\theta}$ are not planer coordinate.

Connection of spherical polar with cartesian



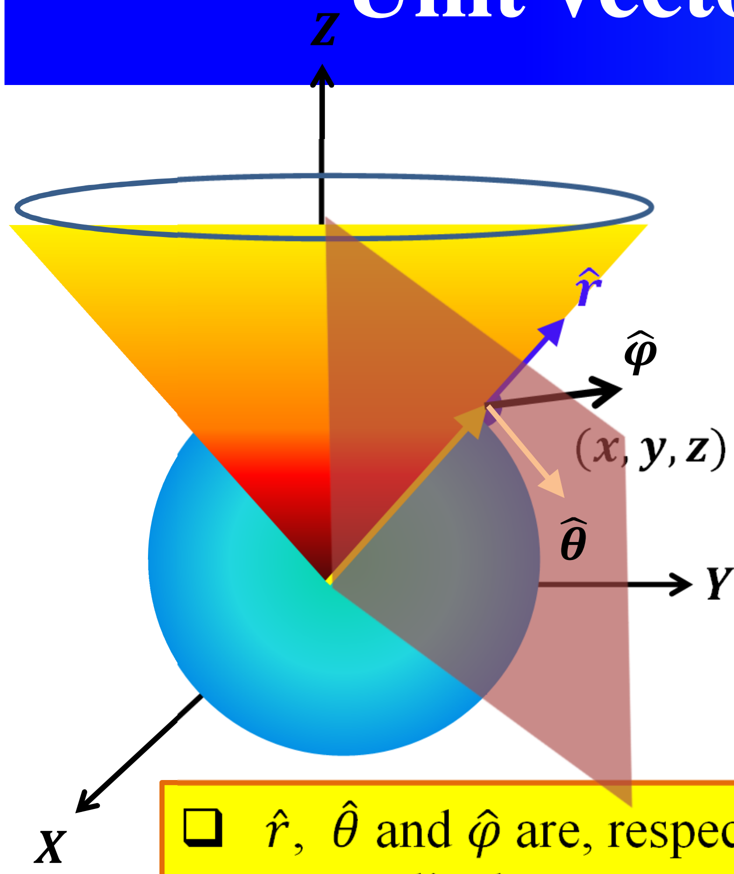
Transformation relations

$$x = r\sin\theta\cos\varphi$$
$$y = r\sin\theta\sin\varphi$$
$$z = r\cos\theta$$

Hence

$$r = \left(x^2 + y^2 + z^2\right)^{1/2}$$
$$\theta = \tan^{-1}\frac{\left(x^2 + y^2\right)^{1/2}}{z}$$
$$\varphi = \tan^{-1}\frac{y}{x}$$

Unit vectors in spherical polar



- Position vector

  $\vec{r} = r \sin\theta \cos\varphi \, \hat{x} + r \sin\theta \sin\varphi \, \hat{y} + r \cos\theta \, \hat{z}$

$$\hat{r} = \frac{\vec{r}}{r}$$

$$\hat{r} = \sin\theta \cos\varphi \, \hat{x} + \sin\theta \sin\varphi \, \hat{y} + \cos\theta \, \hat{z}$$

- $\hat{r}$, $\hat{\theta}$ and $\hat{\varphi}$ are, respectively perpendicular to $r = const.$ $\theta = const.$ $\varphi = const.$

- $\hat{\varphi}$ is the unit vector perpendicular to φ =constant plane (***rz plane***),

I,e, perpendicular to unit vectors $\hat{r}$ and $\hat{z}$

Thus $$\hat{\varphi} = \frac{\hat{r}\times\hat{z}}{|\hat{r}\times\hat{z}|}$$

$$= \frac{(-\hat{x}\sin\theta\sin\varphi + \hat{y}\sin\theta\cos\varphi)}{\sin\theta}$$

$$\hat{\varphi} = -\hat{x}\sin\varphi + \hat{y}\cos\varphi$$

- $$\hat{\theta} = \frac{(\hat{\varphi}\times\hat{r})}{|\hat{\varphi}\times\hat{r}|}$$

$$= \hat{x}\cos\varphi\cos\theta + \hat{y}\sin\varphi\cos\theta - \hat{z}\sin\theta$$

Unit vectors in spherical polar

$$\vec{r} = r\,sin\,\theta\,cos\,\varphi\,\hat{x} + r\,sin\,\theta\,sin\,\varphi\,\hat{y} + r\cos\theta\,\hat{z}$$

$$\hat{r} = \frac{\vec{r}}{r} = \hat{x}\cos\varphi\sin\theta + \hat{y}\sin\varphi\sin\theta + \hat{z}\cos\theta$$

$$\hat{\theta} = \frac{\partial\hat{r}}{\partial\theta} = \hat{x}\cos\varphi\cos\theta + \hat{y}\sin\varphi\cos\theta - \hat{z}\sin\theta$$

$\hat{\varphi} = -\hat{x}\sin\varphi + \hat{y}\cos\varphi$ ($\equiv \hat{\theta}$ *of Plane Polar* $\theta \rightarrow \varphi$!)

Partial differential of unit vectors in spherical polar

Unit vectors in spherical polar coordinate are function of θ and φ only.

$$\frac{\partial \hat{r}}{\partial \theta} = \frac{\partial}{\partial \theta}(\hat{x}\sin\theta\cos\varphi + \hat{y}\sin\theta\sin\varphi + \hat{z}\cos\theta)$$
$$= (\hat{x}\cos\theta\cos\varphi + \hat{y}\cos\theta\sin\varphi - \hat{z}\sin\theta) = \hat{\theta}$$

$$\frac{\partial \hat{r}}{\partial \varphi} = \frac{\partial}{\partial \varphi}(\hat{x}\sin\theta\cos\varphi + \hat{y}\sin\theta\sin\varphi + \hat{z}\cos\theta)$$
$$= (-\hat{x}\sin\theta\sin\varphi + \hat{y}\sin\theta\cos\varphi) = \sin\theta\,\hat{\varphi}$$

Additionally, you may verify:

$$\frac{\partial \hat{\theta}}{\partial \theta} = \frac{\partial}{\partial \theta}(\hat{x}\cos\theta\cos\varphi + \hat{y}\cos\theta\sin\varphi - \hat{z}\sin\theta)$$
$$= (-\hat{x}\sin\theta\cos\varphi - \hat{y}\sin\theta\cos\varphi - \hat{z}\cos\theta) = -\hat{r}$$

$$\frac{\partial \hat{\theta}}{\partial \varphi} = \frac{\partial}{\partial \varphi}(\hat{x}\cos\theta\cos\varphi + \hat{y}\cos\theta\sin\varphi - \hat{z}\sin\theta)$$
$$= (-\hat{x}\cos\theta\sin\varphi + \hat{y}\cos\theta\cos\varphi) = -\cos\theta\,\hat{r}$$

Velocity in spherical polar coordinate

$$\vec{r} = r\sin\theta\cos\varphi\,\hat{x} + r\sin\theta\sin\varphi\,\hat{y} + r\cos\theta\,\hat{z}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{r})$$
$$= \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt}$$
$$= \dot{r}\hat{r} + r\left(\frac{\partial\hat{r}}{\partial\theta}\frac{d\theta}{dt} + \frac{\partial\hat{r}}{\partial\varphi}\frac{d\varphi}{dt}\right)$$
$$= \dot{r}\hat{r} + r\left(\dot{\theta}\hat{\theta} + \sin\theta\dot{\varphi}\,\hat{\varphi}\right)$$

Chain rule

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\dot{\varphi}\,\hat{\varphi}$$

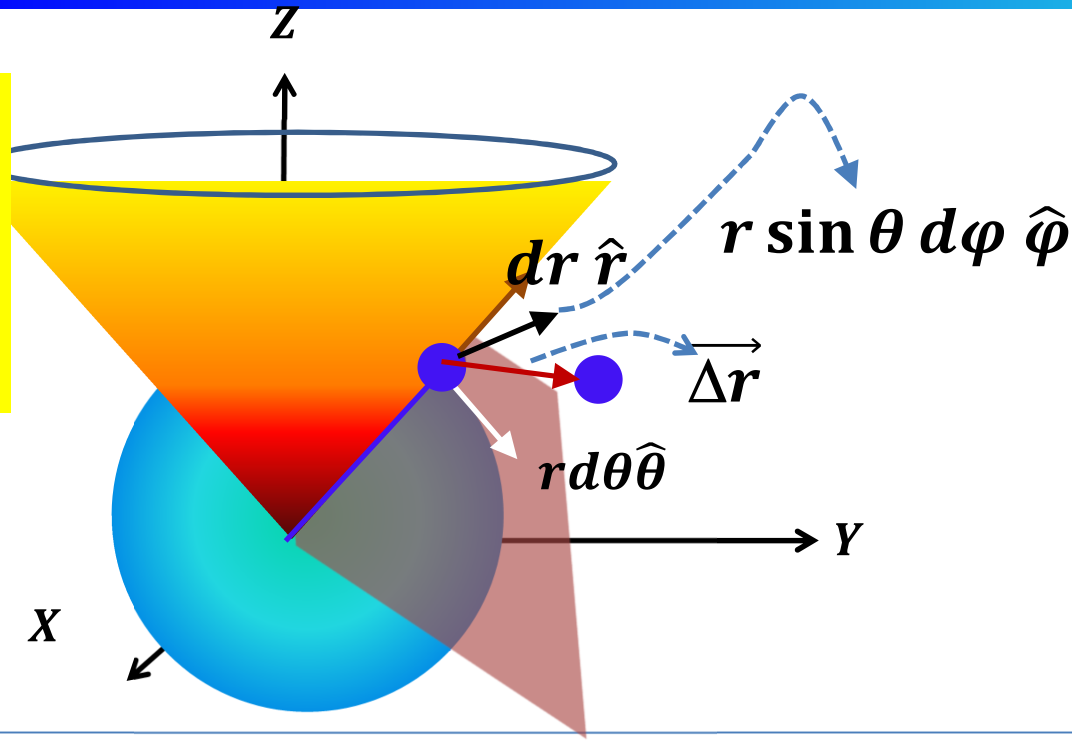
Acceleration

You must try to prove this

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2 sin^2\theta\right)\hat{r} + (r\ddot{\theta} - 2\dot{r}\dot{\theta} - r\dot{\varphi}^2\sin\theta\cos\theta)\hat{\theta} + \left(r\ddot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta + 2\dot{r}\dot{\varphi}\sin\theta\right)\hat{\varphi}$$

Velocity –to remember!

Elementary displacement in arbitrary direction $\overrightarrow{\Delta \boldsymbol{r}}$ in Δt



$$\overrightarrow{\Delta \boldsymbol{r}} = dr\hat{\boldsymbol{r}} + rd\theta\hat{\boldsymbol{\theta}} + r\sin\theta\, d\varphi\, \hat{\boldsymbol{\varphi}}$$

$$\vec{v} = \frac{\overrightarrow{\Delta \boldsymbol{r}}}{\Delta \boldsymbol{t}} = \frac{dr}{\Delta t}\hat{r} + \frac{rd\theta}{\Delta t}\hat{\theta} + \frac{r\sin\theta\, d\varphi}{\Delta t}\hat{\varphi}$$

$$\vec{\boldsymbol{v}} = \dot{\boldsymbol{r}}\hat{\boldsymbol{r}} + \boldsymbol{r}\dot{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}} + \boldsymbol{r}\sin\boldsymbol{\theta}\dot{\boldsymbol{\varphi}}\,\hat{\boldsymbol{\varphi}}$$

Done!

Well, We are done with the necessary mathematical concepts!

Ok, Now in to Physics!