PH101: PHYSICS1

Lecture 6

Euler-Lagrange's equation

The **difficulty** with Newton's Scheme

Newton's 2nd Law: $\vec{\mathrm{F}} = \mathrm{m}\frac{d^2\ddot{\vec{r}}}{dt^2}$

$$m\frac{d^2\ddot{\vec{r}}}{dt^2} = \vec{F}_e + \vec{\boldsymbol{f}}_{\boldsymbol{c}} \qquad \text{where, } \vec{\mathrm{F}} = \vec{F}_e + \vec{f}_c$$

$\vec{\boldsymbol{F}}_{\boldsymbol{e}} = \boldsymbol{vector}$ sum of the external forces ($\boldsymbol{known}$)

$\vec{\boldsymbol{f}}_{\boldsymbol{c}} = \boldsymbol{vector}$ sum of the Constraint forces ($\boldsymbol{unknown}$)

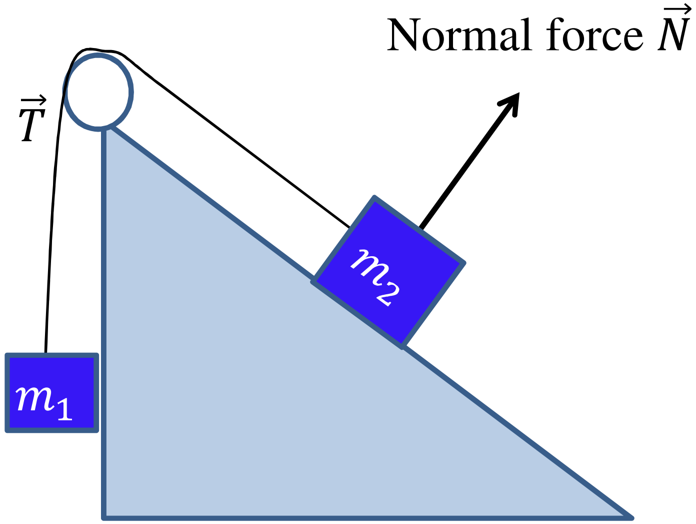
To solve the equation(s), we need to know all the **constrain force(s)** $\vec{\boldsymbol{f}}_{\boldsymbol{c}}$

General difficulty with Newton's Scheme:

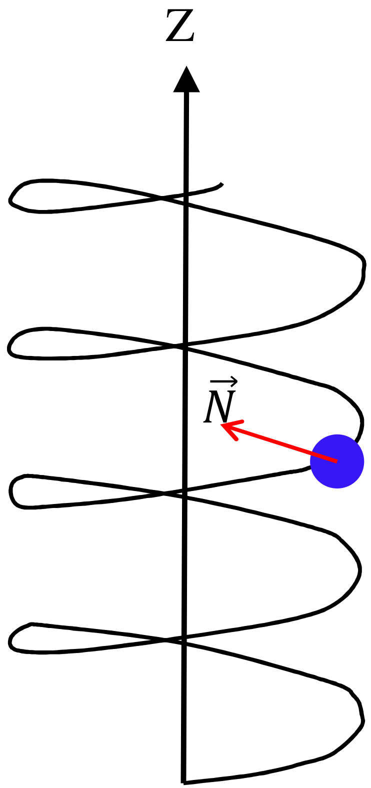
- Constrain equations $f(x,t) = 0$ is known for a problem but constraint force(s) is still unknown. Finding the constraint forces are not always very obvious.

- Handling too many constrain forces and their components for a system of particles is very cumbersome.

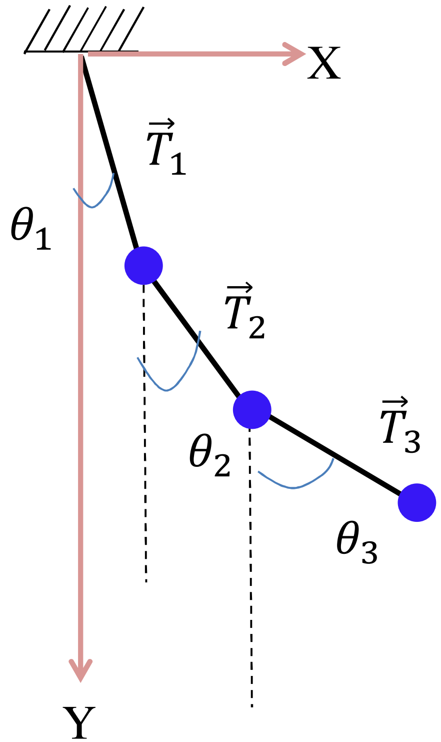
Constraints Forces…the difficulty with Newton's Scheme



Two masses are connected by string of length l. Mass m_2 slides down the inclined frictionless plane.
Different sort of constraint forces involved



A particle is sliding down a spiral: ***Direction of constraint forces changing continuously***



Too many constraint forces and their components to deal with

Alternative to Newton's Scheme

Our life would be easier if
(1) There exists a **new recipe**, **an alternative to Newton's Scheme**, which **does not require** to consider **constraint forces**,
instead **utilize** the **constraint relations.**

(2) Further, it will be ideal if the new recipe does not depend on any specific coordinate system. Then we may be able to utilize the symmetry of the problem, which would simplify the dynamical equations.

Good News

There is one alternative formalism to Newtonian scheme which **does not require** to consider **constraint forces** and also **independent** of the choice of the **coordinate system:**

The **Euler-Largrange** equations**!**

Joseph-Louis Lagrange, engraving
by Robert Hart. British Museum.

Joseph-Louis Lagrange

Born, 1736, Turin, Italy. – Died, 1813, Paris, France.

Known for his contributions to:

Analytical mechanics
Calculus of Variations
Astronomy

Lagrangian formalism

Introduce a new function:

$$\boldsymbol{Lagrangian}:\quad \boldsymbol{L(q_1, \ldots, q_n, \dot{q}_1, \ldots., \dot{q}_n, t)}$$

Defined as,

Potential energy

$$\boldsymbol{L(q_1, \ldots, q_n, \dot{q}_1, \ldots., \dot{q}_n, t) = T - \mathrm{V}}$$

Kinetic energy

Where generally,

$$\boldsymbol{T(q_1, \ldots, q_n, \dot{q}_1, \ldots., \dot{q}_n, t)} \qquad \text{and}$$

$\mathrm{V}(\boldsymbol{q_1, \ldots, q_n})$ - function of positions only(usually)!

$\boldsymbol{q_j}$**-generalized coordinates** and $\boldsymbol{\dot{q}_j}$ -**generalized velocities**

$\boldsymbol{j}$ = 1, 2, …n; n – degree of freedom (n ≤ 3N)

Euler -Lagrange equations

$$L(q_j, \dot{q}_j, t) = T(q_j, \dot{q}_j, t) - \mathrm{V}(q_j)$$

Everything about this system is embodied in this scalar function *L!*

The **Euler-Langange's** equations, or simply **Langange's** equations,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

n- such equations! One equation for each generalized coordinate.

- To define the Largrangian, potential $V(q_1, \dots, q_n)$ must exist, i,e the **forces are conservative.** (***We shall discuss extension to non-conservative forces later!***)
- There is no need to consider constrain forces in Lagrange's formalism
- The form of Lagrange's equations are independent of choice of generalized coordinates chosen.

The recipe of Lagrangian!

I. (a) Recognize, & obtain the **constraint relations**, (b) determine th **DOF**, and (c) choose appropriate **generalized coordinates**!

II. Write down the **total** kinetic energy T and potential energy V of the **whole system** in terms of the **Cartesian** coordinates, **to begin with**!

$$T = \sum_{i=1}^{N} \frac{1}{2} m_i (\dot{x}_i^2 + \dot{y}_i^2 + z_i^2) \qquad \& \qquad V = V(x_i, y_i, z_i) \qquad | \text{ i} = 1, \text{N}$$

III. Obtain appropriate transformation equations (Cartesian -⟶ generalized coordinates) using constraint relations:

$$\boxed{\begin{aligned} x_i &= x_i\,(q_1, \ldots, q_n,\ t) \\ y_i &= y_i\,(q_1, \ldots, q_n,\ t) \\ z_i &= z_i\,(q_1, \ldots, q_n,\ t) \end{aligned}}$$

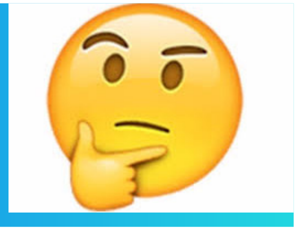
IV. Convert T and V from Cartesian to suitable **generalized -coordinates** ($\boldsymbol{q_j}$) and **generalized velocities** ($\dot{\boldsymbol{q}}_j$) to write L as,

$$\boldsymbol{L(q_j, \dot{q}_j, t) = T(q_j, \dot{q}_j, t) - \mathrm{V}(q_j)} \qquad j = 1, \text{n}$$

V. Now Apply E-L equations:

$$\boldsymbol{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0} \qquad \boldsymbol{for\ each}\ \textbf{j = 1, n!}$$

Questions in your mind?



I guess, there are many unanswered questions which are circulating in your mind

(1) What is the origin of Lagrange's equation?

(2) Is there any **proof** on the validity of the method?

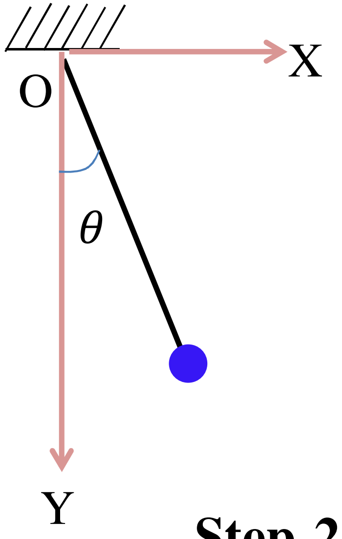
(3) Is their any direct correlation of Lagrange's equation with Newton's laws?

(4) How come constraint forces could be "ignored" in Lagrange's formalism?

We shall answer the questions later, for the moment believe me that Lagrange's equations are correct!

Initially we shall discuss, how to easily solve dynamical problems with Lagrangian formalism!

Solving problem with Lagrange recipe
Example-1: Simple pendulum



Step-1:

a) Obtain the Constrains relations:

$z = 0,$ $\qquad x^2 + y^2 = l^2$

b) Determine the DOF= 1

c) Choice of generalized coordinate: θ

Step-2: *Write T and V in Cartesian*

Kinetic energy $T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2)$

Potential energy $V = -mgy$ (*w.r.to* "O")

Step-3: *Identify the transformation relations,* $\boxed{x = l\ \sin\theta;\ y = l\ \cos\theta}$

$\boxed{\dot{x} = l\ cos\,\theta\ \dot{\theta};\ \dot{y} = -l\ sin\,\theta\ \dot{\theta}}$

Example-1 continued….

Step-4: Convert $\boldsymbol{T}$ *and* $\boldsymbol{V}$ *to generalized coordinates & velocities using Step*3.

$$T = \frac{1}{2}m\left[\left(l\ cos\,\theta\ \dot{\theta}\right)^2 + \left(-l\ sin\,\theta\ \dot{\theta}\right)^2\right] = \frac{1}{2}ml^2\dot{\theta}^2$$

$$V = -mgl\ \cos\theta$$

$$L = T - V$$

$$L = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\,cos\,\theta$$

Step-5: *Employ Lagrange's equation for each generalized coordinates*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0;\ in\ the\ given\ problem\ \boldsymbol{q_j} = \boldsymbol{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \qquad \textbf{-(1)}$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

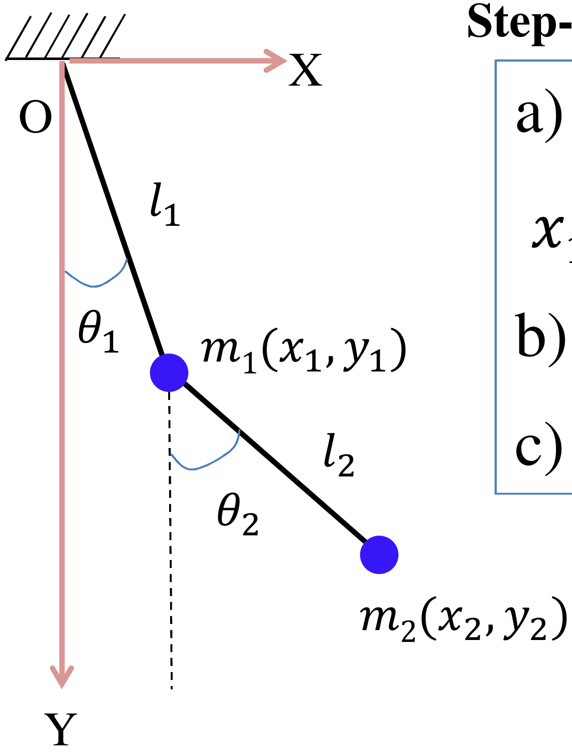
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{d}{dt}\left(ml^2\dot{\theta}\right) = ml^2\ddot{\theta}$$

$$\frac{\partial L}{\partial \theta} = -mgl\sin\theta$$

(1) => $$\boldsymbol{ml^2\ddot{\theta} + mgl\sin\theta = 0}$$

Done!

Example-2 Double Pendulum



Step-1:

a) Obtain the Constrains relations: $z_1 = 0; z_2 = 0;$

$x_1{}^2 + y_1{}^2 = l_1{}^2; \quad (x_2 - x_1)^2 + (y_2 - y_1)^2 = l_2{}^2$

b) Determine the DOF= 2

c) Choice of generalized coordinates: θ_1, θ_2

Step-2: *Write the total* $\boldsymbol{T}$ *and total* $\boldsymbol{V}$ *in Cartesian:*

Kinetic energy $T = \frac{1}{2} m_1(\dot{x}_1{}^2 + \dot{y}_1{}^2) + \frac{1}{2} m_2(\dot{x}_2{}^2 + \dot{y}_2{}^2)$

Potential energy $V = -m_1 g y_1 - m_2 g y_2$ (*w.r.to* "O")

Example-2 continued….

Step-3: *Identify the transformation relations:*

$$x_1 = l_1 \sin\theta_1; \quad y_1 = l_1 \cos\theta_1; \quad \dot{x}_1 = l_1 \, cos\,\theta_1\,\dot{\theta}_1; \quad \dot{y}_1 = -l_1 \, sin\,\theta_1\,\dot{\theta}_1$$

$$x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2; \qquad y_2 = l_1 \cos\theta_1 + l_2 \cos\theta_2;$$

$$\dot{x}_2 = l_1 \, cos\,\theta_1\,\dot{\theta}_1 + l_2\, cos\,\theta_2\,\dot{\theta}_2; \qquad \dot{y}_2 = -l_1 \, sin\,\theta_1\,\dot{\theta}_1 - l_2 \, sin\,\theta_2\,\dot{\theta}_2$$

Step–4:

Convert $\boldsymbol{T}$ *and* $\boldsymbol{V}$ *to generalized coordinates & velocities using Step*3.

$$T = \frac{1}{2}m_1({\dot{x}_1}^2 + {\dot{y}_1}^2) + \frac{1}{2}m_2({\dot{x}_2}^2 + {\dot{y}_2}^2)$$

$$T = \frac{1}{2}m_1\left(l_1\dot{\theta}_1\right)^2 + \frac{1}{2}m_2\left[\left(l_1\dot{\theta}_1\right)^2 + \left(l_2\dot{\theta}_2\right)^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right]$$

$$V = -m_1 g y_1 - m_2 g y_2 = -m_1 g l_1 \cos\theta_1 - m_2 g(l_1 \cos\theta_1 + l_2 \cos\theta_2);$$

Example-2 continued….

…continuing **Step–4**: $L = T - V$

$$L = \frac{1}{2} m_1 (l_1\dot{\theta}_1)^2 + \frac{1}{2} m_2 \; [(l_1\dot{\theta}_1)^2 + (l_2\dot{\theta}_2)^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)]$$

$$+ m_1 g l_1 \cos\theta_1 + m_2 g (l_1 \cos\theta_1 + l_2 \cos\theta_2);$$

Step-5: *Employ Lagrange's equation for each generalized coordinates,* θ_1 *&* θ_2

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0 \quad -[\mathbf{1}] \qquad \text{and} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0; \quad -[\mathbf{2}]$$

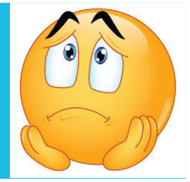
$$\frac{\partial L}{\partial \dot{\theta}_1} = m_1 l_1{}^2 \dot{\theta}_1 + m_2 l_1{}^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = m_1 l_1{}^2 \ddot{\theta}_1 + m_2 l_1{}^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) -$$

$$m_2 l_1 l_2 \dot{\theta}_2 \; (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2)$$

$$\frac{\partial L}{\partial \theta_1} = -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - m_1 g l_1 \sin\theta_1 - m_2 g \, l_1 \sin\theta_1$$

Example-2 continued….



So the first E-L equation, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0$

$$\Rightarrow \quad m_1 l_1{}^2 \ddot{\theta}_1 + m_2 l_1{}^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_2 (\dot{\theta}_1 - \dot{\theta}_2) \sin(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_1 g l_1 \sin\theta_1 + m_2 g l_1 \sin\theta_1 = 0$$

$$\Rightarrow \quad \boxed{(m_1 + m_2) l_1{}^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}_2{}^2 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g l_1 \sin\theta_1 = 0}$$

Similarly, the 2nd E-L equation, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0;$

$$\Rightarrow \quad \boxed{m_2 l_2{}^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1{}^2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin\theta_2 = 0}$$

Solved the problem? **Yes!** As far as **we are concerned**!

But don’t blame it on Lagrange!

The complex nature of these equations are not particular to Lagrange’s Scheme! They are inherent to the system!

Newton’s scheme would also result in equivalent/identical Differential equations!!

Comments!

Let's first note that: the resulting differential equations are,

a) Second order ($\frac{d^2()}{dt^2}$; -in time)

b) Coupled (such as involving, $\dot{\theta}_1\dot{\theta}_2$) &

c) Non-linear in nature (hence more complex in nature!)

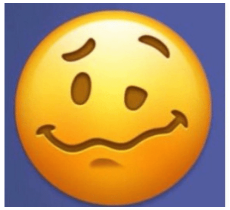
(Linear diff. eq. have the form: $P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x) = 0$)

Often such equations are very hard to solve!

~~Simple~~ Pendulum for example!

So for **most** problems you may **stop at these E-L (differential-) equations,** say in exams/tutorials!

Unless you are asked **explicitly** to "**solve**" the differential equations! (-only in very simple cases!)



Summery

Choice of generalized coordinates and transformation relations automatically includes constrain relations into the problem. **Thus, Lagrange's formalism does not require to consider constraint forces, rather constraint conditions are smartly utilized.**

Lagrangian $L(q_1, \dots, q_n, \dot{q}_1, \dots., \dot{q}_n, t) = T - V$, where T and V **SHOULD** be expressed in terms of **generalized** coordinates & velocities, before the E-L equations are invoked/applied!

Largrangian satisfies equations of motion:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

One equation for each generalized coordinate/DOF.

The form of Lagrange's equations are independent of choice of any set of generalized coordinates. (eg., same for r & θ)

What You **Should** Revise?

- Basic rules of partial differentiation

- Familiarity with different coordinate systems (*Cartesian, Plane polar, Cylindrical, Spherical polar*)

- Conservative forces and potentials

- Motion under constraints:

a) Recognizing and writing down the constraint relations,
b) Determine the degree's of freedom and
c) making proper choice of generalized coordinates (considering also, the **symmetry** of the system).
d) Developing appropriate transformations
   (Cartesian $\rightarrow$ generalized)

Questions?