PH101

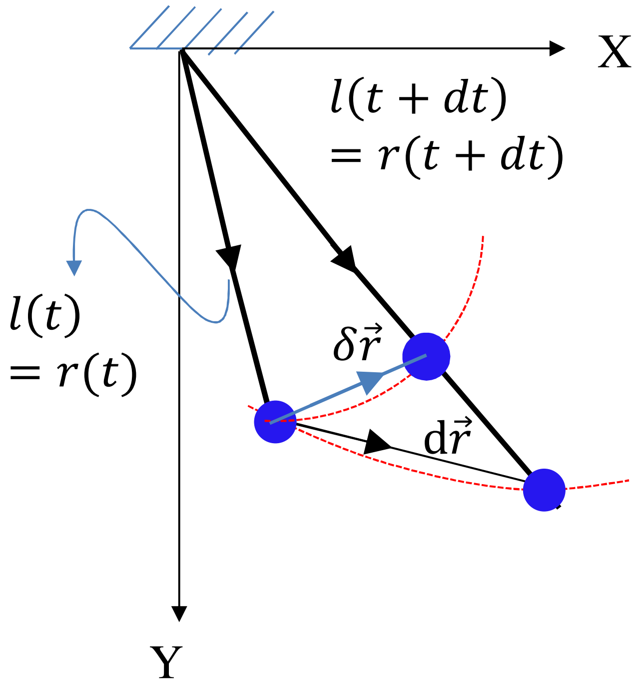
Lecture 8

D’Alembert’s principle of virtual work,

Derivation of Lagrange’s equation from D’Alember’s principle

Real vs Virtual displacement

Simple pendulum with a variable string length $\boldsymbol{l(t)}$
[Time dependent constraint]



Real displacement of the bob in time d$\boldsymbol{t}$ is given by
$\mathrm{d}\vec{r} = \vec{r}(t+dt) - \vec{r}(t)$

Let's **imagine** any **instantaneous arbitrary displacement** at time t (that is, *without allowing time to change,* $dt = 0$)
AND consistent with the constraint relations at time $\boldsymbol{t}$?

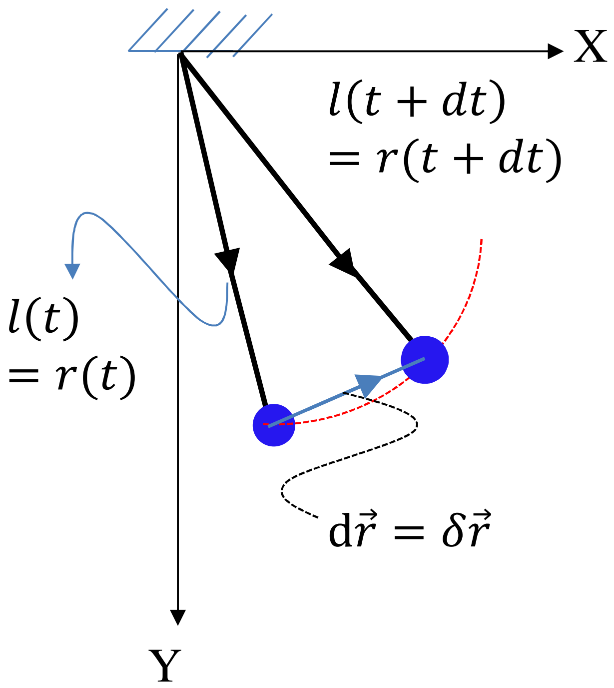
Imaginary, instantaneous displacement which is consistent with the constrain relation at a given instant (i,e. without allowing real time to change) is called *Virtual displacement* and denoted by $\delta\vec{r}$ (for infinitesimal case)

Real vs Virtual displacement

- By definition a virtual infinitesimal displacement is given by

$$\boldsymbol{\delta x_i = dx_i}\Big|_{\boldsymbol{dt=0}}$$

- If the constraint is not time dependent, the real and virtual displacements matches each other.



Virtual displacement in generalized coordinates

- Consider a system of N particles with k constrains, DOF, $\boldsymbol{n = 3N - k}$
- Cartesian coordinates, $\vec{r}_i = \vec{r}_i\,(x_1, y_1, z_1, \ldots x_N, y_N, z_N)\;|\;(i = 1, \ldots \ldots, N)$
- Generalized coordinates $q_j \quad | \;(j = 1, \ldots \ldots, n)$
- Virtual displacements of the particles $\delta\vec{r}_1, \delta\vec{r}_2. \ldots \ldots, \delta\vec{r}_N$
- Virtual displacements of the particles in the generalized coordinates $\delta q_1, \delta q_2. \ldots \ldots, \delta q_n$ can be found from given transformation relations

$$\boldsymbol{\vec{r}_1 = \vec{r}_1(q_1, q_2, \ldots \ldots, q_n, t)}$$
$$\boldsymbol{\vec{r}_2 = \vec{r}_2(q_1, q_2, \ldots \ldots, q_n, t)}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\boldsymbol{\vec{r}_N = \vec{r}_N(q_1, q_2, \ldots \ldots, q_n, t)}$$

$$\delta\vec{r}_i = \sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j}\,\delta q_j$$

$3N$ coordinates, not independent

$n = 3N - k$ generalized coordinates, independent

Note: There is no $\frac{\partial \vec{r}_i}{\partial t}\delta t$, as virtual displacement is instantaneous without allowing time to change, δt=0

Virtual work done

Real work done: Work done due to real displacement ($d\vec{r}$) of a particle acted on by total force $\vec{F}$ is given by

$$\boldsymbol{dW = \vec{F}.d\vec{r}}$$

As you can always **imagine** an instantaneous displacement (without allowing time to change), known as virtual displacement ($\delta\vec{r}$), and hence you can always define a scalar function

$$\delta W = \vec{F}.\delta\vec{r}$$

This scalar function is know called **Virtual work done.**

Note: 'Virtual work' is different from 'Real work', as virtual displacement is imagined without allowing time to change.

Virtual work done for a system of particles

Consider a system of particles and $\vec{F}_1$, $\vec{F}_2$, …, $\vec{F}_N$ are the forces on 1,2 …. N_{th} particles, then
Total virtual work done

$$\boldsymbol{\delta W} = \sum_{i=1}^{N} \vec{\boldsymbol{F}}_{\boldsymbol{i}} . \boldsymbol{\delta} \vec{\boldsymbol{r}}_{\boldsymbol{i}}$$

Here, force on each particle, $\vec{F}_i$ is the sum of external force and also forces of constraints.

$$\vec{\boldsymbol{F}}_{\boldsymbol{i}} = \vec{\boldsymbol{F}}_{\boldsymbol{ie}} + \vec{\boldsymbol{f}}_{\boldsymbol{ic}}$$

Where,
$\vec{\boldsymbol{F}}_{\boldsymbol{ie}}$ is the external applied force on i_{th} particle.
$\vec{\boldsymbol{f}}_{\boldsymbol{ic}}$ is the constraint force

Virtual work for a dynamical system

Newton's second law reads as

$$\boldsymbol{m\ddot{\vec{r}} = \vec{F}}$$

Total force$(\vec{F}) = Applied\ force(\vec{F}_e) + constraint\ force\ (\vec{f}_c)$

$$\boldsymbol{m\ddot{\vec{r}} = \vec{F}_e + \vec{f}_c}$$

Taking dot product with an infinitesimal virtual displacement $\delta\vec{r}$

$$\boldsymbol{m\ddot{\vec{r}} \cdot \delta\vec{r} = (\vec{F}_e + \vec{f}_c) \cdot \delta\vec{r}} \qquad \boxed{1}$$

Now, virtual displacement is instantaneous (frozen in time & imaginary) AND **consistent with ALL the constraint relations.**

As $\delta\vec{r}$ are perpendicular to $\vec{f}_c$, thus virtual work due to constraint force is zero, $\boldsymbol{\vec{f}_c \cdot \delta\vec{r} = 0}$

D'Alembert's principle of virtual work

If virtual work done by the **constraint forces is** ($\vec{f}_c \cdot \delta\vec{r} = 0$) (from eq.-1),

$$\left(\vec{F}_e - m\ddot{\vec{r}}\right) \cdot \delta\vec{r} = 0$$

D'Alembert's principle of Virtual work

Now, for a general system of N particles having virtual displacements, $\delta\vec{r}_1$, $\delta\vec{r}_2$,….., $\delta\vec{r}_N$,

$$\sum_{i=1}^{N} (\vec{F}_{ie} - m_i\ddot{\vec{r}}_i) \cdot \delta\vec{r}_i = 0$$

$\vec{F}_{ie}$ →Applied force on i_{th} particle

Does not necessarily means that individual terms of the summation are zero as $\vec{r}_i$ are not independent, they are connected by constrain relation

Lagrange's equation from D'Alembert's principle

$$\sum_{i=1}^{N} (\vec{F}_{ie} - m_i \ddot{\vec{r}}_i) \cdot \delta \vec{r}_i = 0$$

- Want to express this relation in such a way where all the terms in the summation becomes individually zero.

how to do?

Let's remember:

$u_1\, \delta x_1 + u_2\, \delta x_2 = 0$; does this always mean $u_1 = 0$ and $u_2 = 0$?

If x_1 and x_2 are independent then $u_1 = 0$ and $u_2 = 0$ for all possible variation of x_1 and x_2,

If x_1 and x_2 are not independent, changing one will change the other.

$\sum u_i\, \delta\, x_i = 0, \mathit{then\ all}\ u_i$ will be individually zero for all possible variation of the x_i if they are independent.

Lagrange's equation from D'Alembert's principle

- **D'Alembert's principle,**

$$\sum_{i=1}^{N}(\vec{F}_{ie} - m_i\ddot{\vec{r}}_i)\cdot\delta\vec{r}_i = 0$$

Constraint forces are out of the game!

Now, no need of additional subscript, we shall simply write $\vec{F}_i$ instead of $\vec{F}_{ie}$

But How to express this relation so that individual terms in the summation are zero?

Switch to generalized coordinate system as they are independent!

Let's take the 1st term

$$\sum_i \vec{F}_i\cdot\delta\vec{r}_i = \sum_i \vec{F}_i\cdot\sum_{j=1}^{n}\frac{\partial\vec{r}_i}{\partial q_j}\delta q_j = \sum_{j=1}^{n}\left(\sum_i \vec{F}_i\cdot\frac{\partial\vec{r}_i}{\partial q_j}\right)\delta q_j = \sum_{j=1}^{n} Q_j\delta q_j$$

$$Q_j = \sum_i \vec{F}_i\cdot\frac{\partial\vec{r}_i}{\partial q_j}$$ → **Generalized force**

- Dimensions of Q_j **is not** always of force!
- Dimensions of $Q_j\delta q_j$ **is** always of work!

Lagrange's equation from D'Alembert's principle

- 2nd Term:

$$\sum_i m_i \ddot{\vec{r}}_i \cdot \delta \vec{r}_i = \sum_i m_i \ddot{\vec{r}}_i \cdot \sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j = \sum_{i,j} m_i \ddot{\vec{\boldsymbol{r}}}_{\boldsymbol{i}} \cdot \frac{\boldsymbol{\partial \vec{r}_i}}{\boldsymbol{\partial q_j}} \delta q_j$$

- **Bit of rearrangement in derivatives**

$$\ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \frac{d}{dt}\left(\dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right) - \dot{\vec{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)$$

Time and coordinate derivative can be interchanged!

$$\boldsymbol{\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right) = \left(\frac{\partial \dot{\vec{r}}_i}{\partial q_j}\right)}$$

$$= \frac{d}{dt}\left(\dot{\vec{r}}_i \cdot \frac{\boldsymbol{\partial \dot{\vec{r}}_i}}{\boldsymbol{\partial \dot{q}_j}}\right) - \dot{\vec{r}}_i \cdot \left(\frac{\partial \dot{\vec{\boldsymbol{r}}}_{\boldsymbol{i}}}{\partial q_j}\right)$$

dot cancellation!

$$\frac{\partial \vec{r}_i}{\partial q_j} = \boldsymbol{\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}}$$

$$= \frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}\dot{r}_i{}^2\right)\right\} - \frac{\partial}{\partial q_j}\left(\frac{1}{2}\dot{r}_i{}^2\right)$$

Interchange of order of differential operators

$$\boxed{\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right) = \frac{\partial \dot{\vec{r}}_i}{\partial q_j}}$$

$$\frac{\partial \vec{r}_i}{\partial q_j} = \frac{\partial \vec{r}_i}{\partial q_j}(q_1, \dots q_n; t)$$

$$\vec{r}_i = \vec{r}_i(q_1, \dots, q_n, t)$$

$$\dot{\vec{r}}_i = \frac{d\vec{r}_i}{dt} = \frac{\partial \vec{r}_i}{\partial q_1}\dot{q}_1 + \frac{\partial \vec{r}_i}{\partial q_2}\dot{q}_2 + \ \dots + \frac{\partial \vec{r}_i}{\partial q_n}\dot{q}_n + \frac{\partial \vec{r}_i}{\partial t}$$

$$\text{RHS} = \frac{\partial \dot{\vec{r}}_i}{\partial q_j} = \frac{\partial^2\, \vec{r}_i}{\partial q_j \partial q_1}\dot{q}_1 + \frac{\partial^2 \vec{r}_i}{\partial q_j \partial q_2}\dot{q}_2 + \ \dots + \frac{\partial^2 \vec{r}_i}{\partial q_j \partial q_n}\dot{q}_n + \frac{\partial^2 \vec{r}_i}{\partial q_j \partial t}$$

$$\text{LHS} = \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right) = \frac{\partial}{\partial q_1}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)\frac{dq_1}{dt} + \cdots + \frac{\partial}{\partial q_n}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)\frac{dq_n}{dt} + \frac{\partial}{\partial t}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)$$

$$= \frac{\partial^2 \vec{r}^{\,i}}{\partial q_j \partial q_1}\dot{q}_1 + \frac{\partial^2 \vec{r}_i}{\partial q_j \partial q_2}\dot{q}_2 + \ \dots . + \frac{\partial^2 \vec{r}_i}{\partial q_j \partial q_n}\dot{q}_n + \frac{\partial^2 \vec{r}_i}{\partial q_j \partial t} \quad = \ \text{RHS}$$

$$\boxed{\frac{\partial^2 V}{\partial x \partial y} = \frac{\partial^2 V}{\partial y \partial x}}$$

This true for any *x* & *y*!
ie., even if say, *y*= *t* !

Interchange of order of differential operators

$$\frac{\partial \vec{r}_i}{\partial q_j} = \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}$$

$$\dot{\vec{r}}_i = \dot{\vec{r}}_i(q_1, \dots q_n; \dot{q}_1, \dots \dot{q}_2; t)$$

$$\dot{\vec{r}}_i = \frac{d\vec{r}_i}{dt} = \frac{\partial \vec{r}_i}{\partial q_1}\dot{q}_1 + \frac{\partial \vec{r}_i}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j + \cdots + \frac{\partial \vec{r}_i}{\partial q_n}\dot{q}_n + \frac{\partial \vec{r}_i}{\partial t}$$

Let's look at the dependency=> $\frac{\partial \vec{r}_i}{\partial q_j} = \frac{\partial \vec{r}_i}{\partial q_j}(q_1, \dots q_n; t)$

RHS= $\boxed{\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j}}$ = LHS

Lagrange's equation from D'Alembert's principle

- Thus 2nd term becomes

$$\sum_{i=1}^{N} m_i \ddot{\vec{r}}_i \cdot \delta\vec{r}_i = \sum_{i,j} m_i \left[\frac{d}{dt}\left\{ \frac{d}{d\dot{q}_j}\left(\frac{1}{2}\dot{r}_i^{\,2}\right)\right\} - \frac{\partial}{\partial q_j}\left(\frac{1}{2}\dot{r}_i^{\,2}\right)\right]\delta q_j$$

$$= \sum_{j}\left[\frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_j}\left(\sum_i \frac{1}{2} m_i \dot{r}_i^{\,2}\right)\right\} - \frac{\partial}{\partial q_j}\left(\sum_i \frac{1}{2} m_i \dot{r}_i^{\,2}\right)\right]\delta q_j$$

$$= \sum_j \left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_j}\right\}\delta q_j$$

The 1st term

$$\sum_i \vec{F}_i \cdot \delta\vec{r}_i = \sum_{j=1}^{n} Q_j \delta q_j$$

Lagrange's equation from D'Alembert's principle

❑ D'Alembert's principle in generalized coordinates becomes

$$\sum_j \left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \right\} \delta q_j = \sum_j Q_j \delta q_j$$

$$\sum_j \left[\left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \right\} - Q_j \right] \delta q_j = 0$$

Well, we are very close to Lagrange's equation!

❑ Since generalized coordinates q_j are all independent each term in the summation is zero

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$$

$$-\left(\frac{\partial V_i}{\partial x_i}\hat{\imath} + \frac{\partial V_i}{\partial y_i}\hat{\jmath} + \frac{\partial V_i}{\partial z_i}\hat{k}\right) . \left(\frac{\partial x_i}{\partial q_j}\hat{\imath} + \frac{\partial y_i}{\partial q_j}\hat{\jmath} + \frac{\partial z_i}{\partial q_j}\hat{k}\right)$$

$$= -\left(\frac{\partial V_i}{\partial x_i}\frac{\partial x_i}{\partial q_j} + \frac{\partial V_i}{\partial y_i}\frac{\partial y_i}{\partial q_j} + \frac{\partial V_i}{\partial z_i}\frac{\partial z_i}{\partial q_j}\right)$$

❑ If all the forces are conservative, then $\vec{F}_i = -\vec{\nabla} V_i$

$$Q_j = \sum_i (-\vec{\nabla} V_i) \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\sum_i \frac{\partial V_i}{\partial q_j} = -\frac{\partial}{\partial q_j}\sum_i V_i = -\frac{\partial V}{\partial q_j}$$

Total potential

$$V = \sum_i V_i$$

Lagrange's equation from D'Alembert's principle

Hence,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j = -\frac{\partial V}{\partial q_j}$$

- Assume that $\boldsymbol{V}$ **does not depend on** $\dot{\boldsymbol{q}}_{\boldsymbol{j}}$, then $\frac{\partial \boldsymbol{V}}{\partial \dot{\boldsymbol{q}}_{\boldsymbol{j}}} = \mathbf{0}$

$$\frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_j}(T - V)\right\} - \frac{\partial (T - V)}{\partial q_j} = 0$$

$$\boldsymbol{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0}$$

Where,

$$\boldsymbol{L(q_j, \dot{q}_j, t) = T(q_j, \dot{q}_j, t) - V(q_j, t)}$$

We have reached to Lagrange's equation from D'Alembert's principle.

Review of the steps we followed

- Started from Newton's law

$$m\ddot{\vec{r}} = \vec{F}_e + \vec{f}_c$$

- Taken dot product with virtual displacement to kick out constrain force from the game by using $\vec{f}_c \cdot \delta\vec{r} = 0$; Arrive at D'Alembert's principle $\left(\boldsymbol{\vec{F}_e - m\ddot{\vec{r}} \cdot \delta\vec{r}}\right) \cdot \boldsymbol{\delta\vec{r} = 0}$

- Extended D'Alembert's principle for a system of particles;

$$\sum_{\boldsymbol{i=1}}^{\boldsymbol{N}} (\boldsymbol{\vec{F}_{ie} - m_i\ddot{\vec{r}}_i) \cdot \delta\vec{r}_i = 0}$$

- Converted this expression in generalized coordinate system that *"every"* term of this summation is zero to get

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_j} = Q_j$$

This is a more general expression!

- Now, with the assumptions: i) Forces are conservative, $\vec{F}_i = -\vec{\nabla} V_i$, hence $Q_j = -\frac{\partial V}{\partial q_j}$ and ii) potential does not depend on $\boldsymbol{\dot{q}_j}$, then $\frac{\partial V}{\partial \dot{q}_j} = 0$

We get back our Lagrange's eqn., $$\boldsymbol{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0}$$

QUESTIONS?