Real Analysis

Lecture-Note for M642, July–November 2025

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Updated: March 24, 2026

Contents

Preface and Conventions

These notes were prepared as a self-contained introduction to real analysis, intended to serve both as a first systematic text and as a reference for further study. They are designed to serve simultaneously as a classroom companion, a proof-oriented reference, and a coherent roadmap through the core themes of an advanced undergraduate first course in analysis. Throughout, the emphasis is on precise definitions, logically complete arguments, and instructive examples and counterexamples that clarify the scope of the major theorems.

The presentation follows a progressive structure. We begin with the order and completeness properties of the real line, pass to the abstract setting of metric and normed spaces, and then develop continuity, compactness, connectedness, function spaces, differential calculus, and multiple integration in a manner that makes the underlying conceptual links explicit. Where appropriate, we have also included exercises, remarks, and illustrative constructions intended to sharpen intuition without sacrificing rigor.

Prerequisites. A solid first course in single-variable calculus, together with basic linear algebra, is assumed throughout.

Notation. We write $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ for the standard number systems. We use $\subseteq$ for inclusion and $\subset$ for proper inclusion. If (X, d) is a metric space and $x \in X$, $r > 0$, then

$$B_r(x) := \{y \in X : d(x, y) < r\}$$

denotes the open ball of radius r centred at x. Unless stated otherwise, all vector spaces are over $\mathbb{R}$ or $\mathbb{C}$, according to context.

How to use these notes. The main text is arranged so that each chapter may be read linearly, but the notes are also suitable for selective consultation. In this edition, major chapters are preceded by explicit learning objectives, key theorems are often accompanied by a short proof strategy or conceptual takeaway, and the back-matter problem sets are grouped by theme so that the material may be used equally well for lecture preparation, tutorial work, or independent study. The chapter abstracts indicate the principal themes, while the remarks and examples are designed to clarify not only the statements of the central results but also the role of their hypotheses.

References. Standard references include Rudin's *Principles of Mathematical Analysis*, Royden–Fitzpatrick's *Real Analysis*, and other classical texts in analysis that complement the present exposition.

Outline of Topics

- **Foundations:** Completeness properties of real numbers; countable and uncountable sets; cardinality.

- **Norms and metrics:** Metric spaces; convergence of sequences; completeness; connectedness; sequential compactness.

- **Continuity:** Continuity and uniform continuity.

- **Sequences and series of functions:** Uniform convergence; equicontinuity; Arzelà–Ascoli theorem; Weierstrass approximation theorem; power series.

- **Single-variable calculus:** Differentiability; mean value theorems; Taylor's theorem.

- **Several-variable calculus:** Partial and directional derivatives; differentiability; chain rule; Taylor's theorem; maxima and minima; Lagrange multipliers; inverse function theorem; implicit function theorem.

- **Multiple integration and vector calculus:** Fubini's theorem; line and surface integrals; Green, Gauss (divergence), and Stokes theorems.

Introduction

Real analysis is the rigorous study of limiting processes. Its central questions concern convergence, continuity, compactness, differentiability, and integration; yet the real strength of the subject lies in the way these notions interact across several mathematical settings. What begins as the analysis of sequences and functions on the real line quickly develops into a general theory on metric spaces, normed linear spaces, and function spaces. These notes are therefore organised not merely as a collection of topics, but as a structured progression from the concrete to the abstract and back again.

A recurring theme is that each major theorem reflects a particular structural principle. The least-upper-bound property expresses the completeness of $\mathbb{R}$; compactness turns local information into global control; connectedness prohibits discontinuous jumps in image sets; and differentiability identifies the best linear approximation to a nonlinear map. The later chapters show how these principles culminate in multivariable calculus and the integral theorems of vector analysis.

The notes are written in a proof-oriented style. Whenever possible, the exposition highlights not only what is true, but also why the hypotheses are natural and where the main difficulties lie. Examples and counterexamples are included to prevent formal statements from becoming detached from geometric intuition. Throughout, we aim for a theorem–proof flow in which definitions motivate the statements, proofs make the mechanism transparent, and remarks record the structural lesson that survives beyond a single exercise. In that sense, the present text is intended to be both a teaching document and a durable reference for the foundational syllabus of real analysis.

Chapter-wise organisation

Chapter 1. Foundations on the real line: order, completeness, countability, cardinality, and the Cantor set.

Chapter 2. Metric and normed linear spaces: metrics, norms, convergence, continuity, and standard examples.

Chapter 3. Completeness and compactness: Cauchy sequences, total boundedness, compactness criteria, and related consequences.

Chapter 4. Connectedness and its applications in metric and normed spaces.

Chapter 5. Continuity and function theory on the real line, with special attention to uniform continuity and classical examples.

Chapter 6. Sequences and series of functions, equicontinuity, compactness in function spaces, approximation, and power series.

Chapter 7. Differential calculus of one real variable, including the mean value theorems and Taylor's theorem.

Chapter 8. Differential calculus in several variables, including differentiability, Jacobians, the chain rule, extrema, and inverse/implicit function theorems.

Chapter 9. Multiple integration and vector calculus, culminating in change of variables and the theorems of Green, Gauss, and Stokes.
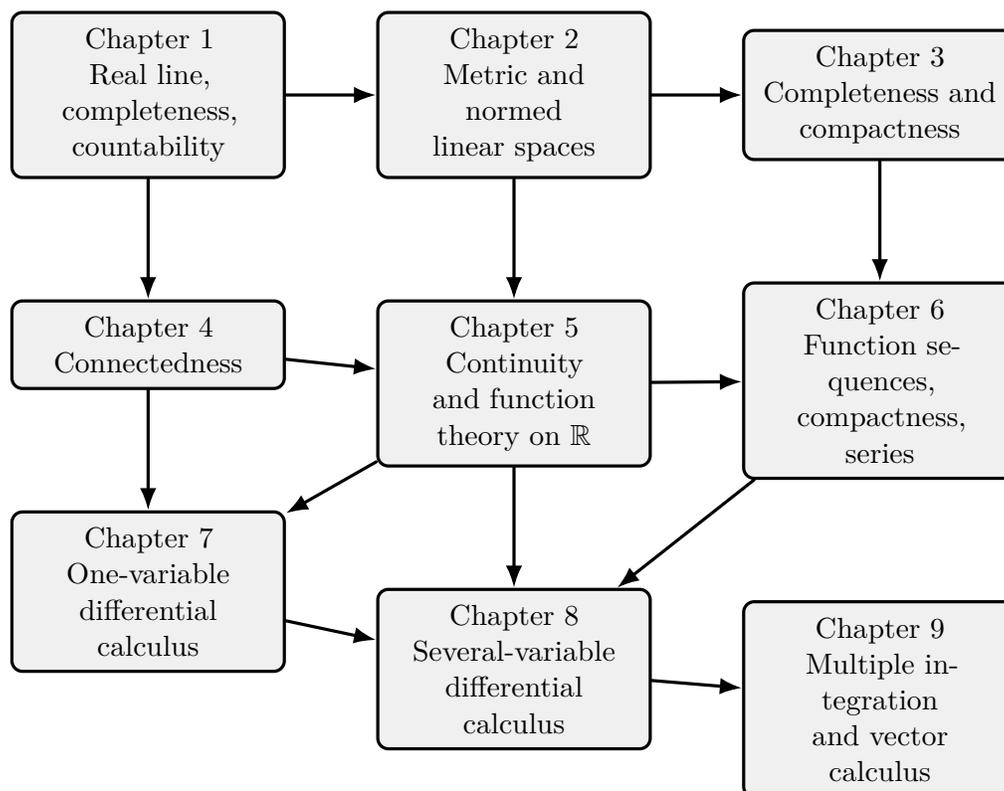
Figure 1. Logical progression of the principal themes in the course.

Chapter 1

Topology of the Real Line

In this chapter we review the order and completeness structure of the real line, and we introduce the basic topological language used throughout real analysis. We develop the least-upper-bound principle, the Archimedean property, and the Nested Interval Theorem. We then study open and closed sets, interior, closure, and dense subsets of $\mathbb{R}$. We conclude with the Cantor set as a central example illustrating how "size" and "topology" can behave in subtle ways.

Learning objectives.

- Understand how order, completeness, and approximation interact on the real line.

- Use the least-upper-bound principle, nested intervals, and Bolzano–Weierstrass as mutually reinforcing forms of completeness.

- Distinguish countable from uncountable phenomena, and use the Cantor set as a guiding example showing that topological size and metric size need not agree.

1.1 The Real Numbers

Section overview.

- We begin with the order structure of $\mathbb{R}$ and the completeness axiom encoded by sup and inf.

- The main goal is to see how approximation statements, monotone convergence, and nested interval arguments all emerge from this single structural principle.

- Keep track of the difference between an upper bound, the least upper bound, and approximating the supremum from within the set.

1.1.1 Preliminary

Recall that the set of rational numbers is

$$\mathbb{Q} := \left\{ \frac{p}{q} : p, q \in \mathbb{Z}, \ q \neq 0 \right\}. \tag{1.1}$$

Not every real number is rational. A classical example is $\sqrt{2}$. Indeed, if $\sqrt{2} = p/q$ with $\gcd(p, q) = 1$, then

$$p^2 = 2q^2.$$

In particular, p is even, so $p = 2m$ for some $m \in \mathbb{Z}$. Substituting into the previous identity gives $q^2 = 2m^2$, so q is also even. This contradicts $\gcd(p, q) = 1$. Therefore $\sqrt{2} \notin \mathbb{Q}$. The elements of $\mathbb{R} \setminus \mathbb{Q}$ are called *irrational numbers*.

Remark 1.1. The rational numbers do not satisfy the least upper bound property, and hence they are not complete.

Definition 1.2 (Upper and lower bounds)**.** Let $A \subseteq \mathbb{R}$. A number $x_0 \in \mathbb{R}$ is an *upper bound* for A if $a \leq x_0$ for every $a \in A$. A number $y_0 \in \mathbb{R}$ is a *lower bound* for A if $y_0 \leq a$ for every $a \in A$.

Definition 1.3 (Supremum and infimum)**.** Assume that $A \subseteq \mathbb{R}$ is nonempty. An upper bound x_0 of A is called the *least upper bound* of A (or the *supremum* of A) if $x_0 \leq x$ for every upper bound x of A. In this case we write $x_0 = \sup A$. Similarly, a lower bound y_0 of A is called the *greatest lower bound* of A (or the *infimum* of A) if $y \leq y_0$ for every lower bound y of A. In this case we write $y_0 = \inf A$.

Example 1.4. $A = \left\{ 1 - \frac{1}{n} : n \in \mathbb{N} \right\}$. Show that $\inf A = 0$ and $\sup A = 1$.

Remark 1.5. The least upper bound principle recorded in the next subsection is the defining completeness axiom for $\mathbb{R}$.

1.1.2 Completeness Property of $\mathbb{R}$

Theorem 1.6 (Least upper bound property)**.** *Every nonempty subset of $\mathbb{R}$ that is bounded above has a least upper bound in $\mathbb{R}$. Equivalently, every nonempty subset of $\mathbb{R}$ that is bounded below has a greatest lower bound in $\mathbb{R}$.*

Remark 1.7. The statement of Theorem 1.6 is the *completeness* of the real line. For a proof, see Chapter 1 of Rudin's *Principles of Mathematical Analysis.*

Takeaway. The least-upper-bound property is the hidden engine behind monotone convergence, nested interval arguments, and the compactness phenomena that first appear on the real line. Later abstract notions of completeness and compactness should be viewed as attempts to recover this same control in broader settings.

Remark 1.8 (Extended real conventions). It is often convenient to work in the extended real line $\overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. Accordingly, if $A \subseteq \mathbb{R}$ is nonempty and unbounded above we write $\sup A = +\infty$, and if $A \subseteq \mathbb{R}$ is nonempty and unbounded below we write $\inf A = -\infty$. We also adopt the standard conventions

$$\sup \varnothing = -\infty, \qquad \inf \varnothing = +\infty.$$

These choices are consistent with the monotonicity properties of inf and sup under inclusion.

Proposition 1.9 (Monotonicity of inf and sup)**.** *If $A \subseteq B \subseteq \mathbb{R}$, then $\inf A \geq \inf B$ and $\sup A \leq \sup B$ (with the extended-real conventions above).*

Example 1.10.

- If $A \neq \varnothing$ and A is not bounded above, we write $\sup A = +\infty$.

- If $B \neq \varnothing$ and B is not bounded below, we write $\inf B = -\infty$.

- If $A = \varnothing$, we write $\inf A = +\infty$ and $\sup A = -\infty$.

Remark 1.11. For instance, since $\varnothing \subseteq \{a\}$, Proposition 1.9 yields $\inf \varnothing \geq \inf\{a\} = a$ for every $a \in \mathbb{R}$, which is consistent with the convention $\inf \varnothing = +\infty$. Similarly, $\sup \varnothing \leq a$ for every $a \in \mathbb{R}$.

Properties 1.12.

- If $A \subseteq B \subseteq \mathbb{R}$, then $\inf A \geq \inf B$ and $\sup A \leq \sup B$.

- If $A \neq \varnothing$, then $\inf A \leq \sup A$.

- $\inf \varnothing = +\infty$ and $\sup \varnothing = -\infty$.

1.1.3 Archimedean property

Theorem 1.13 (Archimedean property)**.** *Let $x > 0$ and let $y \in \mathbb{R}$. Then there exists $n \in \mathbb{N}$ such that $nx > y$. Equivalently, for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$.*

Proof. Suppose, towards a contradiction, that $nx \leq y$ for every $n \in \mathbb{N}$. Then y is an upper bound for the set $S := \{nx : n \in \mathbb{N}\}$. By Theorem 1.6, the least upper bound $\ell := \sup S$ exists in $\mathbb{R}$. Since $\ell - x$ is not an upper bound for S, there exists $n \in \mathbb{N}$ such that $\ell - x < nx \leq \ell$. Consequently $(n + 1)x > \ell$, contradicting the fact that ℓ is an upper bound of S. $\square$

Remark 1.14. The Archimedean property formalizes the fact that the natural numbers are unbounded in $\mathbb{R}$, and it is a basic tool for constructing rational (and irrational) approximations.

Exercise 1.1. Let $A = \{r \in \mathbb{Q} : r^2 < 2, \ r > 0\}$. Show that $\sup A = \sqrt{2}$ (which is not in $\mathbb{Q}$).

Example 1.15. Let $x, y \in \mathbb{R}$ with $x < y$. Then $y - x > 0$, so by the Archimedean property there exists $n \in \mathbb{N}$ such that

$$n(y - x) > 1.$$

Hence the interval (nx, ny) has length greater than 1, and therefore it contains an integer m. Dividing by n gives

$$x < \frac{m}{n} < y.$$

Thus, between any two distinct real numbers there is a rational number.

To produce an irrational number between x and y, choose a rational number r such that

$$\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}.$$

Then

$$x < r\sqrt{2} < y,$$

and $r\sqrt{2}$ is irrational unless $r = 0$. Therefore, between any two distinct real numbers there is also an irrational number.

Example 1.16. Consider the set

$$A = \left\{ \frac{m}{m+n} : m, n \in \mathbb{N} \right\}.$$

Then

$$\inf A = 0, \qquad \sup A = 1.$$

Indeed, for each $n \in \mathbb{N}$ we have

$$\frac{1}{n+1} \in A,$$

and $\frac{1}{n+1} \to 0$, so $\inf A = 0$. Likewise, for each $m \in \mathbb{N}$,

$$\frac{m}{m+1} \in A,$$

and $\frac{m}{m+1} \to 1$, so $\sup A = 1$.

Proposition 1.17. *Let $A \subseteq \mathbb{R}$ be nonempty, and let $\alpha = \inf A$ and $\beta = \sup A$. Then for every $\varepsilon > 0$ there exist $x_0, y_0 \in A$ such that*

$$x_0 < \alpha + \varepsilon \qquad and \qquad y_0 > \beta - \varepsilon.$$

Proof. We prove the first statement; the second is analogous. Suppose there exists $\varepsilon > 0$ such that $x \geq \alpha + \varepsilon$ for every $x \in A$. Then $\alpha + \varepsilon$ is a lower bound for A, contradicting the fact that α is the greatest lower bound. Hence there exists $x_0 \in A$ with $x_0 < \alpha + \varepsilon$.

Similarly, if there were no point of A exceeding $\beta - \varepsilon$, then $\beta - \varepsilon$ would be an upper bound for A, contradicting the definition of $\beta = \sup A$. Thus there exists $y_0 \in A$ with $y_0 > \beta - \varepsilon$. $\square$

Definition 1.18. A *sequence* is a function $f \colon \mathbb{N} \to \mathbb{R}$ or $f \colon \mathbb{N} \to \mathbb{C}$. We usually write it as $(f_n)_{n \geq 1}$ or $\{f_n\}$, where $f_n = f(n)$.

Definition 1.19. A sequence $(a_n) \subseteq \mathbb{R}$ is said to *converge* to $l \in \mathbb{R}$ if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \quad \implies \quad |a_n - l| < \varepsilon.$$

Example 1.20. The sequence $a_n = \frac{1}{n}$ converges to 0. Indeed, given $\varepsilon > 0$, choose $n_0 \in \mathbb{N}$ with $n_0 > \frac{1}{\varepsilon}$. Then for every $n \geq n_0$,

$$\left| \frac{1}{n} - 0 \right| = \frac{1}{n} \leq \frac{1}{n_0} < \varepsilon.$$

Theorem 1.21. *Every convergent sequence is bounded.*

Proof. Let $a_n \to a$. Taking $\varepsilon = 1$, there exists $n_0 \in \mathbb{N}$ such that $|a_n - a| < 1$ for all $n \geq n_0$. Hence $a_n \in (a - 1, a + 1)$ for all $n \geq n_0$.

Let

$$m = \inf\big((a-1, a+1) \cup \{a_1, \ldots, a_{n_0-1}\}\big), \qquad M = \sup\big((a-1, a+1) \cup \{a_1, \ldots, a_{n_0-1}\}\big).$$

Then $m \leq a_n \leq M$ for every n, so the sequence is bounded. $\qquad \square$

Theorem 1.22. *If (a_n) is increasing and bounded above, then (a_n) converges and*

$$\lim_{n \to \infty} a_n = \sup_{n \geq 1} a_n.$$

Similarly, a decreasing sequence that is bounded below converges to its infimum.

Proof. Let

$$\alpha = \sup\{a_n : n \geq 1\}.$$

Fix $\varepsilon > 0$. By Proposition 1.17, there exists n_0 such that $a_{n_0} > \alpha - \varepsilon$. Since (a_n) is increasing,

$$\alpha - \varepsilon < a_{n_0} \leq a_n \leq \alpha \qquad \text{for all } n \geq n_0.$$

Hence $|a_n - \alpha| < \varepsilon$ for all $n \geq n_0$, so $a_n \to \alpha$. The decreasing case is analogous. $\qquad \square$

1.1.4   Nested Interval Theorem

Theorem 1.23 (Nested interval theorem). *Let $(I_n)_{n \geq 1}$ be a nested sequence of nonempty closed intervals in $\mathbb{R}$, say*

$$I_n = [a_n, b_n], \qquad I_{n+1} \subseteq I_n \quad (n \geq 1).$$

If $\ell(I_n) = b_n - a_n \to 0$ as $n \to \infty$, then $\bigcap_{n=1}^{\infty} I_n$ consists of exactly one point.

Proof strategy. Track the left endpoints and right endpoints separately. Monotonicity gives convergence of each endpoint sequence, and the hypothesis $\ell(I_n) \to 0$ forces the two limits to coincide.

Proof. Since $I_{n+1} \subseteq I_n$, the sequence (a_n) is increasing and bounded above by b_1, hence convergent; write $a_n \to a$. Similarly, (b_n) is decreasing and bounded below by a_1, hence convergent; write $b_n \to b$. Taking limits in $0 \leq b_n - a_n \to 0$ gives $b - a = 0$, so $a = b$.

Let $x := a$. Then $a_n \leq x \leq b_n$ for every n, hence $x \in I_n$ for every n, and therefore $x \in \bigcap_{n \geq 1} I_n$.

Conversely, if $y \in \bigcap_{n \geq 1} I_n$, then $a_n \leq y \leq b_n$ for all n. Passing to the limit yields $a \leq y \leq b$, hence $y = a = b = x$. $\qquad \square$

Definition 1.24. Let (x_n) be a sequence. If $n_1 < n_2 < \cdots < n_k < \cdots$ is a strictly increasing sequence of natural numbers, then the sequence (x_{n_k}) is called a *subsequence* of (x_n).

Example 1.25. The sequences $\left(\frac{1}{k^2}\right)$ and $\left(\frac{1}{2^k}\right)$ are subsequences of $\left(\frac{1}{n}\right)$, corresponding respectively to the index choices $n_k = k^2$ and $n_k = 2^k$.

Theorem 1.26 (Bolzano–Weierstrass)**.** *Every bounded sequence in $\mathbb{R}$ admits a convergent subsequence.*

Proof. Let (x_n) be a bounded sequence in $\mathbb{R}$. Choose $a < b$ such that $x_n \in [a, b]$ for all $n \in \mathbb{N}$, and set $I_1 = [a, b]$. Choose $n_1 \in \mathbb{N}$ arbitrarily so that $x_{n_1} \in I_1$.

Suppose that for some $k \geq 1$ we have already chosen a closed interval $I_k = [a_k, b_k]$ containing infinitely many terms of the sequence, together with an index n_k such that $x_{n_k} \in I_k$. Bisect I_k into two closed subintervals of equal length. At least one of these subintervals contains infinitely many terms of the sequence; denote it by I_{k+1}. Since I_{k+1} contains infinitely many terms, we may choose $n_{k+1} > n_k$ such that $x_{n_{k+1}} \in I_{k+1}$.

In this way we obtain a nested sequence of closed intervals

$$I_1 \supseteq I_2 \supseteq \cdots,$$

and a subsequence (x_{n_k}) with $x_{n_k} \in I_k$ for every k. Moreover,

$$\ell(I_k) = \frac{b - a}{2^{k-1}} \xrightarrow[k \to \infty]{} 0.$$

By Theorem 1.23, the intersection $\bigcap_{k=1}^{\infty} I_k$ consists of a single point, say $\{x\}$.

Now fix $\varepsilon > 0$. Choose k_0 such that $\ell(I_{k_0}) < \varepsilon$. Since $x \in I_{k_0}$ and I_{k_0} has length less than ε, it follows that $I_{k_0} \subset (x - \varepsilon, x + \varepsilon)$. For every $k \geq k_0$ we have $x_{n_k} \in I_k \subseteq I_{k_0}$, and therefore $|x_{n_k} - x| < \varepsilon$. Hence $x_{n_k} \to x$ as $k \to \infty$. $\square$

Remark 1.27 (Lower and upper limits). Let (x_n) be a bounded sequence in $\mathbb{R}$. For each $k \geq 1$, define

$$u_k := \inf\{x_n : n \geq k\}, \qquad v_k := \sup\{x_n : n \geq k\}.$$

Then (u_k) is increasing and bounded above, while (v_k) is decreasing and bounded below. Hence both sequences converge.

We define

$$\liminf_{n \to \infty} x_n := \lim_{k \to \infty} u_k = \sup_{k \geq 1} \inf_{n \geq k} x_n,$$

and

$$\limsup_{n \to \infty} x_n := \lim_{k \to \infty} v_k = \inf_{k \geq 1} \sup_{n \geq k} x_n.$$

Since $u_k \leq v_k$ for every k, it follows that

$$\liminf_{n \to \infty} x_n \leq \limsup_{n \to \infty} x_n.$$

Example 1.28. For the sequence $x_n = (-1)^n$, we have

$$\liminf_{n \to \infty} x_n = -1, \qquad \limsup_{n \to \infty} x_n = 1.$$

Exercise 1.2. Show that if $x_n \to x$, then

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = x.$$

Deduce that a bounded sequence converges if and only if its lower and upper limits agree.

Example 1.29. Let $X_n = (x_n, y_n) \in \mathbb{R}^2$ be a bounded sequence. Then both coordinate sequences (x_n) and (y_n) are bounded. By Bolzano–Weierstrass, (x_n) has a convergent subsequence (x_{n_k}). The corresponding subsequence (y_{n_k}) is still bounded, so it has a convergent subsequence $(y_{n_{k_\ell}})$. Hence

$$(x_{n_{k_\ell}}, y_{n_{k_\ell}}) \to (x, y) \in \mathbb{R}^2$$

for suitable $x, y \in \mathbb{R}$.

Remark 1.30. The same diagonal argument works in $\mathbb{R}^n$.

1.2   Countability and Cardinality

In analysis one frequently distinguishes sets not only by their topological properties, but also by their *cardinality.* We recall the basic notions and the standard examples.

1.2.1   Countable and uncountable sets

Definition 1.31. A set A is *countable* if there exists a bijection $A \to \mathbb{N}$. It is *at most countable* if it is finite or countable, equivalently if there exists an injection $A \hookrightarrow \mathbb{N}$. A set that is not at most countable is called *uncountable.*

Example 1.32. The sets $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are countable.

Proof. Clearly $\mathbb{N}$ is countable. The map $n \mapsto (-1)^n \lceil n/2 \rceil$ is a bijection $\mathbb{N} \to \mathbb{Z}$, so $\mathbb{Z}$ is countable.

For $\mathbb{Q}$, write every nonzero rational in reduced form p/q with $p \in \mathbb{Z} \setminus \{0\}$ and $q \in \mathbb{N}$. Consider the set $\mathbb{Z} \times \mathbb{N}$ and enumerate it along diagonals:

$$(0, 1), \ (1, 1), \ (-1, 1), \ (0, 2), \ (1, 2), \ (-1, 2), \ (2, 1), \ (-2, 1), \ldots$$

This gives a surjection $\mathbb{N} \to \mathbb{Z} \times \mathbb{N}$. Composing with $(p, q) \mapsto p/q$ yields a surjection $\mathbb{N} \to \mathbb{Q}$. Since $\mathbb{Q}$ is infinite, it follows that $\mathbb{Q}$ is countable. $\qquad\square$

Theorem 1.33. *A countable union of countable sets is countable. More precisely, if A_n is countable for each $n \in \mathbb{N}$, then $\bigcup_{n\in\mathbb{N}} A_n$ is at most countable.*

Proof. For each $n \in \mathbb{N}$, choose an injection $\iota_n \colon A_n \hookrightarrow \mathbb{N}$. To avoid overlaps among the sets A_n, define

$$B_1 := A_1, \qquad B_n := A_n \setminus \bigcup_{k=1}^{n-1} A_k \quad (n \geq 2).$$

Then the sets B_n are pairwise disjoint, each B_n is at most countable, and

$$\bigcup_{n \in \mathbb{N}} A_n = \bigcup_{n \in \mathbb{N}} B_n.$$

Now define $F \colon \bigcup_{n \in \mathbb{N}} B_n \to \mathbb{N} \times \mathbb{N}$ by

$$F(x) := (n, \iota_n(x)) \qquad \text{whenever } x \in B_n.$$

Because the family (B_n) is pairwise disjoint, the map F is well defined; since each ι_n is injective, F is injective as well. Finally, $\mathbb{N} \times \mathbb{N}$ is countable (for example, by diagonal enumeration), so $\bigcup_{n \in \mathbb{N}} A_n$ is at most countable. $\square$

1.2.2 Cantor's diagonal argument

Theorem 1.34 (Cantor). *The interval $[0,1]$ is uncountable. Consequently, $\mathbb{R}$ is uncountable.*

Proof. Assume, for contradiction, that $[0,1]$ is countable, so there exists an enumeration $(x_n)_{n \geq 1}$ of $[0,1]$. Write each x_n in a decimal expansion

$$x_n = 0.d_{n1}d_{n2}d_{n3}\ldots,$$

choosing, when necessary, the expansion that does *not* terminate in an infinite tail of 9's. Define a new number $y \in [0,1]$ by specifying its decimal digits $(e_k)_{k \geq 1}$ via

$$e_k := \begin{cases} 1, & d_{kk} \neq 1, \\ 2, & d_{kk} = 1. \end{cases}$$

Then $y = 0.e_1e_2e_3\ldots$ differs from x_k at the k-th digit for every k, hence $y \neq x_k$ for all k. This contradicts that (x_n) enumerates $[0,1]$. $\square$

1.2.3 Cardinality and comparison of sets

Definition 1.35. Two sets A and B have the same *cardinality*, written $|A| = |B|$, if there exists a bijection $A \to B$. We write $|A| \leq |B|$ if there exists an injection $A \hookrightarrow B$.

Theorem 1.36 (Schröder–Bernstein). *If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.*

Proof. We omit the standard set-theoretic proof; see any text on set theory or the appendix of a real analysis reference. The result will be used only as a conceptual tool for comparing sizes of infinite sets. $\square$

Remark 1.37. Cantor's theorem shows that for any set A, the power set $\mathcal{P}(A)$ satisfies $|A| < |\mathcal{P}(A)|$. In particular, $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})|$, and one can show that $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$ (the *continuum*).

1.3 The Cantor Set

1.3.1 Cantor Set

The Cantor set is an *uncountable set* in $[0,1]$ having zero length with many striking properties, and it serves as a classical example in elementary topology and measure theory.

Let $C_0 = [0,1]$. Remove the open middle third

$$J_1 := \left(\tfrac{1}{3}, \tfrac{2}{3}\right)$$

and set

$$C_1 := [0, \tfrac{1}{3}] \cup [\tfrac{2}{3}, 1].$$

Next remove the open middle thirds of each component of C_1, namely

$$J_2 := \left(\tfrac{1}{9}, \tfrac{2}{9}\right) \cup \left(\tfrac{7}{9}, \tfrac{8}{9}\right),$$

and define

$$C_2 := [0, \tfrac{1}{9}] \cup [\tfrac{2}{9}, \tfrac{1}{3}] \cup [\tfrac{2}{3}, \tfrac{7}{9}] \cup [\tfrac{8}{9}, 1].$$

Inductively, C_n is a union of 2^n disjoint closed intervals, each of length 3^{-n}. The *middle-third Cantor set* is the nested intersection

$$C := \bigcap_{n=0}^{\infty} C_n.$$
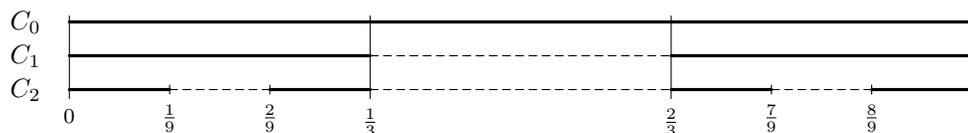
Figure 1.1. The first stages of the Cantor construction. Dashed segments indicate the open intervals removed at each step.

1.3.2 Properties of the Cantor Set

We record the main structural properties of the Cantor set

$$C = \bigcap_{n=0}^{\infty} C_n.$$

1. Each C_n is a nonempty compact subset of $[0,1]$, and the family is nested:

$$C_{n+1} \subseteq C_n \qquad (n \geq 0).$$

Hence C is a nonempty closed subset of $[0,1]$.

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2. Every endpoint of every deleted interval belongs to C.

3. If $\{J_n\}_{n\geq 1}$ denotes the family of open intervals removed in the construction, then

$$C = [0,1] \setminus \bigcup_{n=1}^{\infty} J_n.$$

4. The Cantor set has length zero. Indeed, $C \subseteq C_n$ for every n, and

$$\ell(C) \leq \ell(C_n) = 2^n \cdot \frac{1}{3^n} = \left(\frac{2}{3}\right)^n \longrightarrow 0.$$

Therefore $\ell(C) = 0$.

5. The set C is nowhere dense. If $C^\circ \neq \emptyset$, then there would exist an interval $(x - \varepsilon, x + \varepsilon) \subseteq C$. But then

$$0 < 2\varepsilon = \ell((x - \varepsilon, x + \varepsilon)) \leq \ell(C) = 0,$$

which is impossible. Hence $C^\circ = \emptyset$, and since C is closed, $\overline{C} = C$ also has empty interior.

6. The set C is totally disconnected: every connected subset of C consists of a single point.

7. Every point of C is a limit point of C. Indeed, let $x \in C$. For each n, the point x lies in one of the closed intervals occurring in C_n; denote this interval by $[x_n, y_n]$. Then

$$x \in [x_n, y_n], \qquad y_n - x_n = \frac{1}{3^n},$$

so

$$|x_n - x| \leq \frac{1}{3^n} \to 0.$$

The endpoints x_n and y_n belong to C, and at least one of them is distinct from x for large n. Thus every neighborhood of x contains points of C other than x itself. Therefore C is perfect.

If E denotes the set of all endpoints of the deleted intervals, then $\overline{E} = C$. Since E is countable, the Cantor set is separable.

1.3.3 Representation of Cantor's set

A useful description of the Cantor set is obtained from ternary expansions. Every point $x \in [0,1]$ has a ternary representation

$$x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}, \qquad a_i \in \{0,1,2\}.$$

The points of the Cantor set are precisely those whose ternary expansions involve only the digits 0 and 2.

Indeed, if some ternary digit equals 1, then x lies in one of the middle-third intervals removed at a finite stage of the construction, and therefore $x \notin C$. Conversely, if every ternary digit

belongs to $\{0, 2\}$, then the point survives every step of the construction and hence belongs to C. Therefore

$$C = \left\{ x \in [0,1] : x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}, \ a_i \in \{0,2\} \right\}. \tag{1.2}$$

This characterization already suggests that C is uncountable, because there are infinitely many possible choices of digit sequences with values in $\{0, 2\}$.

1.3.4 Representation is Unique

For points of the Cantor set, the ternary expansion using only the digits 0 and 2 is unique.

Proposition 1.38. *For every $x \in C$, there exists a unique sequence (a_i) with $a_i \in \{0,2\}$ such that*

$$x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}. \tag{1.3}$$

Proof. Existence is exactly the content of (1.2). For uniqueness, suppose that x also admits a representation

$$x = \sum_{i=1}^{\infty} \frac{b_i}{3^i}, \qquad b_i \in \{0,2\}. \tag{1.4}$$

Assume that the two digit sequences are different, and let i_0 be the first index for which $a_{i_0} \neq b_{i_0}$. Then $a_i = b_i$ for $1 \leq i < i_0$, while at the index i_0 one sequence has the digit 0 and the other has the digit 2. Subtracting the common initial part

$$\sum_{i=1}^{i_0-1} \frac{a_i}{3^i} = \sum_{i=1}^{i_0-1} \frac{b_i}{3^i},$$

and multiplying by 3^{i_0-1}, we reduce to the case $i_0 = 1$. In that case one expansion places x in the interval $[0, 1/3]$, whereas the other places x in $[2/3, 1]$, which is impossible. This contradiction shows that no such index i_0 can exist. Therefore $a_i = b_i$ for all i, and the representation is unique. $\square$

Exercise 1.3. Complete the last step of the proof without reducing to the case $i_0 = 1$.

Proposition 1.39. *The Cantor set is uncountable.*

Proof. Define a map $f \colon C \to [0,1]$ as follows. If

$$x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}, \qquad a_i \in \{0,2\},$$

set

$$f(x) := \sum_{i=1}^{\infty} \frac{a_i/2}{2^i}.$$

Since $a_i/2 \in \{0,1\}$, the image lies in $[0,1]$. The uniqueness of the ternary expansion in (1.3) shows that f is well defined.

Now let

$$y = \sum_{i=1}^{\infty} \frac{b_i}{2^i} \in [0,1], \qquad b_i \in \{0,1\}.$$

Define

$$x = \sum_{i=1}^{\infty} \frac{2b_i}{3^i}.$$

Then $x \in C$ and $f(x) = y$. Thus f is surjective. Since $[0,1]$ is uncountable, the Cantor set must also be uncountable. $\square$

Remark 1.40 (The Cantor function). The map f above is the starting point for the construction of the Cantor function. One extends it from C to all of $[0,1]$ by declaring it constant on each deleted interval. The resulting function is continuous, nondecreasing, and maps $[0,1]$ onto $[0,1]$.

1.4 Basic Topology on $\mathbb{R}$

1.4.1 Open Sets and Closed Sets

Definition 1.41 (Open set in $\mathbb{R}$)**.** A set $O \subseteq \mathbb{R}$ is **open** if for every $x \in O$ there exists $\varepsilon > 0$ such that

$$(x - \varepsilon, x + \varepsilon) \subseteq O.$$

Equivalently, every point of O is an interior point of O.

Remark 1.42. A (finite or countable) union of open intervals is an open set. Conversely, every open subset of $\mathbb{R}$ can be written as a countable union of pairwise disjoint open intervals.

A countable union of open intervals is an open set. On the other hand, any open set in $\mathbb{R}$ can be written as a countable union of disjoint open intervals.

Theorem 1.43 (Decomposition of open sets in $\mathbb{R}$)**.** *Let $O \subset \mathbb{R}$ be open. Then there exists a countable family of pairwise disjoint open intervals $\{I_n\}_{n \geq 1}$ such that*

$$O = \bigcup_{n=1}^{\infty} I_n.$$

Moreover, this representation is unique up to a permutation of the intervals.

Proof. Fix $x \in O$. Since O is open, there exists an open interval (a,b) with $x \in (a,b) \subset O$. Define

$$a_x := \inf\{a \in \mathbb{R} : (a,x] \subset O\}, \qquad b_x := \sup\{b \in \mathbb{R} : [x,b) \subset O\},$$

and set $I_x := (a_x, b_x)$.

Step 1: $I_x \subset O$. Let $z \in (a_x, b_x)$. Choose $\eta > 0$ so small that $a_x + \eta < z < b_x - \eta$. By the definition of a_x as an infimum, there exists $a < a_x + \eta$ such that $(a,x] \subset O$; hence $(a_x + \eta, x] \subset O$. Similarly, by the definition of b_x as a supremum, there exists $b > b_x - \eta$ such that $[x,b) \subset O$; hence $[x, b_x - \eta) \subset O$. Therefore $(a_x + \eta, b_x - \eta) \subset O$, and in particular $z \in O$. Thus $I_x \subset O$.

Step 2: maximality and disjointness. By construction, I_x is an open interval containing x and contained in O. Moreover, it is maximal with respect to these properties. Indeed, if J is any open interval with $x \in J \subset O$, then every left endpoint of J belongs to the set whose infimum defines a_x, and every right endpoint of J belongs to the set whose supremum defines b_x; hence $J \subset I_x$. Now if $x, y \in O$ and $I_x \cap I_y \neq \varnothing$, then $I_x \cup I_y$ is again an open interval contained in O. By maximality, this forces $I_x = I_y$. Consequently, the family $\{I_x : x \in O\}$ consists of pairwise disjoint open intervals and

$$O = \bigcup_{x \in O} I_x.$$

Step 3: countability. Every nonempty open interval contains a rational number. Choose $q_x \in I_x \cap \mathbb{Q}$ for each $x \in O$. If $I_x \neq I_y$, then $I_x \cap I_y = \varnothing$, so necessarily $q_x \neq q_y$. Thus the assignment $I_x \mapsto q_x$ is injective into $\mathbb{Q}$. Since $\mathbb{Q}$ is countable, the family $\{I_x : x \in O\}$ is countable. Renaming these intervals as $\{I_n\}_{n \geq 1}$ gives

$$O = \bigcup_{n=1}^{\infty} I_n.$$

Step 4: uniqueness. Suppose also that

$$O = \bigcup_{n=1}^{\infty} I_n = \bigcup_{m=1}^{\infty} J_m,$$

where both families consist of pairwise disjoint open intervals. Fix n. Since $I_n \subset O$, we have

$$I_n = I_n \cap O = \bigcup_{m=1}^{\infty} (I_n \cap J_m).$$

Each set $I_n \cap J_m$ is an open subset of the interval I_n, and the family is pairwise disjoint. Because I_n is connected, at most one of these intersections can be nonempty. Hence $I_n \subset J_{m_0}$ for some m_0. Applying the same argument with the roles of I_n and J_{m_0} reversed yields $J_{m_0} \subset I_n$. Therefore $I_n = J_{m_0}$. This proves uniqueness up to a permutation of the intervals. $\qquad\square$

Definition 1.44 (Closed set in $\mathbb{R}$). A set $F \subseteq \mathbb{R}$ is **closed** if it contains the limits of all convergent sequences of its points, that is, whenever $(x_n) \subseteq F$ and $x_n \to x$ in $\mathbb{R}$, we have $x \in F$.

Theorem 1.45. *A set $F \subseteq \mathbb{R}$ is closed if and only if its complement F^c is open.*

Proof. Assume first that F is closed, and let $x \in F^c$. If F^c were not open at x, then for every $n \in \mathbb{N}$ we could choose
$$x_n \in \left(x - \frac{1}{n},\, x + \frac{1}{n}\right) \cap F.$$
Then $x_n \to x$ with $x_n \in F$ for all n, so closedness of F would imply $x \in F$, a contradiction. Thus F^c is open.

Conversely, assume F^c is open and let $(x_n) \subseteq F$ with $x_n \to x$. If $x \notin F$, then $x \in F^c$, so there exists $r > 0$ such that $(x - r, x + r) \subseteq F^c$. For n large enough we have $x_n \in (x - r, x + r) \subseteq F^c$, contradicting $x_n \in F$. Therefore $x \in F$, and hence F is closed. $\qquad\square$

Remark 1.46. The same definitions make sense in $\mathbb{R}^n$ (and, more generally, in metric spaces).

Example 1.47. Let
$$A := \{(x, y) \in \mathbb{R}^2 : y = \sin(1/x), \ x \neq 0\}.$$

Then A is neither open nor closed in $\mathbb{R}^2$ (with the Euclidean metric). Indeed, the sequence
$$\left(\frac{1}{n\pi}, 0\right) \in A$$

converges to $(0,0) \notin A$, so A is not closed. On the other hand, no Euclidean ball centered at a point of A can lie entirely in A, because A is the graph of a function on $\mathbb{R} \setminus \{0\}$ and therefore has empty interior in $\mathbb{R}^2$. Hence A is not open.

1.4.2 Interior of a set

Definition 1.48 (Interior). Let $A \subseteq \mathbb{R}$. The **interior** of A, denoted A°, is the union of all open sets contained in A:
$$A^\circ := \bigcup \{ O \subseteq A : O \text{ is open in } \mathbb{R} \}.$$

Equivalently, A° is the largest open subset of A, in the sense that A° is open, $A^\circ \subseteq A$, and whenever O is open with $O \subseteq A$, we have $O \subseteq A^\circ$.

Example 1.49. We have
$$\mathbb{N}^\circ = \varnothing, \qquad \mathbb{Q}^\circ = \varnothing, \qquad (\mathbb{R} \setminus \mathbb{Q})^\circ = \varnothing.$$

Moreover, in $\mathbb{R}^2$ equipped with the Euclidean metric,
$$\{(x, y) \in \mathbb{R}^2 : y = \sin(1/x), \ x \neq 0\}^\circ = \varnothing.$$

1.4.3 Closure of a set

Definition 1.50 (Closure). Let $A \subseteq \mathbb{R}$. The **closure** of A, denoted $\overline{A}$, is the smallest closed set containing A, that is,
$$\overline{A} := \bigcap \{ F \subseteq \mathbb{R} : F \text{ is closed and } A \subseteq F \}.$$

Equivalently (in a metric space), $\overline{A}$ is the set of all limits of sequences in A:
$$\overline{A} = \{ x \in \mathbb{R} : \exists (x_n) \subseteq A \text{ with } x_n \to x \}.$$

Example 1.51. Let
$$A := \{(x, \sin(1/x)) \in \mathbb{R}^2 : x \neq 0\}.$$

Then
$$\overline{A} = A \cup (\{0\} \times [-1, 1]).$$

Proof. Let $(x_n, \sin(1/x_n)) \in A$ be a convergent sequence in $\mathbb{R}^2$ and write

$$(x_n, \sin(1/x_n)) \to (x, y).$$

If $x \neq 0$, then $x_n \to x$ with $x_n \neq 0$ for all sufficiently large n, and $t \mapsto \sin(1/t)$ is continuous at x. Hence $\sin(1/x_n) \to \sin(1/x)$, so $(x, y) = (x, \sin(1/x)) \in A$.

If $x = 0$, then automatically $y \in [-1, 1]$ because each $\sin(1/x_n) \in [-1, 1]$ and $[-1, 1]$ is closed. Thus every limit point of A lies in $A \cup (\{0\} \times [-1, 1])$, proving

$$\overline{A} \subseteq A \cup (\{0\} \times [-1, 1]).$$

For the reverse inclusion, fix $y \in [-1, 1]$ and choose $\theta \in [-\pi/2, \pi/2]$ such that $\sin \theta = y$. Define

$$x_n := \frac{1}{2\pi n + \theta} \qquad (n \geq 1).$$

Then $x_n \neq 0$, $x_n \to 0$, and

$$\sin(1/x_n) = \sin(2\pi n + \theta) = \sin \theta = y.$$

Hence $(x_n, \sin(1/x_n)) \to (0, y)$, showing $(0, y) \in \overline{A}$. Since $A \subseteq \overline{A}$, we conclude $A \cup (\{0\} \times [-1, 1]) \subseteq \overline{A}$. $\qquad \square$

1.4.4 Compactness

Definition 1.52 (Open cover). Let $K \subseteq \mathbb{R}^n$. A family $\{U_i\}_{i \in I}$ of open subsets of $\mathbb{R}^n$ is called an **open cover** of K if

$$K \subseteq \bigcup_{i \in I} U_i.$$

Definition 1.53 (Compact set). A set $K \subseteq \mathbb{R}^n$ is **compact** if every open cover of K admits a finite subcover.

Theorem 1.54 (Heine–Borel theorem). *A subset $K \subseteq \mathbb{R}^n$ (with the Euclidean metric) is compact if and only if it is closed and bounded.*

Idea. In Euclidean space, compactness has two complementary interpretations: a set must prevent sequences from escaping to infinity, and it must also retain the limits of sequences that remain trapped. Boundedness addresses the first issue, closedness the second.

Remark 1.55. In metric spaces, compactness (Definition 1.53) is equivalent to *sequential compactness* (every sequence has a convergent subsequence with limit in the set). In $\mathbb{R}^n$, Theorem 1.54 provides the convenient characterization "compact $\iff$ closed and bounded".

Remark 1.56. In $\mathbb{R}$ one may equivalently say that $K \subseteq \mathbb{R}$ is compact if and only if every open cover of K by open intervals admits a finite subcover. This equivalence follows from the general theory of compact metric spaces together with the Bolzano–Weierstrass theorem. Similar statements hold for compact subsets of $\mathbb{R}^n$.

Exercise 1.4. Show that the set

$$\overline{\{(x,y) \in \mathbb{R}^2 : y = \sin(1/x), \ x \neq 0\}}$$

is closed but not bounded.

Example 1.57. A subset $F \subseteq \mathbb{R}$ is closed if and only if it contains all of its adherent points; equivalently,

$$((x - \varepsilon, x + \varepsilon) \cap F \neq \varnothing \text{ for every } \varepsilon > 0) \implies x \in F.$$

Proof. Assume first that F is closed, and suppose that every interval $(x - \varepsilon, x + \varepsilon)$ meets F. For each $n \in \mathbb{N}$, choose

$$x_n \in \left(x - \frac{1}{n}, x + \frac{1}{n} \right) \cap F.$$

Then $|x_n - x| < 1/n$ for every n, so $x_n \to x$. Since F is closed and each x_n belongs to F, we conclude that $x \in F$.

Conversely, assume that F satisfies the displayed condition, and let $(x_n) \subseteq F$ be a sequence with $x_n \to x$. Fix $\varepsilon > 0$. For sufficiently large n we have $x_n \in (x - \varepsilon, x + \varepsilon)$, so $(x - \varepsilon, x + \varepsilon) \cap F \neq \varnothing$. By hypothesis, this implies $x \in F$. Therefore F is closed. $\square$

1.4.5 Dense Set

Definition 1.58. A subset $A \subseteq \mathbb{R}$ is called *dense in* $\mathbb{R}$ if $\overline{A} = \mathbb{R}$. Equivalently, A is dense in $\mathbb{R}$ if every nonempty open interval in $\mathbb{R}$ contains a point of A.

Example 1.59. Let $x \in \mathbb{R}$. Choose a decimal expansion

$$x = x_0 + \frac{x_1}{10} + \frac{x_2}{10^2} + \cdots, \qquad x_0 \in \mathbb{Z}, \ x_k \in \{0, 1, \ldots, 9\} \ (k \geq 1). \tag{1.5}$$

For $n \geq 0$, define the rational truncations

$$S_n := x_0 + \sum_{k=1}^{n} \frac{x_k}{10^k} \in \mathbb{Q}.$$

Then $S_n \to x$ as $n \to \infty$, so $\overline{\mathbb{Q}} = \mathbb{R}$.

Similarly, $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$. Indeed, fix $x \in \mathbb{R}$ and define

$$u_n := x + \frac{\sqrt{2}}{n}, \qquad v_n := x + \frac{\sqrt{3}}{n} \qquad (n \geq 1).$$

For each n, at least one of u_n or v_n is irrational: if both were rational, then their difference $(\sqrt{2} - \sqrt{3})/n$ would be rational, which is impossible. Choosing, for each n, an irrational number among $\{u_n, v_n\}$ produces a sequence of irrationals converging to x. Hence $\overline{\mathbb{R} \setminus \mathbb{Q}} = \mathbb{R}$.

Finally, the representation (1.5) is not unique; for instance, $0.5 = 0.4999\ldots$.

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Theorem 1.60. *Let $p \in \mathbb{Z}$ with $p \geq 2$, and let $x \in [0,1]$. Then there exists a sequence of integers $(a_n)_{n \geq 1}$ such that $0 \leq a_n \leq p - 1$ for every n and*

$$x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}.$$

Proof. We construct the digits inductively. Suppose that integers $a_1, \ldots, a_{n-1}$ have already been chosen so that

$$0 \leq x - \sum_{k=1}^{n-1} \frac{a_k}{p^k} \leq \frac{1}{p^{n-1}}.$$

Multiply this inequality by p^n and set

$$r_n := p^n \left(x - \sum_{k=1}^{n-1} \frac{a_k}{p^k} \right).$$

Then $0 \leq r_n \leq p$. Choose $a_n \in \{0, 1, \ldots, p - 1\}$ so that

$$a_n \leq r_n < a_n + 1.$$

Such a choice is possible because the interval $[0, p]$ is covered by the half-open intervals $[j, j + 1)$ for $j = 0, 1, \ldots, p - 1$, together with the endpoint p, for which we may take $a_n = p - 1$. It follows that

$$0 \leq r_n - a_n < 1,$$

and hence, after dividing by p^n,

$$0 \leq x - \sum_{k=1}^{n} \frac{a_k}{p^k} < \frac{1}{p^n}.$$

Thus the partial sums

$$S_n := \sum_{k=1}^{n} \frac{a_k}{p^k}$$

satisfy $0 \leq x - S_n < p^{-n}$ for every n. Since $p^{-n} \to 0$, we obtain $S_n \to x$. Therefore

$$x = \sum_{n=1}^{\infty} \frac{a_n}{p^n},$$

as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Exercise 1.5. Show that $\left\{ \frac{k}{2^n} : k = 0, 1, 2, \ldots, 2^n ; n = 1, 2, \ldots \right\}$ is dense in $[0, 1]$.
(Hint: Use binary expansion)

Exercises

Exercise 1.6. Prove the Archimedean property: for every $x \in \mathbb{R}$ there exists $n \in \mathbb{N}$ with $n > x$. Deduce that for every $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < \varepsilon$.

Exercise 1.7. Let $(I_n)_{n\geq 1}$ be a nested sequence of nonempty closed intervals with $I_{n+1} \subset I_n$ and $\ell(I_n) \to 0$. Prove that $\bigcap_{n=1}^{\infty} I_n$ consists of a single point (Nested Interval Theorem).

Exercise 1.8. Show that a subset $A \subset \mathbb{R}$ is open if and only if it is a (possibly empty) countable disjoint union of open intervals.

Exercise 1.9. Let $A \subset \mathbb{R}$. Prove that $x \in \overline{A}$ if and only if every open interval containing x meets A.

Exercise 1.10. Show that $\mathbb{Q}$ is dense in $\mathbb{R}$, and that $\mathbb{R} \setminus \mathbb{Q}$ is also dense in $\mathbb{R}$.

Exercise 1.11. Let C be the middle-third Cantor set. Prove that C is closed and has empty interior.

Exercise 1.12. Prove that every $x \in C$ admits a ternary expansion using only digits 0 and 2, and conversely. Deduce that C is uncountable.

Exercise 1.13. Show that C is perfect: it is closed and every point of C is a limit point of C.

Exercise 1.14. Prove that if $A \subset \mathbb{R}$ is closed and bounded, then A is compact (Heine–Borel in $\mathbb{R}$).

Exercise 1.15. Let $A \subset \mathbb{R}$ be nonempty and bounded above. Prove that $\sup A$ is characterized by: (i) $a \leq \sup A$ for all $a \in A$; (ii) for every $\varepsilon > 0$ there exists $a \in A$ with $\sup A - \varepsilon < a \leq \sup A$.

Chapter 2

Metric and Normed Linear Spaces

We extend the basic ideas from $\mathbb{R}$ to abstract metric spaces and normed linear spaces. After introducing metrics, open balls, and the induced topology, we discuss convergence, Cauchy sequences, and continuity in the metric setting. We then turn to normed spaces and the classical inequalities (Young, Hölder, Minkowski) that underpin the geometry of ℓ^p and L^p spaces. These notions prepare the ground for the study of completeness and compactness in later chapters.

Learning objectives.

- Move fluently between metric language, topological language, and norm-induced geometry.

- Prove continuity and convergence statements directly from the definitions and compare them across inequivalent metrics.

- Recognize the role of classical inequalities in turning linear structure into quantitative analytic control.

2.1   Metric Spaces and Topology

Section overview.

- This section builds the abstract language in which convergence, continuity, and compactness will later be phrased.

- The key task is to learn which arguments depend only on the metric and which use additional linear structure.

- As you read, compare every new definition with its familiar version on $\mathbb{R}^n$.

We introduce metric spaces as the common framework for convergence, continuity, and compactness. After developing the topology induced by a metric (open and closed sets, closure, density), we study continuous maps between metric spaces and the basic invariance properties they enjoy.

2.1.1   Metric spaces

Metrics and basic examples

Definition 2.1 (Metric space)**.** Let X be a non-empty set. A function $d\colon X \times X \to [0, \infty)$ is called a *metric* on X if, for all $x, y, z \in X$,

(i) $d(x, y) = 0$ if and only if $x = y$.

(ii) $d(x, y) = d(y, x)$ (symmetry).

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

The pair (X, d) is called a *metric space.*

Example 2.2 (Standard metrics on $\mathbb{R}^n$)**.** For $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, define

$$d_1(x, y) := \sum_{i=1}^{n} |x_i - y_i|,$$

$$d_2(x, y) := \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{1/2},$$

$$d_\infty(x, y) := \max_{1 \leq i \leq n} |x_i - y_i|.$$

Each of these functions defines a metric on $\mathbb{R}^n$.

Exercise 2.1. Let (X, d) be a metric space. Prove that $d'(x, y) := \min\{1, d(x, y)\}$ defines a metric on X.

Example 2.3 (Supremum metric on $C[0, 1]$)**.** Let $X = C[0, 1]$, the space of continuous real-valued functions on $[0, 1]$. For $f, g \in X$, set

$$d_\infty(f, g) := \sup_{0 \leq t \leq 1} |f(t) - g(t)|.$$

Since every continuous function on $[0, 1]$ is bounded, the supremum is finite. The triangle inequality follows from

$$|f(t) - h(t)| \leq |f(t) - g(t)| + |g(t) - h(t)|, \qquad t \in [0, 1].$$

Thus d_∞ is a metric on $C[0, 1]$.

Example 2.4 (Discrete metric)**.** Let $X \neq \emptyset$. Define, for $x, y \in X$,

$$d_0(x, y) := \begin{cases} 1, & x \neq y, \\ 0, & x = y. \end{cases}$$

Then d_0 is a metric on X. The metric space (X, d_0) is called the *discrete metric space.* In particular, every non-empty set admits at least one metric.

Remark 2.5. For the discrete metric d_0, the triangle inequality is immediate. Indeed, if $x = z$, then $d_0(x, z) = 0$. If $x \neq z$, then $d_0(x, z) = 1$, while $d_0(x, y) + d_0(y, z) \geq 1$. Thus $d_0(x, z) \leq d_0(x, y) + d_0(y, z)$ in every case.

Example 2.6. If (X, d) is a metric space, then

$$\tilde{d}(x, y) := \frac{d(x, y)}{1 + d(x, y)}$$

is also a metric on X.

Proof. Consider the function $f(t) = \frac{t}{1+t}$ on $[0, \infty)$. It is increasing, satisfies $f(0) = 0$, and one checks that

$$f(s + t) \leq f(s) + f(t) \qquad (s, t \geq 0).$$

Therefore $\tilde{d} = f \circ d$ is a metric by Proposition 2.7. $\qquad\square$

Proposition 2.7 (Changing a metric by a subadditive function). *Let (X, d) be a metric space and let $f \colon [0, \infty) \to [0, \infty)$ be an increasing function such that*

$$f(s + t) \leq f(s) + f(t) \qquad \text{for all } s, t \geq 0, \tag{2.1}$$

and $f(t) = 0$ if and only if $t = 0$. Then $d_f := f \circ d$ defines a metric on X.

Proof. Non-negativity and symmetry are immediate. If $d_f(x, y) = 0$, then $f(d(x, y)) = 0$, hence $d(x, y) = 0$ and $x = y$. Finally, by the triangle inequality for d and the monotonicity and subadditivity (2.1) of f,

$$\begin{aligned}
d_f(x, z) = f(d(x, z)) \\
\leq f(d(x, y) + d(y, z)) \\
\leq f(d(x, y)) + f(d(y, z)) \\
= d_f(x, y) + d_f(y, z).
\end{aligned}$$

$\qquad\square$

Example 2.8. Let H^∞ denote the set of all sequences $x = (x_n)$ with $|x_n| \leq 1$ for every n. Define

$$d(x, y) := \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n}.$$

Then d is a metric on H^∞.

Proof. Since $|x_n - y_n| \leq 2$ for every n, the series converges absolutely and

$$d(x, y) \leq \sum_{n=1}^{\infty} \frac{2}{2^n} < \infty.$$

Positivity and symmetry are immediate.

If $d(x, y) = 0$, then each summand must be zero, so $x_n = y_n$ for all n; hence $x = y$.

Finally, for $x, y, z \in H^\infty$ and each k,

$$\sum_{n=1}^{k} \frac{|x_n - z_n|}{2^n} \leq \sum_{n=1}^{k} \frac{|x_n - y_n|}{2^n} + \sum_{n=1}^{k} \frac{|y_n - z_n|}{2^n}.$$

Letting $k \to \infty$ gives

$$d(x, z) \leq d(x, y) + d(y, z).$$

Therefore d is a metric. $\qquad\square$

Exercise 2.2. Prove that $d(x, y) = \left| \frac{1}{x} - \frac{1}{y} \right|$ defines a metric on $(0, \infty)$.

Open balls and open sets

Definition 2.9 (Open and closed balls)**.** Let (X, d) be a metric space, let $x_0 \in X$, and let $r > 0$. The set

$$B_r(x_0) := \{y \in X : d(x_0, y) < r\}$$

is called the *open ball* of radius r centred at x_0. The set

$$\overline{B_r(x_0)} := \{y \in X : d(x_0, y) \leq r\}$$

is called the *closed ball* of radius r centred at x_0.
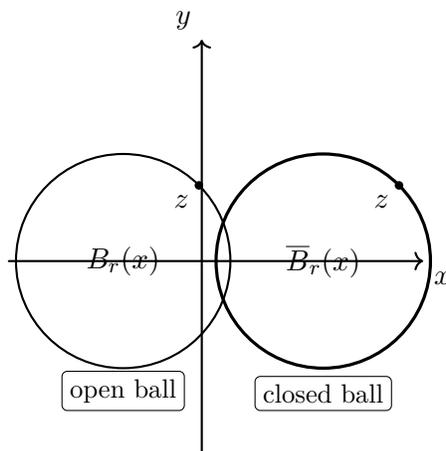
Figure 2.1. Open and closed metric balls (schematic in $\mathbb{R}^2$).

Definition 2.10 (Open sets)**.** A subset $O \subseteq X$ is called *open* if for every $x \in O$ there exists $r > 0$ such that

$$B_r(x) \subseteq O.$$

Proposition 2.11. *Let $\{O_i : i \in I\}$ be a family of open subsets of a metric space (X, d). Then:*

(i) $\bigcup_{i \in I} O_i$ *is open.*

(ii) $\bigcap_{i=1}^{n} O_i$ *is open for every finite collection* $O_1, \ldots, O_n$.

Arbitrary intersections of open sets need not be open.

Example 2.12. In $(\mathbb{R}, |\cdot|)$,

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\},$$

and the singleton $\{0\}$ is not open.

Example 2.13. Let $f \colon \mathbb{R} \to \mathbb{R}$ be continuous. Then the set

$$A := \{x \in \mathbb{R} : f(x) > 0\}$$

is open.

Proof. Fix $x \in A$. Then $f(x) > 0$. By continuity of f at x, with $\varepsilon = f(x)$, there exists $\delta > 0$ such that

$$|y - x| < \delta \quad \Longrightarrow \quad |f(y) - f(x)| < f(x).$$

For such y we obtain

$$f(y) > f(x) - |f(y) - f(x)| > 0.$$

Hence $(x - \delta, x + \delta) \subseteq A$. Since $x \in A$ was arbitrary, A is open. $\qquad\square$

Open Sets in $\mathbb{R}$***:***

A countable union of open intervals is an open set. On the other hand, any open set in $\mathbb{R}$ can be written as a countable union of disjoint open intervals.

Theorem 2.14 (Decomposition of open sets in $\mathbb{R}$)**.** *Let $O \subset \mathbb{R}$ be open. Then there exists a countable family of pairwise disjoint open intervals, possibly unbounded, $\{I_n\}_{n\geq 1}$ such that*

$$O = \bigcup_{n=1}^{\infty} I_n.$$

If $O = \varnothing$, this family is empty. Moreover, this representation is unique up to a permutation of the intervals.

Proof. If $O = \varnothing$, there is nothing to prove. Assume $O \neq \varnothing$.

For each $x \in O$, define

$$I_x := \bigcup \{J \subseteq O : J \text{ is an open interval and } x \in J\}.$$

This union is nonempty because O is open, so there exists $\delta > 0$ such that $(x - \delta, x + \delta) \subseteq O$. Since I_x is a union of open sets, it is open; clearly $x \in I_x \subseteq O$.

Step 1: I_x is an interval. Let $u, v \in I_x$ with $u < v$. Then there exist open intervals $J_u, J_v \subseteq O$ such that

$$x, u \in J_u, \qquad x, v \in J_v.$$

Because both intervals contain x, they intersect. Hence $J_u \cup J_v$ is an open interval contained in O. Since $u, v \in J_u \cup J_v$, every t with $u < t < v$ also lies in $J_u \cup J_v \subseteq I_x$. Therefore I_x is an open interval.

Step 2: maximality and disjointness. The interval I_x is maximal among open intervals contained in O and containing x: if $J \subseteq O$ is an open interval with $x \in J$, then J appears in the defining union, so $J \subseteq I_x$. If $x, y \in O$ and $I_x \cap I_y \neq \varnothing$, then $I_x \cup I_y$ is an open interval contained in O that contains both x and y. By maximality, $I_x \subseteq I_y$ and $I_y \subseteq I_x$, hence $I_x = I_y$. Consequently, the distinct intervals among $\{I_x : x \in O\}$ are pairwise disjoint, and

$$O = \bigcup_{x \in O} I_x.$$

Step 3: countability. Every nonempty open interval in $\mathbb{R}$ contains a rational number. For each $q \in \mathbb{Q} \cap O$, consider the interval I_q. Since $\mathbb{Q} \cap O$ is countable, the family $\{I_q : q \in \mathbb{Q} \cap O\}$ is at most countable. Conversely, if I_x is one of the intervals above, choose $q \in I_x \cap \mathbb{Q}$. Then $q \in O$ and $I_q = I_x$. Hence every distinct interval occurs as I_q for some rational $q \in O$. Therefore the collection of distinct intervals is countable. Renaming these intervals as $\{I_n\}_{n \geq 1}$ gives the desired representation.

Step 4: uniqueness. Suppose

$$O = \bigcup_{n=1}^{\infty} I_n = \bigcup_{m=1}^{\infty} J_m,$$

where $\{I_n\}$ and $\{J_m\}$ are pairwise disjoint families of open intervals. Fix n. Then

$$I_n = I_n \cap O = \bigcup_{m=1}^{\infty} (I_n \cap J_m).$$

Each set $I_n \cap J_m$ is open in the relative topology of I_n, and the family is pairwise disjoint. Since I_n is an interval, it is connected. Hence only one of these intersections can be nonempty. Thus $I_n \subseteq J_{m_0}$ for some m_0. By the same argument with the roles reversed, $J_{m_0} \subseteq I_n$, and therefore $I_n = J_{m_0}$. This proves uniqueness up to a permutation. $\square$

Sequences in metric spaces

Definition 2.15 (Convergent sequence)**.** A sequence (x_n) in a metric space (X, d) converges to $x_0 \in X$ if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n \geq N \quad \implies \quad d(x_n, x_0) < \varepsilon.$$

Definition 2.16 (Cauchy sequence)**.** A sequence (x_n) in (X, d) is a *Cauchy sequence* if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$m, n \geq N \quad \implies \quad d(x_n, x_m) < \varepsilon.$$

Example 2.17. In the metric space $((0,1), |\cdot|)$, the sequence $x_n = \frac{1}{n}$ is Cauchy, but it does not converge in $(0,1)$, because its limit in $\mathbb{R}$ is $0 \notin (0,1)$.

Every convergent sequence is Cauchy.

Definition 2.18 (Bounded set). A subset $A \subseteq X$ is called *bounded* if there exist $x_0 \in X$ and $M > 0$ such that

$$d(a, x_0) \leq M \qquad \text{for all } a \in A.$$

Example 2.19. The set

$$\left\{ \left(x, \sin \frac{1}{x} \right) : 0 < |x| \leq 1 \right\} \cup (\{0\} \times [-1, 1])$$

is bounded in $\mathbb{R}^2$.

Proposition 2.20. *Every Cauchy sequence is bounded.*

Proof. Let (x_n) be a Cauchy sequence. Taking $\varepsilon = 1$, there exists N such that

$$d(x_n, x_m) < 1 \qquad \text{whenever } m, n \geq N.$$

In particular,

$$d(x_n, x_N) < 1 \qquad \text{for all } n \geq N.$$

Set

$$M := \max\{1, d(x_1, x_N), \ldots, d(x_{N-1}, x_N)\}.$$

Then $d(x_n, x_N) \leq M$ for every n, so the sequence is bounded. □

The converse need not hold: the sequence $(-1)^n$ is bounded in $\mathbb{R}$ but not Cauchy.

Proposition 2.21. *Let (x_n) be a Cauchy sequence in (X, d). If a subsequence (x_{n_k}) converges to $x \in X$, then $x_n \to x$.*

Proof. Fix $\varepsilon > 0$. Since (x_n) is Cauchy, there exists N_1 such that

$$d(x_n, x_m) < \frac{\varepsilon}{2} \qquad \text{for all } m, n \geq N_1.$$

Since $x_{n_k} \to x$, there exists K such that

$$d(x_{n_k}, x) < \frac{\varepsilon}{2} \qquad \text{for all } k \geq K.$$

Choose k so large that $n_k \geq N_1$ and $k \geq K$. Then for every $n \geq N_1$,

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \varepsilon.$$

Hence $x_n \to x$. □

In particular, a Cauchy sequence can have at most one limit.

Closed sets in metric spaces

Definition 2.22. A subset $F \subseteq X$ is called *closed* if its complement $F^c = X \setminus F$ is open. Equivalently, F is closed if whenever $x \in X$ satisfies

$$B_\varepsilon(x) \cap F \neq \emptyset \qquad \text{for every } \varepsilon > 0,$$

then necessarily $x \in F$.

Example 2.23. The set

$$A = \left\{ (x, y) : y = \sin \frac{1}{x}, \ x \neq 0 \right\} \subset \mathbb{R}^2$$

is neither open nor closed. It is not closed because

$$\left(\frac{1}{n\pi}, 0 \right) \in A \qquad \text{and} \qquad \left(\frac{1}{n\pi}, 0 \right) \to (0, 0) \notin A.$$

It is not open because no Euclidean ball centred at a point of A can be contained in the graph.

Theorem 2.24. *Let (X, d) be a metric space and let $F \subseteq X$. The following are equivalent:*

(i) F is closed.

(ii) For every $x \in X$, if $B_\varepsilon(x) \cap F \neq \emptyset$ for every $\varepsilon > 0$, then $x \in F$.

(iii) Whenever $(x_n) \subseteq F$ and $x_n \to x$ in X, we have $x \in F$.

Proof. (i) $\Rightarrow$ (ii). Assume F is closed, so F^c is open. If $x \notin F$, then $x \in F^c$, and therefore there exists $\varepsilon_0 > 0$ such that

$$B_{\varepsilon_0}(x) \subseteq F^c.$$

Hence $B_{\varepsilon_0}(x) \cap F = \emptyset$, contradicting (ii). So $x \in F$.

(ii) $\Rightarrow$ (iii). Let $(x_n) \subseteq F$ and suppose $x_n \to x$. Given $\varepsilon > 0$, there exists N such that $x_n \in B_\varepsilon(x)$ for all $n \geq N$. Thus $B_\varepsilon(x) \cap F \neq \emptyset$ for every $\varepsilon > 0$, and (ii) implies $x \in F$.

(iii) $\Rightarrow$ (i). Suppose F is not closed. Then F^c is not open, so there exists $x \in F^c$ such that for every $n \in \mathbb{N}$ one can choose $x_n \in F$ with

$$d(x_n, x) < \frac{1}{n}.$$

Then $x_n \to x$, and by (iii) we conclude that $x \in F$, a contradiction. Therefore F is closed. $\square$

Example 2.25. If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then the zero set

$$A = \{ x \in \mathbb{R} : f(x) = 0 \}$$

is closed. Indeed, if $x_n \in A$ and $x_n \to x$, then continuity of f gives

$$f(x) = \lim_{n \to \infty} f(x_n) = 0,$$

so $x \in A$.

Interior points and interior of a set

Let $A \subseteq X$. The *interior* of A, denoted A° or $\text{Int}(A)$, is the largest open set contained in A:

$$A^\circ = \bigcup \{O \subseteq X : O \text{ is open and } O \subseteq A\}.$$

Equivalently, $x \in A^\circ$ if and only if there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A$.

Closure and limit points

Let $A \subseteq (X, d)$. The *closure* of A, denoted $\overline{A}$, is the smallest closed set containing A:

$$\overline{A} = \bigcap \{F \subseteq X : F \text{ is closed and } A \subseteq F\}.$$

Definition 2.26 (Limit point / accumulation point)**.** A point $x \in X$ is called a *limit point* of A if for every $r > 0$,

$$(B(x, r) \setminus \{x\}) \cap A \neq \emptyset.$$

The set of all limit points of A is denoted by A'.

Remark 2.27. If x is a limit point of A, then for each $n \in \mathbb{N}$ one can choose

$$x_n \in (B(x, 1/n) \setminus \{x\}) \cap A.$$

Then $x_n \to x$.

Exercise 2.3. Show that every finite subset of a metric space has no limit points.

Exercise 2.4. Let

$$A = \left\{ \left(n, \frac{1}{n} \right) : n \in \mathbb{N} \right\} \subset \mathbb{R}^2.$$

Show that $\overline{A} = A$ and $A^\circ = \emptyset$.

Example 2.28. 1. If

$$A = \{(x, y) : |x| < 1, \ |y| < 1\},$$

then

$$\overline{A} = \{(x, y) : |x| \leq 1, \ |y| \leq 1\}.$$

2. If

$$A = \left\{ (x, y) : y = \sin \left(\frac{1}{x} \right), \ x \neq 0 \right\},$$

then

$$\overline{A} = \left\{ (x, y) : y = \sin \left(\frac{1}{x} \right), \ x \neq 0 \right\} \cup (\{0\} \times [-1, 1]).$$

Proposition 2.29. *Let $A \subseteq (X, d)$. Then $x \in \overline{A}$ if and only if*

$$B_\varepsilon(x) \cap A \neq \varnothing \qquad \text{for every } \varepsilon > 0.$$

Proof. If $x \in \overline{A}$ and there existed $\varepsilon_0 > 0$ such that $B_{\varepsilon_0}(x) \cap A = \emptyset$, then $A \subseteq (B_{\varepsilon_0}(x))^c$. Since $(B_{\varepsilon_0}(x))^c$ is closed, minimality of $\overline{A}$ would imply

$$\overline{A} \subseteq (B_{\varepsilon_0}(x))^c,$$

contradicting $x \in \overline{A}$.

Conversely, if every ball centred at x meets A, then every ball centred at x meets the closed set $\overline{A}$. Hence $x \in \overline{A}$. $\qquad \square$

Proposition 2.30. *For every subset $A \subseteq X$,*

$$\overline{A} = A \cup A'.$$

Proof. If $x \in A'$, then every ball centred at x meets A, so $x \in \overline{A}$ by Proposition 2.29. Thus $A \cup A' \subseteq \overline{A}$.

Conversely, let $x \in \overline{A}$. If $x \in A$, then clearly $x \in A \cup A'$. If $x \notin A$, then every ball centred at x meets A at a point different from x, so $x \in A'$. Hence $\overline{A} \subseteq A \cup A'$. $\qquad \square$

Proposition 2.31. *A point x belongs to $\overline{A}$ if and only if there exists a sequence (x_n) in A such that $x_n \to x$.*

Proof. If $x \in \overline{A}$, then for each n we may choose $x_n \in B_{1/n}(x) \cap A$. Then $d(x_n, x) < 1/n$, so $x_n \to x$.

Conversely, if $x_n \in A$ and $x_n \to x$, then every neighborhood of x contains x_n for all sufficiently large n. Thus every ball centred at x meets A, so $x \in \overline{A}$. $\qquad \square$

Dense subsets

A set $A \subseteq X$ is called *dense* in X if $\overline{A} = X$. Equivalently, A is dense in X if every nonempty open ball meets A.

Example 2.32. The set $\mathbb{Q}$ is dense in $\mathbb{R}$. Indeed, every real number can be approximated by its finite decimal truncations, which are rational.

Example 2.33. If $1 \leq p < \infty$, then $\overline{c_{00}} = \ell^p$. For $x = (x_1, x_2, \dots) \in \ell^p$, define

$$X^{(n)} := (x_1, \dots, x_n, 0, 0, \dots) \in c_{00}.$$

Then

$$\|x - X^{(n)}\|_p^p = \sum_{k=n+1}^{\infty} |x_k|^p \to 0.$$

Hence $X^{(n)} \to x$ in ℓ^p.

Example 2.34. We also have $\overline{c_{00}} = c_0$ in the supremum norm. If $x = (x_n) \in c_0$, define

$$X^{(n)} := (x_1, \ldots, x_n, 0, 0, \ldots).$$

Since $x_n \to 0$,

$$\|x - X^{(n)}\|_\infty = \sup_{k \geq n+1} |x_k| \to 0.$$

Therefore $X^{(n)} \to x$ in c_0.

Remark 2.35. Thus $\overline{c_{00}} = c_0 \subsetneq \ell^\infty$, so c_{00} is not dense in ℓ^∞.

2.1.2 Continuous maps between metric spaces

A function $f \colon (X, d) \to (\mathbb{R}, u)$ is said to be *continuous at $x_0 \in X$* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d(x_0, y) < \delta \quad \implies \quad |f(y) - f(x_0)| < \varepsilon.$$

Equivalently,

$$f(B_\delta(x_0)) \subseteq (f(x_0) - \varepsilon, \; f(x_0) + \varepsilon).$$

Example 2.36. Fix $x_0 \in X$ and define $f(y) = d(x_0, y)$. Then f is continuous on X. Indeed, the triangle inequality gives

$$|d(x_0, y) - d(x_0, z)| \leq d(y, z) \qquad \text{for all } y, z \in X.$$

Thus f is in fact Lipschitz continuous with constant 1.

Theorem 2.37. *Let $f : (X, d) \to (\mathbb{R}, u)$. Then the following are equivalent:*

(i) f is continuous on X (with $\varepsilon - \delta$ definition).

(ii) For any sequence $x_n \in X$ such that $x_n \to x \implies f(x_n) \to f(x)$.

(iii) $f^{-1}(O)$ is open in (X, d), for every open set $O \subseteq \mathbb{R}$.

(iv) $f^{-1}(F)$ is closed in (X, d), for every closed set $F \subseteq \mathbb{R}$.

Proof. Recall that f is *continuous at $x \in X$* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d(x, y) < \delta \implies |f(y) - f(x)| < \varepsilon.$$

The map f is *continuous on X* if it is continuous at every $x \in X$. **(i) $\Rightarrow$ (ii).** Assume that f is continuous on X and let $(x_n) \subset X$ satisfy $x_n \to x$. Fix $\varepsilon > 0$. By continuity of f at x, there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $|f(y) - f(x)| < \varepsilon$. Since $x_n \to x$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $d(x_n, x) < \delta$. Hence for all $n \geq N$,

$$|f(x_n) - f(x)| < \varepsilon,$$

so $f(x_n) \to f(x)$. **(ii) $\Rightarrow$ (i).** Assume (ii). Fix $x \in X$ and suppose, for contradiction, that f is not continuous at x. Then there exists $\varepsilon_0 > 0$ such that for every $\delta > 0$ there exists $y \in X$ with

$$d(x, y) < \delta \quad \text{and} \quad |f(y) - f(x)| \geq \varepsilon_0.$$

For each $n \in \mathbb{N}$, apply this with $\delta = \frac{1}{n}$ to choose $y_n \in X$ satisfying

$$d(x, y_n) < \frac{1}{n}, \qquad |f(y_n) - f(x)| \geq \varepsilon_0.$$

Then $y_n \to x$ in X, but $f(y_n) \not\to f(x)$ in $\mathbb{R}$, contradicting (ii). Therefore f is continuous at x. Since x was arbitrary, f is continuous on X. **(i) $\Rightarrow$ (iii).** Assume f is continuous on X and let $O \subset \mathbb{R}$ be open. Take $x \in f^{-1}(O)$, so $f(x) \in O$. Since O is open, there exists $\varepsilon > 0$ such that

$$(f(x) - \varepsilon, \ f(x) + \varepsilon) \subset O.$$

By continuity of f at x, there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $|f(y) - f(x)| < \varepsilon$, that is $f(y) \in (f(x) - \varepsilon, f(x) + \varepsilon) \subset O$. Thus $y \in f^{-1}(O)$ whenever $d(x, y) < \delta$, so

$$B_\delta(x) \subset f^{-1}(O).$$

Hence $f^{-1}(O)$ is open in X. **(iii) $\Rightarrow$ (i).** Assume (iii). Fix $x \in X$ and $\varepsilon > 0$. Let

$$O := (f(x) - \varepsilon, \ f(x) + \varepsilon),$$

which is open in $\mathbb{R}$. By (iii), $f^{-1}(O)$ is open in X, and plainly $x \in f^{-1}(O)$. Therefore, there exists $\delta > 0$ such that $B_\delta(x) \subset f^{-1}(O)$. If $d(x, y) < \delta$, then $y \in f^{-1}(O)$, so $f(y) \in O$, that is $|f(y) - f(x)| < \varepsilon$. This proves continuity of f at x. Since x was arbitrary, f is continuous on X. **(iii) $\Rightarrow$ (iv).** Assume (iii) and let $F \subset \mathbb{R}$ be closed. Then $\mathbb{R} \setminus F$ is open, so $f^{-1}(\mathbb{R} \setminus F)$ is open in X by (iii). But

$$f^{-1}(\mathbb{R} \setminus F) = X \setminus f^{-1}(F).$$

Hence $X \setminus f^{-1}(F)$ is open, which implies $f^{-1}(F)$ is closed. **(iv) $\Rightarrow$ (iii)** is similar to the above. $\qquad\qquad\square$

Remark 2.38. More generally, if (X, d) and (Y, ρ) are metric spaces, then a map $f \colon X \to Y$ is continuous at $x \in X$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d(x, y) < \delta \quad \Longrightarrow \quad \rho\big(f(x), f(y)\big) < \varepsilon.$$

This is the usual ε–δ formulation of continuity in metric spaces.

Uniform continuity

Definition 2.39. A function $f \colon A \subseteq (X, d) \to (Y, \rho)$ is said to be *uniformly continuous* on A if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d(x, y) < \delta \quad \implies \quad \rho(f(x), f(y)) < \varepsilon \qquad \text{for all } x, y \in A.$$

The essential point is that δ depends only on ε, not on the location of the points.

Example 2.40. For a fixed $x_0 \in X$, the function $f(x) = d(x, x_0)$ is uniformly continuous on X, because

$$|f(x) - f(y)| = |d(x, x_0) - d(y, x_0)| \leq d(x, y).$$

Example 2.41. If $A \subseteq X$, define

$$d(x, A) := \inf\{d(x, a) : a \in A\}.$$

Then the map $x \mapsto d(x, A)$ is uniformly continuous, since

$$|d(x, A) - d(y, A)| \leq d(x, y) \qquad \text{for all } x, y \in X.$$

Example 2.42. The function $f(x) = 1/x$ is continuous on $(0, 1)$ but not uniformly continuous there. Indeed, if

$$x_n = \frac{1}{n}, \qquad y_n = \frac{1}{n+1},$$

then $|x_n - y_n| \to 0$, but

$$|f(x_n) - f(y_n)| = 1$$

for every n.

Theorem 2.43. *Let $f \colon A \subseteq (X, d) \to \mathbb{R}$. Then f is uniformly continuous on A if and only if for every pair of sequences (x_n) and (y_n) in A satisfying $d(x_n, y_n) \to 0$, one has*

$$|f(x_n) - f(y_n)| \to 0.$$

Proof. Assume first that f is uniformly continuous. Given $\varepsilon > 0$, choose $\delta > 0$ such that

$$d(x, y) < \delta \quad \implies \quad |f(x) - f(y)| < \varepsilon$$

for all $x, y \in A$. If $d(x_n, y_n) \to 0$, then $d(x_n, y_n) < \delta$ for all sufficiently large n, and hence $|f(x_n) - f(y_n)| < \varepsilon$ eventually. Therefore $|f(x_n) - f(y_n)| \to 0$.

Conversely, suppose the sequential criterion holds, but f is not uniformly continuous. Then there exists $\varepsilon_0 > 0$ such that for every $n \in \mathbb{N}$ one can choose $x_n, y_n \in A$ with

$$d(x_n, y_n) < \frac{1}{n} \qquad \text{and} \qquad |f(x_n) - f(y_n)| \geq \varepsilon_0.$$

Thus $d(x_n, y_n) \to 0$, while $|f(x_n) - f(y_n)|$ does not tend to 0, a contradiction. $\qquad \square$

Example 2.44. A uniformly continuous function sends Cauchy sequences to Cauchy sequences. Indeed, if (x_n) is Cauchy in X and f is uniformly continuous, then for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d(x, y) < \delta \quad \Longrightarrow \quad |f(x) - f(y)| < \varepsilon.$$

Since (x_n) is Cauchy, we have $d(x_n, x_m) < \delta$ for all sufficiently large m, n, and therefore

$$|f(x_n) - f(x_m)| < \varepsilon$$

for all sufficiently large m, n. Hence $(f(x_n))$ is Cauchy in $\mathbb{R}$.

Theorem 2.45. *Let $f \colon [a, b] \to \mathbb{R}$ be continuous. Then f is uniformly continuous on $[a, b]$.*

Proof. If f were not uniformly continuous, then there would exist $\varepsilon_0 > 0$ and sequences (x_n), (y_n) in $[a, b]$ such that

$$|x_n - y_n| < \frac{1}{n} \qquad \text{but} \qquad |f(x_n) - f(y_n)| \geq \varepsilon_0.$$

By compactness of $[a, b]$, a subsequence (x_{n_k}) converges to some $x \in [a, b]$. Since $|x_{n_k} - y_{n_k}| \to 0$, we also have $y_{n_k} \to x$. By continuity,

$$f(x_{n_k}) \to f(x) \qquad \text{and} \qquad f(y_{n_k}) \to f(x),$$

so $|f(x_{n_k}) - f(y_{n_k})| \to 0$, contradicting $|f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon_0$. $\qquad \square$

Theorem 2.46 (Uniform continuity on compact sets). *Let K be compact and let $f : K \to Y$ be continuous between metric spaces. Then f is uniformly continuous on K.*

Proof strategy. Negate uniform continuity to produce two nearby sequences whose images stay uniformly separated. Compactness lets us extract a convergent subsequence; continuity then forces the two image sequences to collide, giving the contradiction.

Proof. Assume f is not uniformly continuous. Then there exists $\varepsilon_0 > 0$ and sequences $x_n, y_n \in K$ such that

$$d(x_n, y_n) \to 0 \quad \text{and} \quad \rho\big(f(x_n), f(y_n)\big) \geq \varepsilon_0 \text{ for all } n.$$

By compactness, there exists a subsequence $x_{n_k} \to x \in K$. Since $d(x_{n_k}, y_{n_k}) \to 0$, it follows that $y_{n_k} \to x$. By continuity, $f(x_{n_k}) \to f(x)$ and $f(y_{n_k}) \to f(x)$, contradicting $\rho(f(x_{n_k}), f(y_{n_k})) \geq \varepsilon_0$. Hence f is uniformly continuous. $\qquad \square$

Example 2.47. Let $f : (a, b] \to \mathbb{R}$ and $f : (b, c) \to \mathbb{R}$ be uniformly continuous. Then $f : (a, c) \to \mathbb{R}$ is uniformly continuous.

Proof. Since f is uniformly continuous on $(a, b]$ and (b, c), for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $x, y \in (a, b]$ or $x, y \in (b, c)$ with $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. Now, let $x, y \in (a, c)$, with $|x - y| < \delta$. Then $|x - b| < \delta$ and $|y - b| < \delta$. Hence,

$$|f(x) - f(y)| < |f(x) - f(b)| + |f(b) - f(y)| < 2\varepsilon.$$

Thus, f is uniformly continuous on (a, c). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Example 2.48. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous such that $\lim_{|x|\to\infty} f(x) = 0$. Then f is uniformly continuous.

Proof. For $\varepsilon > 0$, there exists $[-a, a]$ such that $|f(x)| < \varepsilon/2$ if $x \in [-a, a]^c$. Hence, if $x, y \in [-a, a]^c$, then

$$|f(x) - f(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \tag{2.2}$$

Since f is uniformly continuous on $[-a, a]$. For $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

Since 2.2 holds true for x, y with $|x - y| < \delta$. It follows that for $\varepsilon > 0$, we get $\delta > 0$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$ (for any $x, y \in \mathbb{R}$). Hence, f is uniformly continuous on $\mathbb{R}$. $\quad\square$

Notice that if $f \in C_0(\mathbb{R})$, that is f is continuous and $\lim_{|x|\to\infty} f(x) = 0$ and hence f is uniformly continuous. But if f is continuous and bounded, then f need not be uniformly continuous on $\mathbb{R}$.

Takeaway. Compactness of the domain is the decisive hypothesis. Boundedness of the range, by itself, places almost no local restriction on the oscillation of the function.

Example 2.49. $f(x) = \sin x^2$, which is continuous and bounded but not uniformly continuous on $\mathbb{R}$. (Hint: Take $x^2 = n\pi$ and $y^2 = n\pi + \frac{1}{2}\pi$.)

Example 2.50. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and its derivative is bounded. Then f is uniformly continuous on $\mathbb{R}$. For any $x, y \in \mathbb{R}$, by the Mean Value Theorem,

$$|f(x) - f(y)| = |f'(t)(x - y)| \leq M|x - y|,$$

where t is between x and y, and M is an upper bound for $|f'(t)|$. However, $f(x) = \sqrt{x}$ for $x \in (0, \infty)$ is uniformly continuous, but its derivative is $f'(x) = \frac{1}{2\sqrt{x}}$, is not bounded.

Example 2.51. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function. If f is monotone, then f is uniformly continuous on $\mathbb{R}$. Since f is bounded, let $\inf f(x) = L, \quad \sup f(x) = M$. For $\varepsilon > 0$, there exist $x_0, y_0 \in \mathbb{R}$ such that $f(x_0) < L + \varepsilon$ and $f(y_0) > M - \varepsilon$.
If f is monotone increasing, then for $x, y \in [x_0, y_0]^c$ and $x, y \geq y_0$

$$f(y) - f(x) \leq M - f(y_0) < M - (M - \varepsilon) = \varepsilon.$$

Similarly, if $x, y \leq x_0$ then

$$f(y) - f(x) \leq L + \varepsilon - f(x_0) < L + \varepsilon - L = \varepsilon.$$

Thus, for $x, y \in [x_0, y_0]^c$, we get

$$|f(x) - f(y)| < \varepsilon. \tag{2.3}$$

Since f is continuous on $[x_0, y_0]$, f is uniformly continuous on $[x_0, y_0]$. For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x, y \in [x_0, y_0], \ |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

Notice that 2.3 also holds for $x, y \in [x_0, y_0]^c$ with $|x - y| < \delta$. Thus, we get a single $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

We observe that a uniformly continuous function can be extended uniformly to the closure of the set.

Theorem 2.52 (Uniformly continuous extension to the closure). *Let (X, d) be a metric space, let $A \subset X$, and let (Y, ρ) be a complete metric space. If $f \colon A \to Y$ is uniformly continuous, then there exists a unique uniformly continuous map $\widetilde{f} \colon \overline{A} \to Y$ such that $\widetilde{f}|_A = f$.*

Proof. Fix $x \in \overline{A}$. Choose a sequence $(x_n) \subset A$ with $x_n \to x$. Then (x_n) is Cauchy in X, hence $(f(x_n))$ is Cauchy in Y because f is uniformly continuous. Since Y is complete, the limit $\lim_{n \to \infty} f(x_n)$ exists. Define

$$\widetilde{f}(x) := \lim_{n \to \infty} f(x_n).$$

Well-definedness. If $(y_n) \subset A$ is another sequence with $y_n \to x$, then $d(x_n, y_n) \to 0$. By uniform continuity, $\rho(f(x_n), f(y_n)) \to 0$. Since both $(f(x_n))$ and $(f(y_n))$ converge, their limits must coincide, so $\widetilde{f}$ is well defined. *Extension property.* If $x \in A$, choose the constant sequence $x_n = x$. Then $\widetilde{f}(x) = f(x)$. *Uniform continuity of $\widetilde{f}$.* Let $\varepsilon > 0$. By uniform continuity of f, there exists $\delta > 0$ such that

$$d(u, v) < \delta \implies \rho(f(u), f(v)) < \varepsilon/3 \qquad (u, v \in A).$$

Set $\delta' := \delta/3$. Take $x, y \in \overline{A}$ with $d(x, y) < \delta'$. Choose sequences $(x_n), (y_n) \subset A$ such that $x_n \to x$ and $y_n \to y$. Pick N so large that

$$d(x_N, x) < \delta', \qquad d(y_N, y) < \delta', \qquad \rho(\widetilde{f}(x), f(x_N)) < \varepsilon/3, \qquad \rho(\widetilde{f}(y), f(y_N)) < \varepsilon/3.$$

Then

$$d(x_N, y_N) \le d(x_N, x) + d(x, y) + d(y, y_N) < \delta' + \delta' + \delta' = \delta,$$

hence $\rho(f(x_N), f(y_N)) < \varepsilon/3$. Therefore,

$$\rho(\widetilde{f}(x), \widetilde{f}(y)) \le \rho(\widetilde{f}(x), f(x_N)) + \rho(f(x_N), f(y_N)) + \rho(f(y_N), \widetilde{f}(y)) < \varepsilon.$$

This proves that $\widetilde{f}$ is uniformly continuous on $\overline{A}$. *Uniqueness.* If $g\colon \overline{A} \to Y$ is another uniformly continuous extension of f, then g is continuous. For $x \in \overline{A}$ and any sequence $(x_n) \subset A$ with $x_n \to x$, continuity gives

$$g(x) = \lim_{n \to \infty} g(x_n) = \lim_{n \to \infty} f(x_n) = \widetilde{f}(x),$$

so $g = \widetilde{f}$. $\hfill\square$

Exercise 2.5 (Why completeness is needed)**.** Show that completeness of the target space cannot be omitted in Theorem 2.52. More precisely, consider $A = \mathbb{Q} \subset \mathbb{R}$ with the usual metric and the function $f\colon \mathbb{Q} \to \mathbb{Q}$ given by $f(q) = q$. Prove that f is uniformly continuous on $\mathbb{Q}$, but there is no continuous (hence no uniformly continuous) extension $\widetilde{f}\colon \mathbb{R} \to \mathbb{Q}$.

Next, we shall see that a uniformly continuous function grows at most linearly.

Theorem 2.53. *Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous. Then there exist constants $A, B \geq 0$ such that $|f(x)| \leq A|x| + B$ for all $x \in \mathbb{R}$.*

Proof. For any $\varepsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < 1$. We divide the proof into two parts: one is near "0" and other is away from "0". Let $a > 0$. Then $|f(x)| \leq A < \infty$ for $x \in [-a, a]$. Now, consider $f : [a, \infty) \to \mathbb{R}$. Then for $x \in [a, \infty)$, we can find $n \in \mathbb{N}$ such that $x \in [a + n\delta,\, a + (n+1)\delta]$. Then,

$$f(x) - f(a) = f(x) - f(a + n\delta) + f(a + n\delta) - f(a)$$

$$= f(x) - f(a + n\delta) + \sum_{j=1}^{n} [f(a + j\delta) - f(a + (j+1)\delta)]$$

$$\Rightarrow |f(x)| < 1 + n + |f(a)|$$

$$\Rightarrow \left| \frac{f(x)}{x} \right| < \frac{(n+1) + |f(a)|}{a + n\delta} < \frac{(n+1) + |f(a)|}{n\delta} < \left(1 + \frac{1}{n}\right)\frac{1}{\delta} + \frac{|f(a)|}{n\delta} \leq B < \infty.$$

Notice that B is independent of n, hence B is independent of x. That is, $|f(x)| \leq B|x|$ if $x > a$. Therefore, $|f(x)| \leq B|x| + A$ for all $x \in \mathbb{R}$. $\hfill\square$

Example 2.54. Notice that $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$, as it does not satisfy the conclusion of the above theorem. On the other hand, for $x_n = n$ and $y_n = n + \frac{1}{n}$, we have $|f(x_n) - f(y_n)| = 2$.

2.2 Normed Linear Spaces and Classical Inequalities

2.2.1 Normed Linear Space

A *normed linear space* is a vector space equipped with a notion of length that is compatible with its linear structure.

Let $(X, +, \cdot)$ be a vector space over the field F (that is, $F = \mathbb{R}$ or $F = \mathbb{C}$). Our aim is to endow X with a topology that reflects the algebraic operations naturally. In other words, we seek a notion of distance on X that interacts well with vector addition and scalar multiplication.
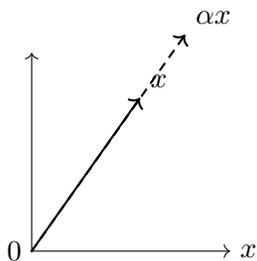
The linear structure of X is encoded by the two basic operations

$$
\begin{aligned}
(i) \quad & (x, y) \mapsto x + y \quad && (X \times X \to X), \\
(ii) \quad & (\alpha, x) \mapsto \alpha x \quad && (F \times X \to X).
\end{aligned}
$$

A topology on X is compatible with the vector-space structure when these two maps are continuous. A vector space endowed with such a topology is called a *topological vector space.*

If J denotes the topology on X and U the usual topology on F, then the relevant product topologies are $J \times J$ on $X \times X$ and $U \times J$ on $F \times X$. Open sets in these product spaces are unions of basic rectangles of the form $O_1 \times O_2$.

In normed spaces, the topology comes from a function that measures the size of a vector. Motivated by linearity and homogeneity, we seek a notion of distance from the origin satisfying the following properties:

$$
\begin{aligned}
(i) \quad & \text{dist}(0, \alpha x) = |\alpha| \, \text{dist}(0, x) \\
(ii) \quad & \text{dist}(0, x + y) \le \text{dist}(0, x) + \text{dist}(0, y) \\
(iii) \quad & \text{when } \alpha = 0, \ \text{dist}(0, 0) = 0
\end{aligned}
$$

Let $p := \text{dist} : X \to [0, \infty)$ be defined by $p(x) = \text{dist}(0, x)$. Then

$$
\begin{aligned}
(i) \quad & p(x) = 0 \quad \text{for } x = 0 \\
(ii) \quad & p(\alpha x) = |\alpha| \, p(x) \quad \text{(absolute homogeneity)} \\
(iii) \quad & p(x + y) \le p(x) + p(y) \quad \text{(triangle inequality)}
\end{aligned}
$$

Such a function p is called a *seminorm.* It satisfies the basic algebraic properties of a norm, except that nonzero vectors may still have seminorm 0.

Example 2.55.
$$p : \mathbb{R}^2 \to [0, \infty), \quad p(x_1, x_2) = |x_1|.$$

Then p is a seminorm and $p(0, 1) = 0$.

Thus every point on the y-axis has seminorm 0. This shows precisely why a seminorm need not define an honest metric on the underlying vector space.

Let $\| \cdot \| : X \to [0, \infty)$ be a map such that

(i) $\|x\| \geq 0$ for each $x \in X$, and $\|x\| = 0$ if and only if $x = 0$.

(ii) $\|\alpha x\| = |\alpha| \, \|x\|$ for each $(\alpha, x) \in F \times X$ (absolute homogeneity).

(iii) $\|x + y\| \leq \|x\| + \|y\|$ for each $x, y \in X$ (triangle inequality).

The map $\| \cdot \|$ is called a *norm* on X.

Every norm induces a metric on X by

$$d(x, y) = \|x - y\|.$$

This metric in turn determines a topology on X. For $r > 0$ and $x \in X$, the corresponding open ball is

$$B_r(x) = \{y \in X : \|x - y\| < r\}.$$

Open sets are then defined in the usual metric sense.

Not every metric on a vector space arises from a norm. For example, the discrete metric on a vector space is not induced by any norm, since it fails the absolute homogeneity property.

For $x, y \in X$, define

$$d_0(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

If we write $\|x\| = d(0, x)$, then for $\alpha \in \mathbb{F}$, $\|\alpha x\| \neq |\alpha| \|x\|$ ($x \neq 0$) unless $|\alpha| = 1$.

However, if d is a metric on a linear space X such that $d(x, y) = d(x - y, 0)$ and $d(\alpha x, \alpha y) = |\alpha| d(x, y)$, then $d(x, 0) = \|x\|$ defines a norm on X.

1. $\|x\| = 0 \iff d(x, 0) = 0 \iff x = 0$.

2. $\|\alpha x\| = d(\alpha x, 0) = |\alpha| d(x, 0) = |\alpha| \|x\|$.

3. $\|x + y\| = d(x + y, 0) = d(x, -y) \leq d(x, 0) + d(-y, 0) = \|x\| + \|y\|$.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be convex if

$$f(t_1 x_1 + \cdots + t_n x_n) \leq t_1 f(x_1) + \cdots + t_n f(x_n)$$

where $0 \leq t_i \leq 1$ and $x_i \in \mathbb{R}^n$.

Example 2.56. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function satisfying $f(\alpha x) = \alpha f(x)$ for all $\alpha \in \mathbb{R}$, for all $x \in \mathbb{R}^n$. Prove that

(i) $f(x + y) \leq f(x) + f(y)$.

(ii) $f(0) = 0$.

(iii) $f(-x) \geq -f(x)$.

(iv) $f(t_1 x_1 + \cdots + t_n x_n) \leq t_1 f(x_1) + \cdots + t_n f(x_n)$.

Further, what requires to make f a norm on $\mathbb{R}^n$?

2.2.2 Convergence of Sequence in Metric Space

A sequence (x_n) in a metric space (X, d) is said to be converging to $x \in X$, if for any $\epsilon > 0$, $\exists N_0 \in \mathbb{N}$ such that $n \geq N_0 \implies d(x_n, x) < \epsilon$.

Example 2.57. Let $X = (0, \infty)$ and $d(x, y) = \left| \frac{1}{x} - \frac{1}{y} \right|$. Then $x_n = n$ does not converge to any point of X. However, this sequence is not so bad as $x_n = n \to 0$, which is not in X. Such sequences can be classified as Cauchy sequences.

2.2.3 Cauchy Sequences

Definition 2.58. A sequence (x_n) in (X, d) is said to be a *Cauchy sequence* if for any $\epsilon > 0$, there exist $N_0 \in \mathbb{N}$ such that $\forall m, n \geq N_0$, $d(x_n, x_m) < \epsilon$.

Example 2.59. Show that every Cauchy sequence in a metric space is bounded.

Proof. Let (x_n) be a Cauchy sequence in (X, d). Choose $n_0 \in \mathbb{N}$ such that

$$d(x_n, x_m) < 1, \qquad m, n \geq n_0.$$

In particular, $d(x_n, x_{n_0}) < 1$ for every $n \geq n_0$, so the tail of the sequence lies in $B_1(x_{n_0})$. Set

$$r := \max\{1, d(x_{n_0}, x_1), \ldots, d(x_{n_0}, x_{n_0 - 1})\}.$$

Then $d(x_n, x_{n_0}) \leq r$ for every $n \in \mathbb{N}$, and hence $\{x_n : n \in \mathbb{N}\} \subseteq B_r(x_{n_0})$. Thus every Cauchy sequence is bounded. $\qquad\square$

2.2.4 Young's Inequality

Let $1 < p < \infty$ and $a, b > 0$. Then for $\frac{1}{p} + \frac{1}{q} = 1$,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \tag{2.4}$$

Proof: Let $y = x^{p-1}$. Since $\frac{1}{p} + \frac{1}{q} = 1$, we have $p - 1 = \frac{1}{q-1}$ and therefore $x = y^{q-1}$. Hence

$$ab \leq \int_0^a x^{p-1} \, dx + \int_0^b y^{q-1} \, dy = \frac{a^p}{p} + \frac{b^q}{q}$$

Note that equality in (2.4) holds if and only if $a^p = b^q$ (or $a = b^{q-1}$). For this, consider

$$ab = \frac{a^p}{p} + \frac{b^q}{q}, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

Replace $a \to a^{\frac{1}{p}}$, $b \to b^{\frac{1}{q}}$ and $\frac{1}{p} = \alpha$. Then, we get

$$a^\alpha b^{1-\alpha} = \alpha a + (1 - \alpha)b$$

or

$$t^\alpha - \alpha t - (1 - \alpha) = 0 \quad \text{if} \quad t = a/b.$$

Let

$$f(t) = t^\alpha - \alpha t - (1 - \alpha), \quad t \in (0, \infty).$$

Then $f(1) = 0$ and

$$f'(t) = \alpha t^{\alpha-1} - \alpha = \alpha(t^{\alpha-1} - 1) = 0 \iff t = 1.$$

Since $f'(t) > 0$ for $0 < t < 1$ and $f'(t) < 0$ for $t > 1$, the function f is strictly increasing on $(0, 1)$ and strictly decreasing on $(1, \infty)$. Thus $t = 1$ is the unique point at which f attains its maximum, so $f(t) \leq f(1) = 0$ for all $t > 0$. This yields the inequality again, and equality holds if and only if $t = 1$, equivalently, if and only if $a^p = b^q$.

Example 2.60. Let $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Write

$$\|x\|_1 = \sum_{i=1}^{n} |x_i|$$

Then $(\mathbb{R}^n, \|\cdot\|_1)$ is a normed linear space (n.l.s.). If

$$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2}$$

then by Cauchy-Schwarz inequality, $(\mathbb{R}^n, \|\cdot\|_2)$ is a n.l.s. For

$$\|x\|_\infty = \sup_i |x_i|$$

$(\mathbb{R}^n, \|\cdot\|_\infty)$ is a normed linear space.
For $1 \leq p < \infty$, write

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

Then $\ell_n^p := (\mathbb{R}^n, \|\cdot\|_p)$ will be a normed linear space.
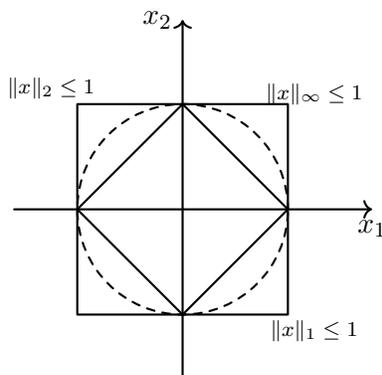
Figure 2.2. Unit balls in $\mathbb{R}^2$ for the norms $\|\cdot\|_1$, $\|\cdot\|_2$, and $\|\cdot\|_\infty$.

2.2.5 Space of Sequences

Let $1 \le p < \infty$ and let ℓ^p denote the space of all sequences that satisfy

$$\sum_{i=1}^{\infty} |x_i|^p < \infty; \qquad x = (x_1, x_2, \ldots, x_n, \ldots)$$

Then $(\ell^p, \|\cdot\|_p)$ or simply ℓ^p, will be a normed linear space.
If $p = \infty$,

$$\|x\|_\infty = \sup_{1 \le i < \infty} |x_i| < \infty,$$

then $(\ell^\infty, \|\cdot\|_\infty)$ is a normed linear space (follows from definition of supremum).
For $1 \le p < \infty$, showing ℓ^p is a normed linear space required the following inequalities.

2.2.6 Hölder's Inequality

Let $1 \le p \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then for $x \in \ell^p$ and $y \in \ell^q$, it follows that

$$x \cdot y (= x_1 y_1 + \ldots + x_n y_n + \ldots) \in \ell^1,$$

and

$$\|x \cdot y\|_1 \le \|x\|_p \|y\|_q \qquad \cdots (2.4)$$

(where $\frac{1}{\infty} = 0$ adopted.) When $p = 1$, $q = \infty$. In this case (2.4),

$$\|x \cdot y\|_1 = \sum_{i=1}^{\infty} |x_i y_i| \le \sum |x_i| \cdot \sup |y_i| = \|x\|_1 \|y\|_\infty$$

Now, let $1 < p < \infty$, then $1 < q < \infty$. Substitute $a = a_j = \frac{|x_j|}{\|x\|_p}$ and $b = b_j = \frac{|y_j|}{\|y\|_q}$ in the Young's Inequality. Then

$$\sum_{i=1}^{n} \frac{|x_j y_j|}{\|x\|_p \|y\|_q} \le \sum_{i=1}^{n} \left(\frac{|x_j|^p}{p \|x\|_p^p} + \frac{|y_j|^q}{q \|y\|_q^q} \right) \le \left(\frac{\|x\|_p^p}{p \|x\|_p^p} + \frac{\|y\|_p^p}{q \|y\|_q^q} \right) = \frac{1}{p} + \frac{1}{q} = 1$$

That is,

$$\sum_{j=1}^{n} |x_j y_j| \leq \|x\|_p \|y\|_q, \quad \text{for all} \quad n \geq 1$$

Since LHS is an increasing sequence which is bounded above, hence

$$\|x \cdot y\|_1 \leq \|x\|_p \|y\|_q$$

Notice that if $\|x\|_p = 1 = \|y\|_q$, then $\|x \cdot y\|_1 \leq 1$, and equality holds if and only if $|y_j|^p = |x_j|^q$, $\forall j$. This follows from Young's equality. For

$$ab = \frac{a^p}{p} + \frac{b^q}{q},$$

we must have $a^p = b^q$.

2.2.7 Minkowski's Inequality

Let $1 \leq p \leq \infty$. Then for $x, y \in \ell^p$, $x + y \in \ell^p$, and $\|x + y\|_p \leq \|x\|_p + \|y\|_p$

Proof. For $b = 1$ or ∞, the proof is trivial. Let $1 < p < \infty$ .Then

$$\|x + y\|_p = \left(\sum_{i=1}^{\infty} |x_i + y_i|^p \right)^{1/p} \tag{2.5}$$
$$\leq \left(\sum_{i=1}^{\infty} (|x_i| + |y_i|)^p \right)^{1/p}$$

Since

$$(|x_i| + |y_i|)^p = (|x_i| + |y_i|)(|x_i| + |y_i|)^{p-1}$$

By Hölder's inequality,

$$\sum (|x_i| + |y_i|)^{p-1}|x_i| \leq \left(\sum (|x_i| + |y_i|)^{(p-1)q} \right)^{1/q} \left(\sum |x_i|^p \right)^{1/p}$$

Thus,

$$\sum (|x_i| + |y_i|)^p \leq \left(\sum (|x_i| + |y_i|)^p \right)^{1/q} (\|x\|_p + \|y\|_p)$$

That is

$$\left(\sum (|x_i| + |y_i|)^p \right)^{1 - \frac{1}{q}} \leq \|x\|_p + \|y\|_p$$

From (2.5), we get

$$\|x + y\|_p \leq \left(\sum (|x_i| + |y_i|)^p \right)^{1/p} \leq \|x\|_p + \|y\|_p$$

□

Remark 2.61. Equality in $\|x + y\|_p \leq \|x\|_p + \|y\|_p$ holds if and only if $x = \frac{\|x\|_p}{\|y\|_p} y$. (*Hint.* Consider $\|x\|_p = 1 = \|y\|_p$ etc.)

Example 2.62. Since we know that any convergent sequence is bounded, it follows that the space c of all convergent sequences is a normed linear space under the norm

$$\|x\| = \sup |x_i| < \infty;$$

where $x = (x_1, x_2, \ldots, x_n, \ldots)$.

Further, the space c_0 of all sequences converging to "zero" is also a normed linear space. That is, $x = (x_1, x_2, \ldots, x_n, \ldots)$,

$$\lim_{n \to \infty} |x_n| = 0.$$

Thus, $(c_0, \| \cdot \|_\infty)$ is a linear subspace of $(c, \| \cdot \|_\infty)$.

Exercise 2.6. Show that the following strict inclusions hold:

$$\ell^1 \subsetneq \ell^2 \subsetneq c_0 \subsetneq c \subsetneq \ell^\infty$$

(*Hint.* $x = (x_n) \in \ell^1$, then $\lim x_n = 0 \implies x \in \ell^\infty$, $\sum |x_n|^2 \le \sum \|x\|_\infty |x_n| \implies \|x\|_2^2 \le \|x\|_\infty \|x\|_1$.)

Exercise 2.7. For $x = (x_1, ..., x_n) \in \mathbb{R}^n$ (or $\mathbb{C}^n$), show that:

$$\|x\|_\infty \le \|x\|_1 \le \sqrt{n} \|x\|_2 \le n \|x\|_\infty$$

2.2.8   Geometry of spheres in $(\mathbb{R}^n, \| \cdot \|_p)$

For $0 < p \le \infty$ and $x \in \mathbb{R}^n$, define

$$\|x\|_p = \left(\sum |x_i|^p \right)^{1/p}.$$

Then $\| \cdot \|_p$ is a norm for $1 \le p < \infty$. For $0 < p < 1$, the quantity $\|x\|_p^p = d_p(0, x)$, where

$$d_p(x, y) = \|x - y\|_p^p,$$

defines a metric. We return to this point later.

Let $S_1^p(0) = \{x : d_p(0, x) = 1\}$. The following figure illustrates the corresponding unit spheres for several values of p ($0 < p < \infty$ and $p = \infty$).
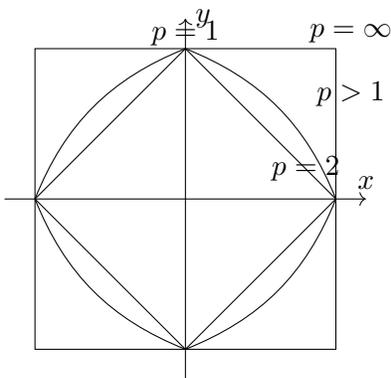
Shapes for $0 < p < 1$ would look like star-shaped curves (not shown).

2.2.9   Closed sets in (X, d)

Definition 2.63. A set $F \subset (X, d)$ is said to be closed if F^c is open. that is, for all $x \in F^c = X \backslash F$, $\exists \, \epsilon > 0$ such that $B_\epsilon(x) \subseteq F^c$.
On the other hand, if for each $\epsilon > 0$, $B_\epsilon(x) \cap F \neq \emptyset \implies x \in F$.

Theorem 2.64. *Let (X, d) be a metric space and $F \subset X$. Then the following are equivalent (F.A.E):*

1. F is a closed set (F^c open).

2. $\forall \epsilon > 0$, $B_\epsilon(x) \cap F \neq \emptyset \implies x \in F$.

3. $\forall$ sequence $(x_n) \in F$ such that $x_n \to x \implies x \in F$.

.

Proof. (2.5) $\implies$ (1.4): Suppose F is closed. *Claim:* $B_\epsilon(x) \cap F \neq \emptyset$, $\forall \epsilon > 0 \implies x \in F$.
Notice that if $x \notin F \implies x \in F^c$ and F^c is open $\implies \exists \epsilon_0 > 0$ s.t.

$$B_{\epsilon_0}(x) \subset F^c \implies B_{\epsilon_0}(x) \cap F = \emptyset,$$

which is a contradiction.
     (2) $\implies$ (3): Let $(x_n) \subset F$ and $x_n \to x$. Then for each $\epsilon > 0$, $x_n \in B_\epsilon(x)$ for all $n \geq n_0$.

$$\implies x_n \in B_\epsilon(x) \cap F \neq \emptyset, \quad \forall \epsilon > 0 \implies x \in F$$

(3) $\implies$ (2.5): Claim: F^c is open. Let $x \in F^c$.
Then $x \notin F$. By (3), $\exists \epsilon_0 > 0$ such that

$$B_{\epsilon_0}(x) \cap F = \emptyset \implies B_{\epsilon_0}(x) \subset F^c.$$

$\square$

Example 2.65. Let $f : (X, d) \to \mathbb{R}$ be a function. Then f is continuous at $x \in X$ if and only if for every sequence (x_n) in X with $x_n \to x$, we have $f(x_n) \to f(x)$.

Proof. Assume first that f is continuous at x, and let (x_n) be a sequence in X such that $x_n \to x$. Given $\epsilon > 0$, continuity at x yields $\delta > 0$ such that

$$d(y, x) < \delta \implies |f(y) - f(x)| < \epsilon.$$

Since $x_n \to x$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $d(x_n, x) < \delta$. Hence $|f(x_n) - f(x)| < \epsilon$ for all $n \geq N$, so $f(x_n) \to f(x)$.

Conversely, assume that for every sequence (x_n) in X with $x_n \to x$, we have $f(x_n) \to f(x)$. Suppose, for contradiction, that f is not continuous at x. Then there exists $\epsilon_0 > 0$ such that for every $\delta > 0$ there exists $y \in X$ with

$$d(x, y) < \delta \quad \text{but} \quad |f(y) - f(x)| \geq \epsilon_0.$$

For each $n \in \mathbb{N}$, choose $y_n \in X$ such that

$$d(x, y_n) < \frac{1}{n} \quad \text{but} \quad |f(y_n) - f(x)| \geq \epsilon_0.$$

Then $y_n \to x$, but $f(y_n)$ does not converge to $f(x)$, a contradiction. $\qquad\square$

Exercise 2.8. If $f : (X, d) \to \mathbb{R}$ is continuous and $f(x_0) \neq 0$ for some $x_0 \in X$, then $\exists \delta > 0$ such that

$$f(x) \neq 0 \ \forall x \in B_\delta(x).$$

(*Hint.* take $\epsilon_0 = \frac{1}{2}|f(x_0)| > 0$, $\exists \delta > 0$ etc.)

Example 2.66. Show that if $f : (X, d) \to \mathbb{R}$ is continuous, then $A = \{x : f(x) > 0\}$ is open (without using the complement should be closed).
(*Hint.* Let $x \in A$, then for $\epsilon = \frac{1}{2}f(x) > 0$, $\exists \delta > 0$ such that $d(x, y) < \delta \implies |f(x) - f(y)| < \epsilon$)

2.2.10 Interior in (X, d)

Let $A \subset X$. Then interior(A) or Int(A) or A° is the largest open set contained in A. That is,

$$A^\circ = \bigcup \{O \subset X : O \text{ open}, O \subseteq A\}$$

$$= \bigcup \{B_\epsilon(x) \subset A : \text{ for } x \in A \text{ and some } \epsilon > 0\} = \text{ union of all open balls contained in } A.$$

2.2.11 Closure in (X, d)

The closure of set $A \subset X$ is the smallest closed set containing A. That is,

$$\overline{A} = \bigcap \{F \subset X : F \text{ closed and } A \subset F\}$$

$$= \{x \in X : \exists x_n \in A \text{ with } x_n \to x\}$$

= collection of limits of all convergent sequences in A (limit need not be in the set A).

Example 2.67. $A = \left\{(n, \frac{1}{n}) : n \in \mathbb{N}\right\}$. Then closure of A in $(\mathbb{R}, u)$ is $\overline{A} = A$ and $A^\circ = \emptyset$ (Why?).

Result: Let $A \subset (X, d)$. Then $x \in \overline{A} \iff B_\epsilon(x) \cap A \neq \emptyset, \quad \forall \epsilon > 0$.

Proof. Assume first that $x \in \overline{A}$. If there were $\epsilon_0 > 0$ such that $B_{\epsilon_0}(x) \cap A = \emptyset$, then $A \subseteq (B_{\epsilon_0}(x))^c$. Since $(B_{\epsilon_0}(x))^c$ is closed and contains A, the minimality of $\overline{A}$ would give

$$\overline{A} \subseteq (B_{\epsilon_0}(x))^c,$$

which is impossible because $x \in \overline{A}$ but $x \notin (B_{\epsilon_0}(x))^c$.

   Conversely, suppose that $B_\epsilon(x) \cap A \neq \emptyset$ for every $\epsilon > 0$. If $x \notin \overline{A}$, then, since $\overline{A}$ is closed, its complement is open. Hence there exists $\epsilon_0 > 0$ such that $B_{\epsilon_0}(x) \subseteq X \setminus \overline{A}$. This implies $B_{\epsilon_0}(x) \cap A = \emptyset$, a contradiction. Therefore $x \in \overline{A}$.   □

Result: $x \in \overline{A}$ if and only if there exists a sequence (x_n) with $x_n \in A$ such that $x_n \to x$.

Proof. Assume that $x \in \overline{A}$. Then $B_{1/n}(x) \cap A \neq \emptyset$ for every $n \in \mathbb{N}$. Choose $x_n \in B_{1/n}(x) \cap A$. Then

$$d(x_n, x) < \frac{1}{n}, \qquad n \in \mathbb{N},$$

and therefore $x_n \to x$.

   Conversely, suppose there exists a sequence (x_n) in A such that $x_n \to x$. Let $\epsilon > 0$ be arbitrary. Choose n_0 such that $d(x_n, x) < \epsilon$ for all $n \geq n_0$. Then $x_{n_0} \in B_\epsilon(x) \cap A$, so every open ball centred at x meets A. By the previous characterization, $x \in \overline{A}$.   □

Definition 2.68. A set $A \subset (X, d)$ is said to be *dense* in X if $\overline{A} = X$.

2.2.12 Space of Finite Sequences

Space of finite sequences play a vital role similar to the space of all polynomials.

$$P(x) = a_0 + a_1 x + \ldots + a_n x^n$$

$$\Rightarrow (a_0, a_1, \ldots, a_n) \sim (a_0, a_1, \ldots, a_n, 0, 0, 0, \ldots)$$

Let

$$c_{00} = \{x = (x_1, x_2, \ldots, x_n, 0, 0, \ldots) : x_i \in F\}.$$

Then obviously, x is a bounded sequence, and

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i| < \infty$$

defines a norm on c_{00}.

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Notice that the space of all finite sequences c_{00} is dense in all ℓ^p; $1 \le p < \infty$ (as we shall see later). However, the closure of c_{00} is c_0, which is a closed proper subspace of ℓ^∞. For

$$x_n = \left(1, \frac{1}{2}, \ldots, \frac{1}{n}, 0, 0, \ldots\right) \in c_{00},$$

and

$$x = \left(1, \frac{1}{2}, \ldots, \frac{1}{n}, \frac{1}{n+1}, \ldots\right),$$

then

$$\|x - x_n\|_\infty = \sup_{k \ge n} \frac{1}{k+1} = \frac{1}{n+1} \to 0,$$

but $x \notin c_{00}$. Hence c_{00} is not a closed subspace of ℓ^∞. In addition c_{00} is not open in ℓ^∞. For this, let $\epsilon > 0$,be arbitrarily small. Then for $B_\epsilon(0) \subset \ell^\infty$, $(\epsilon/2, \epsilon/2, \ldots) \in B_\epsilon(0)$ but $(\epsilon/2, \epsilon/2, \ldots) \notin c_{00}$. Hence, $B_\epsilon(0) \not\subseteq c_{00}$ for any $\epsilon > 0$.

For $1 \le p < \infty$, $c_{00} \subsetneq \ell^p$ and c_{00} is neither closed nor open in ℓ^p. For this, let

$$x_n = \left(\frac{\epsilon^p}{2^{n+1}}\right)^{1/p}, \ 1 \le p < \infty,$$

and consider $x = (x_1, x_2, \ldots)$. Then $x \in B_\epsilon(0) \subset \ell^p$, but $x \notin c_{00}$. Now write $x_n = (x_1, \ldots, x_n, 0, \ldots) \in c_{00}$. Then

$$\|x - x_n\|_p^p = \sum_{k=n+1}^\infty \frac{\epsilon^p}{2^{k+1}} \to 0,$$

but $x \notin c_{00}$.

Example 2.69. Let M be a non-open subspace of a normed linear space (n.l.s.) X. Show that $M = X$.

(*Hint.* $O \in M \implies B_\epsilon(0) \subset M \subset X$. Since M is linear, $\alpha B_\epsilon(0) \subset M \subset X$ for all $\epsilon > 0$ $\implies B_{\epsilon_1}(0) \subset M$ for some $\epsilon_1 > 0$. If $y \in X$, then $y \in B_{\epsilon_1}(0) \subset M \subset X$ for some $\epsilon_1 > 0$.)

Notice that for $x = (x_1, x_2, \ldots, x_n, \ldots) \in \ell^p$; $1 \le p < \infty$, $x_n = (x_1, \ldots, x_n, 0, \ldots) \in c_{00}$. And

$$\|x - x_n\|_p^p = \sum_{k=n+1}^\infty |x_k|^p \longrightarrow 0,$$

because $x \in \ell^p$. Hence $x_n \to x$ in ℓ^p, and therefore $\overline{c_{00}} = \ell^p$.

However, c_{00} is not dense in ℓ^∞, but $\overline{c_{00}} = c_0$. For this, let $X = (x_1, x_2, \ldots) \in c_0$. Then $\lim_{n \to \infty} x_n = 0$. For $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $n \ge n_0$ implies $|x_n| < \epsilon/2$. Now, write $X_n = (x_1, \ldots, x_n, 0, 0, \ldots)$. Then $X_n \in c_{00}$ and for $n \ge n_0$,

$$\|X - X_n\|_\infty = \sup_{i \ge n+1} |x_i| < \epsilon/2.$$

Therefore, $X_n \to X$.

Remark 2.70. $\overline{c_{00}} = c_0 \subsetneq \ell^\infty$. That is, c_{00} is not dense in ℓ^∞.

Example 2.71. Let $f \colon \mathbb{R} \to \mathbb{R}$ (or $\mathbb{C}$) be a continuous function. Suppose $\lim_{|x| \to \infty} f(x) = 0$. Then for $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x)| < \epsilon$ for $|x| > \frac{1}{\delta}$.

Since f is continuous, it follows that f is bounded. Let $\|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)| < \infty$. Then

$$C_0 = \left\{ f \colon \mathbb{R} \to \mathbb{R} \text{ is continuous, } \lim_{|x| \to \infty} |f(x)| = 0 \right\}$$

is a normed linear space.

For any function $f \colon \mathbb{R} \to \mathbb{R}$, define

$$\mathrm{supp}(f) = \overline{\{x \in \mathbb{R} : f(x) \neq 0\}}$$

called the support of f.

Let

$$C_C = \{ f \colon \mathbb{R} \to \mathbb{R} \text{ continuous} \quad \text{and } \mathrm{supp}(f) \text{ is compact} \}.$$

Then $f \in C_c$ is a bounded function and

$$\|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)| = \sup_{x \in \mathrm{supp}(f)} |f(x)| < \infty.$$

Let $K = \mathrm{supp}(f)$ be compact. Then $(C_C, \| \cdot \|_\infty)$ is a dense subspace of $(C_0, \| \cdot \|_\infty)$.

For this, let $f \in C_0$, then for $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x)| < \epsilon$ for $|x| > \frac{1}{\delta}$. Write $K = \{x : |x| \leq \frac{1}{\delta}\}$.

Let O be a bounded open set with $K \subset O$. Define

$$g(x) = \frac{d(x, O^c)}{d(x, O^c) + d(x, K)}$$

Then g is continuous on $\mathbb{R}$, $0 \leq g(x) \leq 1$ and $g(x) = 1$ for $x \in K$ and $g(O^c) = \{0\}$.

Let $h = f \cdot g$. Then $h \in C_C$ and

$$\|f - h\|_\infty = \|f(1 - g)\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|(1 - g(x)) \leq \epsilon.$$

Hence, C_C is dense in C_0.

Note that $d(x, A) = \inf_{y \in A} |x - y|$.

2.2.13 Complete Metric Spaces

We have seen that there are Cauchy sequences whose limits need not necessarily belong to the space.

For example, the sequence $\frac{1}{n} \in ((0,1), u)$ under the usual metric, is a Cauchy sequence but the limit $\frac{1}{n} \to 0 \notin (0,1)$.

It is always possible to enlarge the space so that limits of all Cauchy sequences can be accommodated. This process is known as the completion of metric spaces, we shall see later. However, there are many spaces which do accommodate limits of their Cauchy sequences.

Definition 2.72. A metric space (X, d) is called complete if every Cauchy sequence in X has its limit in X.

Example 2.73. $(\mathbb{R}, u)$ is a complete space.

Let (x_n) be a Cauchy sequence in $\mathbb{R}$. Then it is bounded. And by the Bolzano–Weierstrass theorem, there exists a subsequence $x_{n_k} \to x \in \mathbb{R}$. For any $\epsilon > 0$, there exists a natural number k_0 such that

$$|x_{n_k} - x| < \epsilon \quad \text{for all} \quad k \geq k_0 \tag{2.6}$$

But the sequence (x_n) is Cauchy, so for all $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|x_n - x_m| < \epsilon$ for all $n, m \geq n_0$. Let $m \geq n_0$ and $m \geq n_{k_0}$. Then

$$|x_n - x_{n_k}| < \epsilon \quad \text{for any } n \geq n_0 \text{ and } k \geq k_0. \tag{2.7}$$

From (2.6) and (2.7), it follows that:

$$|x_n - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| < 2\epsilon$$

for $n \geq n_0$ and $n_k \geq n_{k_0}$. Thus, for $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \implies |x_n - x| < \epsilon.$$

Notice that the above discussion can be used to prove the following result.

Result: Let (x_n) be a Cauchy sequence in a metric space (X, d). If (x_n) has a convergent subsequence $x_{n_k} \to x$, then $x_n \to x$. (Proof is similar to the above.)

Example 2.74. $(\mathbb{R}^n, \| \cdot \|_p)$ is complete for $1 \leq p \leq \infty$.

Let $1 \leq p < \infty$, and $x^k = (x_1^k, \ldots, x_n^k)$ be a Cauchy sequence in $(\mathbb{R}^n, \| \cdot \|_p)$. Then for $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that for all $k, l \geq k_0$,

$$\|x^k - x^l\|_p = \left(\sum_{j=1}^{n} |x_j^k - x_j^l|^p \right)^{1/p} < \epsilon$$

$$\implies |x_j^k - x_j^l| < \epsilon \quad \text{for all } k, l \geq k_0$$

$$\implies (x_j^k) \text{ is a Cauchy sequence in } (\mathbb{R}, u).$$

Hence $x_j^k \to x_j$ for all j. Then for $\epsilon > 0$, there exists $m_j \in \mathbb{N}$ such that $k \geq m_j \implies |x_j^k - x_j| < \epsilon$. Let $m_0 = \max_j \{m_j\}$. Then, for $x = (x_1, \ldots, x_n)$,

$$\|x^k - x\|_p < \epsilon \qquad \text{for } k \geq m_0.$$

Notice that the case $p = \infty$ is similar. We skip its proof here.

Example 2.75. Let $1 \leq p \leq \infty$. Then $(\ell^p, \|\cdot\|_p)$ is complete.

     Let $1 \leq p < \infty$, and let $x^k = (x_1^k, x_2^k, \dots)$ be a Cauchy sequence in $(\ell^p, \|\cdot\|_p)$. Then for $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\forall k, l \geq n_0 \implies \|x^k - x^l\|_p < \epsilon$

$$\implies \sum_{j=1}^{n} |x_j^k - x_j^l|^p < \epsilon^p \tag{2.8}$$

For each fixed n, this reduces to $(\mathbb{R}^n, \|\cdot\|_p)$, which we know is complete. Hence $x_j^k \to x_j$; $j = 1, 2, \dots, n$. Thus, letting $k \to \infty$ in (2.8), it follows that

$$\sum_{j=1}^{n} |x_j^l - x_j|^p < \epsilon^p, \quad \forall l \geq n_0 \tag{2.9}$$

But the left-hand side of (2) is an increasing sequence and bounded above, hence, letting $n \to \infty$, we get

$$\sum_{j=1}^{\infty} |x_j^l - x_j|^p < \epsilon^p$$

$$\|x^l - x\|_p \leq \epsilon, \quad \forall l \geq n_0$$

where $x = (x_1, x_2, \dots, x_n, \dots)$. Notice that

$$\|x\|_p \leq \|x - x^{n_0}\|_p + \|x^{n_0}\|_p < \epsilon + \|x^{n_0}\|_p < \infty \implies x \in \ell^p.$$

Result: Every closed subset of a complete metric space is complete.

Proof. Let F be a closed subset of a complete metric space (X, d). Then $(x_n) \subset F$ is a Cauchy sequence, it follows that (x_n) is a Cauchy sequence in X. Hence $x_n \to x \in X$. But F is closed, it implies that $x \in F$.

In fact, if (X, d) is complete, then F is closed if and only if F is complete. (*Hint.* it follows easily.) $\qquad \square$

Example 2.76. Show that $(c_0, \|\cdot\|_\infty)$ is a proper closed subspace of $(\ell^\infty, \|\cdot\|_\infty)$.

     We know that $c_0 \subsetneq \ell^\infty$. Now, let $x^k = (x_1^k, \dots, x_j^k, \dots)$ be a sequence in c_0 such that $x^k \to x = (x_1, \dots, x_j, \dots)$. That is, for every $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that $\forall k > k_0 \implies \|x^k - x\|_\infty < \epsilon$ which implies

$$|x_j^k - x_j| < \epsilon \quad \text{for each } j \geq 1 \text{and} \quad \forall k > k_0. \tag{2.10}$$

Since $x_j^k \in c_0 \implies \lim_{j \to \infty} x_j^k = 0$ for each k. For $\epsilon > 0$, there exists $j_0 \in \mathbb{N}$ such that

$$|x_j^k| < \epsilon \quad \forall j \geq j_0 \quad \text{and} \quad k \geq k_0. \tag{2.11}$$

It follows from (2.10) and (2.11) that

$$|x_j| < |x_j^{k_0} - x_j| + |x_j^{k_0}| < 2\epsilon \quad \forall j > J_0,$$

that is, $|x_j| < 2\epsilon$ for all $j > J_0$, which means $\lim_{j \to \infty} x_j = 0$. Hence c_0 is a closed subspace of ℓ^∞. Thus, c_0 is complete in its own right.

Example 2.77. The space $(C[a,b], \| \cdot \|_\infty)$ is a complete normed linear space.

Let (f_n) be a Cauchy sequence in $(C[a,b], \| \cdot \|_\infty)$. Then for $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\forall n, m \geq n_0 \implies \|f_n - f_m\|_\infty < \epsilon$ which implies

$$|f_n(t) - f_{n_0}(t)| < \epsilon \quad \forall n \geq n_0, \forall t \in [a,b]. \tag{2.12}$$

So $(f_n(t))$ is a Cauchy sequence in $(\mathbb{R}, u)$ for each fixed $t \in [a,b]$. Hence $f_n(t) \to f(t)$. Letting $n \to \infty$ in (2.12), we get $|f(t) - f_{n_0}(t)| \leq \epsilon \quad \forall t \in [a,b]$. (Notice that n_0 is free of choice of t). Since f_{n_0} is continuous, for each fixed t and $\epsilon > 0$, there exists $\delta > 0$ such that $|s - t| < \delta$ implies $|f_{n_0}(s) - f_{n_0}(t)| < \epsilon$. Hence,

$$|f(s) - f(t)| < |f(s) - f_{n_0}(s)| + |f_{n_0}(s) - f_{n_0}(t)| + |f_{n_0}(t) - f(t)|$$
$$< 3\epsilon$$

So f is continuous on $[a,b]$.

However, the space $(C[a,b], \| \cdot \|_1)$ is not complete. For this, we consider the following: Consider

$$f_n(t) = \begin{cases} nt & 0 \leq t < \frac{1}{n} \\ 1 & \frac{1}{n} \leq t \leq 1 \end{cases}$$

It is easy to see that for $\frac{1}{m} < \frac{1}{n}$,

$$\|f_n - f_m\|_1 = \left(\int_0^{1/m} + \int_{1/m}^{1/n} + \int_{1/n}^1 \right) |f_n(t) - f_m(t)| dt$$

$$= \int_0^{1/m} (mt - nt) dt + \int_{1/m}^{1/n} (1 - nt) dt + \int_{1/n}^1 (1 - 1) dt$$

$$= \frac{1}{2} \left(\frac{1}{m} - \frac{1}{n} \right) \to 0 \text{ as } n < m \to \infty$$

Thus (f_n) is a Cauchy sequence in $(C[0,1], \| \cdot \|_1)$. But the pointwise limit:

$$f(t) = \lim_{n \to \infty} f_n(t) = \begin{cases} 1 & 0 < t \leq 1 \\ 0 & t = 0 \end{cases}$$

(*Hint.* $f_n(0) = 0$ and $f_n(1) = 1$ for all n, so $f(0) = 0$ and $f(1) = 1$. For $0 < t_0 < 1$, we can find large n such that $0 < \frac{1}{n} < t_0 < 1$. Hence $f_n(t_0) = 1$ for large n. Thus $f(t_0) = 1$.) However, f is not continuous, hence $(C[0,1], \|\cdot\|_1)$ is not complete.

2.2.14 Sequences of Functions

Notice that in the previous exercises, we have seen that $(C([0,1]), \|\cdot\|_\infty)$ is complete. That is, if $\|f_n - f_m\|_\infty \to 0$, then there exists $f \in C([0,1])$ such that $\|f_n - f\|_\infty \to 0$. But then,

$$|f_n(t) - f(t)| < \|f_n - f\|_\infty \to 0, \quad \forall t \in [0,1],$$

that is, $f_n(t) \to f(t)$ for each $t \in [0,1]$. We say that $f_n \to f$ uniformly if

$$\sup_t |f_n(t) - f(t)| \to 0.$$

But there are sequence of functions which converge pointwise but not uniformly.

Example 2.78. Let $f_n(t) = t^n$, $t \in [0,1]$. Then,

$$f(t) = \lim_{n \to \infty} f_n(t) = \begin{cases} 0 & 0 \le t < 1 \\ 1 & t = 1 \end{cases}$$

So,

$$\sup_t |f_n(t) - f(t)| = 1 \nrightarrow 0.$$

Example 2.79. Let $f_n : \mathbb{R} \to \mathbb{R}$ be given by

$$f_n(t) = e^{-nt^2}, \quad n \in \mathbb{N}$$

Then,

$$f(t) = \lim_{n \to \infty} f_n(t) = \begin{cases} 1 & t = 0 \\ 0 & |t| > 0 \end{cases}$$

Notice that for $t = 0$, $|f_n(0) - f(0)| = |1 - 1| = 0 < \epsilon$, $\quad \forall n \in \mathbb{N}$ If $|t_0| > 0$, $t_0^2 > 0$. Then for $|f_n(t_0) - 0| < \epsilon$, we get

$$e^{-nt_0^2} < \epsilon \implies n > \frac{\log \frac{1}{\epsilon}}{t_0^2}$$

Let $n_0 = \left\lceil \frac{\log \frac{1}{\epsilon}}{t_0^2} \right\rceil + 1$. Then, $|f_n(t_0) - f(t_0)| < \epsilon$ for $n \ge n_0$

Notice that $n_0 = n_0(\epsilon, t_0)$ and n_0 is large for $|t_0|$ close to zero. Thus, n_0 cannot be free from t_0. Therefore, $f_n \to f$ pointwise but not uniformly. Also,

$$\|f_n - f\|_\infty = \sup_{t \in \mathbb{R}} e^{-nt^2} = 1 \nrightarrow 0$$

If $f_n(t) = e^{-nt}$ for $t \in [1, \infty)$, then

$$\sup_t |f_n(t) - 0| = e^{-n} \to 0 \implies e^{-nt} \xrightarrow[\,[1,\infty)\,]{\text{unif.}} 0$$

Exercise 2.9. Let $f_n, f : A(\subseteq \mathbb{R}) \to \mathbb{R}$ be such that $f_n \to f$ uniformly on A. Then for $|f_n(t)| \le M_n$ (that is f_n's are bounded), that implies f is bounded.
(*Hint.* $|f(t)| \le |f_{n_0}(t) - f(t)| + |f_{n_0}(t)| < \epsilon + M_{n_0} < \infty \quad \forall t \in A$)

We shall see later that uniform convergent sequences is a good carrier for many underline properties.

Result: Let $f, f_n : A(\subset \mathbb{R}) \to \mathbb{R}$ be such that $f_n \to f$ uniformly. Then f is continuous if f_n's are continuous (that is the uniform limit of a sequence of continuous functions is continuous).

Proof. For $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\sup_{t \in A} |f_{n_0}(t) - f(t)| < \epsilon$ Thus,

$$|f_{n_0}(t) - f(t)| < \epsilon, \quad \forall t \in A$$

Since f_{n_0} is continuous on A, for fixed t and for $\epsilon > 0$, there exists $\delta > 0$ such that if $|t - s| < \delta \implies |f_{n_0}(t) - f_{n_0}(s)| < \epsilon$. Thus,

$$|f(s) - f(t)| < |f(s) - f_{n_0}(s)| + |f_{n_0}(s) - f_{n_0}(t)| + |f_{n_0}(t) - f(t)| < 3\epsilon$$

$\square$

Result: Let $\mathcal{R}[a, b]$ denote the space of all Riemann integrable functions on $[a, b]$. Let $f_n, f \in \mathcal{R}[a, b]$ and $f_n \to f$ uniformly. Then,

$$\int_a^b f_n \to \int_a^b f$$

that is,

$$\lim_{n \to \infty} \int_a^b f_n = \int_a^b \lim_{n \to \infty} f_n$$

Proof.

$$\left| \int_a^b (f_n - f) \right| \le \int_a^b |f_n - f| \le \|f_n - f\|_\infty (b - a) \to 0$$

$\square$

Corollary 2.80. *If $f_n \in \mathcal{R}[a, b]$ such that $S_n = f_1 + f_2 + \ldots + f_n$ converges uniformly to S, then*

$$\int_a^b \sum_{n=1}^\infty f_n = \sum_{n=1}^\infty \int_a^b f_n$$

(Obvious from the previous result)

Result: Let $f_n \in C^1[a,b]$ be such that $f_n' \to g$ uniformly. If there exists $x_0 \in [a,b]$ such that $f_n(x_0)$ converges, then there exists $f \in C^1[a,b]$ such that $f_n \to f$ uniformly and $f' = g$.

Proof. Since $f_n' \to g$ uniformly and f_n is continuous, g will be continuous. Define

$$f : [a,b] \to \mathbb{R} \quad \text{by} \quad f(x_0) = \lim_{n\to\infty} f_n(x_0)$$

and

$$f(x) = \begin{cases} f(x_0) + \int_{x_0}^{x} g(t)\, dt, & \text{if } x > x_0 \\ f(x_0) - \int_{x}^{x_0} g(t)\, dt, & \text{if } x < x_0 \end{cases}$$

Then $f'(x) = g(x)$ for every $x \in [a,b]$. Hence, $f \in C^1[a,b]$. Now,

$$f_n(x) - f_m(x) = f_n(x) - f_m(x) - (f_n(x_0) - f_m(x_0)) + (f_n(x_0) - f_m(x_0))$$
$$= (x - x_0)(f_n'(t) - f_m'(t)) + (f_n(x_0) - f_m(x_0))$$

Therefore,

$$\|f_n - f_m\|_\infty \le (b-a)\|f_n' - f_m'\|_\infty + |f_n(x_0) - f_m(x_0)| \to 0,$$

as $n, m \to \infty$. Hence, (f_n) is a Cauchy sequence in $(C[a,b], \|\cdot\|_\infty)$. Therefore, f_n converges uniformly. Again, since $f_n' \to g = f'$ uniformly, it follows that

$$\int_{x_0}^{x} f_n'(t)\, dt \to \int_{x_0}^{x} f'(t)\, dt.$$

$$\lim_{n\to\infty} [f_n(x) - f_n(x_0)] = f(x) - f(x_0).$$

Since $f_n(x_0) \to f(x_0)$ as well, it follows that $f_n(x) \to f(x)$ for every $x \in [a,b]$. $\qquad\square$

Remark 2.81. Convergence of $(f_n(x_0))$ is necessary in the above result. Consider

$$f_n(t) = \sqrt{t + n}, \qquad t \in [0,1]$$

Then f_n does not converge at any point of $[0,1]$, but

$$f_n'(t) = \frac{1}{2\sqrt{t+n}} \xrightarrow{\text{unif.}} 0$$

Since

$$\sup_{t\in[0,1]} |f_n'(t) - 0| = \sup_{t\in[0,1]} \frac{1}{2\sqrt{t+n}} = \frac{1}{2\sqrt{n}} \to 0.$$

Exercise 2.10. Let $f_n : \mathbb{R} \to \mathbb{R}$. Check for uniform convergence of f_n to some f:

1. $f_n(t) = \frac{\sin(nt)}{\sqrt{n}}$.

2. $f_n(t) = n^2 t(1 - t^2)^n$.

3. $f_n(t) = te^{-nt}$.

Also, verify for term-by-term integration and differentiation for each of the above.

Theorem 2.82. *Let $E \subseteq \mathbb{R}$, and $f_n \to f$ uniformly on E. For a limit point x of E. Suppose*

$$\lim_{t \to x} f_n(t) = A_n \quad \text{(finite)} \tag{2.13}$$

Then (A_n) is convergent and

$$\lim_{t \to x} f(t) = \lim_{n \to \infty} A_n.$$

That is,

$$\lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t)$$

Proof. Since $f_n \to f$ uniformly on E. For each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$|f_n(t) - f_m(t)| < \epsilon, \quad \forall n, m \geq n_0, \ \forall t \in E \tag{2.14}$$

By (2.14), it implies that $|A_n - A_m| < \epsilon, \quad \forall n, m \geq n_0$. So (A_n) is Cauchy, hence convergent $\implies A_n \to A$ (Say). Now,

$$\begin{aligned}
|f(t) - A| &= |f(t) - f_n(t) + f_n(t) - A_n + A_n - A| \\
&\leq |f(t) - f_n(t)| + |f_n(t) - A_n| + |A_n - A| \\
&< \epsilon + \epsilon + \epsilon
\end{aligned}$$

for $t \in (x - \delta, x + \delta) \setminus x$ and $n \geq n_0$ (free of t)

$$\lim_{t \to x} f(t) = A = \lim_{n \to \infty} A_n$$

$$\text{Thus,} \quad \lim_{t \to x} \lim_{n \to \infty} f_n(t) = \lim_{n \to \infty} \lim_{t \to x} f_n(t)$$

$$\square$$

Theorem 2.83. *Let $f_n : [a, b] \to \mathbb{R}$ be such that (f_n') converges uniformly. If there exists $x_0 \in [a, b]$ such that $(f_n(x_0))$ is convergent, then (f_n) is uniformly convergent, and*

$$\lim_{n \to \infty} f_n'(x) = \left(\lim_{n \to \infty} f_n(x) \right)'$$

(that is limit and derivative commute).

Proof. As in the previous theorem, the Mean Value Theorem gives

$$|f_n(x) - f_m(x)| \leq (b - a)\|f_n' - f_m'\|_\infty + |f_n(x_0) - f_m(x_0)|$$

for all $x \in [a, b]$. Since (f_n') converges uniformly and $(f_n(x_0))$ converges, the sequence (f_n) is uniformly Cauchy, hence converges uniformly on $[a, b]$ to some function f.

Fix $x \in [a, b]$. For $t \neq x$, define

$$\varphi_n(t) = \frac{f_n(x) - f_n(t)}{x - t}, \qquad \varphi(t) = \frac{f(x) - f(t)}{x - t}.$$

Because $f_n \to f$ uniformly, we have $\varphi_n(t) \to \varphi(t)$ for each $t \neq x$. Moreover, by the Mean Value Theorem,

$$|\varphi_n(t) - \varphi_m(t)| = \left| \frac{(f_n - f_m)(x) - (f_n - f_m)(t)}{x - t} \right| = |(f_n - f_m)'(\xi_{n,m,t})| \leq \|f_n' - f_m'\|_\infty$$

for some $\xi_{n,m,t}$ between x and t. Hence (φ_n) is uniformly Cauchy on $[a, b] \setminus \{x\}$, so $\varphi_n \to \varphi$ uniformly there.

For each n, the limit of $\varphi_n(t)$ as $t \to x$ exists and equals $f_n'(x)$. The preceding theorem on interchanging limits therefore yields

$$\lim_{n \to \infty} f_n'(x) = \lim_{n \to \infty} \lim_{t \to x} \varphi_n(t) = \lim_{t \to x} \lim_{n \to \infty} \varphi_n(t) = \lim_{t \to x} \varphi(t) = f'(x).$$

Since x was arbitrary, f is differentiable on $[a, b]$ and

$$\lim_{n \to \infty} f_n'(x) = \left(\lim_{n \to \infty} f_n(x) \right)' \qquad (x \in [a, b]).$$

$\square$

2.2.15 Term-by-term differentiation

Let $S_n = f_1 + f_2 + \cdots + f_n$, where each $f_i : [a, b] \to \mathbb{R}$ such that $S_n' \xrightarrow{\text{unif}} S$ and $S_n(x_0) \to L$. Then, $\lim(S_n') = (\lim S_n)'$. That is,

$$f_1' + f_2' + \cdots + f_n' + \cdots = (f_1 + f_2 + \cdots + f_n + \cdots)'.$$

This raises a very fundamental question: When does

$$\left(\int_a^x f(t)\, dt \right)' = \int_a^x f'(t) dt \tag{2.15}$$

hold? Notice that if f' is continuous then for

$$F(x) = \int_a^x f'(t)\, dt,$$

by the Fundamental Theorem of Calculus, $F'(x) = f'(x)$.

$$(F - f)' = 0$$

By the Mean Value Theorem, $F - f$ is constant. So $F(x) = f(x) - f(a)$ ($\because F(a) = 0$). However, if f' is not continuous, that is $f' \in \mathcal{R}[a, b]$, then (2.15) need not be true.

2.2.16 Uniform continuity

Definition 2.84. A function $f : A(\subset (X, d)) \to \mathbb{R}$ is said to be *uniformly continuous* on A if for each $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in A$,

$$d(x, y) < \delta \implies |f(x) - f(y)| < \epsilon$$

Notice that δ is free of choice of locations of points $x, y \in A$; it only depends on their separation.

Example 2.85. For $x_0 \in X$, let $f(x) = d(x, x_0)$. Then f is uniformly continuous on X. (Hint: $d(x, x_0) \le d(x, y) + d(y, x_0) \implies f(x) - f(y) < d(x, y)$.) Similarly, by replacing x with y, it follows.

Example 2.86. For $x \in X$, $A \subset X$, define $d(x, A) = \inf\{d(x, a) : a \in A\}$, which is called the *distance of A from x*, and is uniformly continuous as a function of x. (Hint: $d(x, a) \le d(x, y) + d(y, a)$.) Thus, $d(x, A) \le d(x, y) + d(y, A)$ and so,

$$|f(x) - f(y)| \le d(x, y) \quad (\because x \leftrightarrow y)$$

Example 2.87. The function $f : (0, 1) \to \mathbb{R}$ given by $f(x) = \frac{1}{x}$ is continuous on $(0, 1)$, but not uniformly continuous.

2.2.17 Pointwise continuity of f

Let $x_0 \in (0, 1)$. Then for $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $(x_0 - \frac{\epsilon}{n}, x_0 + \frac{\epsilon}{n}) \subset (0, 1)$. Suppose $\left|\frac{1}{x_0} - \frac{1}{y}\right| < \epsilon$ for $y \in (x_0 - \frac{\epsilon}{n}, x_0 + \frac{\epsilon}{n}) =: I_{x_0}$. Then $|x_0 - y| < \epsilon x_0 y$. Let $\delta = \min_{y \in I_{x_0}}\{\epsilon x_0 y\} = \epsilon x_0(x_0 - \epsilon/n) > 0$. If $|x_0 - y| < \delta$. Then

$$\left|\frac{1}{x_0} - \frac{1}{y}\right| = \frac{|x_0 - y|}{x_0 y} < \frac{\delta}{x_0 y} \le \frac{\epsilon x_0(x_0 - \epsilon/n)}{x_0 y} < \epsilon$$

Hence, f is continuous at each $x_0 \in (0, 1)$.

f is not uniformly continuous: Let $\epsilon = \frac{1}{2}$, $x = \frac{1}{n}$, $y = \frac{1}{n+1}$, $n \in \mathbb{N}$. Then for any $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$|x - y| = \left|\frac{1}{n} - \frac{1}{n+1}\right| < \delta$$

but

$$|f(x) - f(y)| = 1 \not< \frac{1}{2}.$$

Hence, f is not uniformly continuous on $(0, 1)$. From the above argument, we can prove the following result.

Theorem 2.88. *Let $f : A(\subset (X, d)) \to \mathbb{R}$. Then f is uniformly continuous on A if and only if for every pair of sequences $x_n, y_n \in A$ with $d(x_n, y_n) \to 0$, implies $|f(x_n) - f(y_n)| \to 0$.*

Proof. Suppose f is uniformly continuous on A. Then for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$d(x, y) < \delta \implies |f(x) - f(y)| < \epsilon. \tag{2.16}$$

Let $x_n, y_n \in A$ such that $d(x_n, y_n) \to 0$. Then for $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$d(x_n, y_n) < \delta \implies |f(x_n) - f(y_n)| < \epsilon \quad \text{(from (2.16))},$$

That is, if $d(x_n, y_n) \to 0$, then $|f(x_n) - f(y_n)| \to 0$. Conversely, suppose that f is not uniformly continuous. Then there exists $\epsilon_0 > 0$ such that for every $\delta > 0$ there exist $x, y \in A$ with $d(x, y) < \delta$ but $|f(x) - f(y)| \geq \epsilon_0$. Now, let $\delta = \frac{1}{n}$ for $n \in \mathbb{N}$. Then there exist $x_n, y_n \in A$ such that

$$d(x_n, y_n) < \frac{1}{n}, \forall n \in \mathbb{N}, \quad \text{but} \quad |f(x_n) - f(y_n)| \geq \epsilon_0.$$

That is, $d(x_n, y_n) \to 0$ but $\underline{\lim}|f(x_n) - f(y_n)| \geq \epsilon_0$, is a contradiction. Hence, f is uniformly continuous. $\square$

Exercise 2.11. Show that a uniformly continuous function on a metric space (X, d) sends Cauchy sequences to Cauchy sequences. (*Hint.* If $f : (X, d) \to \mathbb{R}$ is uniformly continuous, so for $d(x_n, x_m) \to 0 \implies |f(x_n) - f(x_m)| \to 0$.)

Result: Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f is uniformly continuous.

Proof. Suppose, on the contrary, that f is not uniformly continuous on $[a, b]$. Then there exists $\epsilon_0 > 0$ such that for every $\delta > 0$, there exist $x, y \in [a, b]$ with $|x - y| < \delta$ but $|f(x) - f(y)| \geq \epsilon_0$. For $\delta = \frac{1}{n}$, there exist $x_n, y_n \in [a, b]$ such that $|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \epsilon_0$. By the Bolzano–Weierstrass theorem, x_n, y_n have convergent subsequences, say $x_{n_k} \to x$ and $y_{n_k} \to y$. Now,

$$|x - y| = \lim_{k \to \infty} |x_{n_k} - y_{n_k}| \leq \lim_{k \to \infty} \frac{1}{n_k} = 0,$$

so $x = y$. Since f is continuous, $f(x_{n_k}) - f(y_{n_k}) \to f(x) - f(y) = 0$, but $|f(x_{n_k}) - f(y_{n_k})| \geq \epsilon_0$, contradiction. $\square$

Example 2.89. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous such that $\lim_{|x| \to \infty} f(x) = 0$. Then f is uniformly continuous.

Proof. For $\epsilon > 0$, there exists $[-a, a]$ such that $|f(x)| < \epsilon/2$ if $x \in [-a, a]^c$. Hence, if $x, y \in [-a, a]^c$, then

$$|f(x) - f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \qquad (1)$$

Since f is uniformly continuous on $[-a, a]$. For $\epsilon > 0$, there exists $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon \qquad (2)$$

Since (2.16) holds true for x, y with $|x - y| < \delta$. It follows that for $\epsilon > 0$, we get $\delta > 0$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \epsilon$ (for any $x, y \in \mathbb{R}$). Hence, f is uniformly continuous on $\mathbb{R}$. $\square$

Notice that if $f \in C_0(\mathbb{R})$, that is f is continuous and $\lim_{|x| \to \infty} f(x) = 0$ and hence f is uniformly continuous. But if f is continuous and bounded, then f need not be uniformly continuous on $\mathbb{R}$.

Example 2.90. $f(x) = \sin x^2$, which is continuous and bounded but not uniformly continuous on $\mathbb{R}$.

Example 2.91. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function. If f is monotone, then f is uniformly continuous on $\mathbb{R}$. Since f is bounded, let $\inf f(x) = L$, $\sup f(x) = M$. For $\epsilon > 0$, there exist $x_0, y_0 \in \mathbb{R}$ such that $f(x_0) < L + \epsilon$ and $f(y_0) > M - \epsilon$.

If f is monotone increasing, then for $x, y \in [x_0, y_0]^c$ and $x, y \geq y_0$

$$f(y) - f(x) \leq M - f(y_0) < M - (M - \varepsilon) = \varepsilon.$$

Similarly, if $x, y \leq x_0$ then

$$f(y) - f(x) \leq L + \varepsilon - f(x_0) < L + \varepsilon - L = \varepsilon.$$

Thus, for $x, y \in [x_0, y_0]^c$, we get $|f(x) - f(y)| < \varepsilon$ (1).

Since f is continuous on $[x_0, y_0]$, f is uniformly continuous on $[x_0, y_0]$. For any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$x, y \in [x_0, y_0], \ |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon \qquad (2)$$

Notice that (1) also holds for $x, y \in [x_0, y_0]^c$ with $|x - y| < \delta$. Thus, we get single $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

Exercise 2.12. If $f : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function then for f monotone, it follows that

$$\lim_{x \to -\infty} f(x) = \text{finite}, \qquad \lim_{x \to +\infty} f(x) = \text{finite}.$$

(*Hint.* For any sequence $x_n \to \infty$, $f(x_n)$ is bounded and $\lim_{n \to \infty} f(x_n) = \sup_n f(x_n)$, for f is increasing.)

Example 2.92. Let $f : (a, b] \to \mathbb{R}$ and $f : (b, c) \to \mathbb{R}$ be uniformly continuous. Then $f : (a, c) \to \mathbb{R}$ is uniformly continuous.

Proof. Since f is uniformly continuous on $(a, b]$ and (b, c), for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $x, y \in (a, b]$ or $x, y \in (b, c)$ with $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$. Now, let $x, y \in (a, c)$, with $|x - y| < \delta$. Then $|x - b| < \delta$ and $|y - b| < \delta$. Hence,

$$|f(x) - f(y)| < |f(x) - f(b)| + |f(b) - f(y)| < 2\varepsilon.$$

Thus, f is uniformly continuous on (a, c). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

We see that a uniformly continuous function can be extended uniformly to the closure of the set.

Theorem 2.93. *Let $f : A(\subset \mathbb{R}) \to \mathbb{R}$ be uniformly continuous on A. Then f can be extended uniformly to $\overline{A}$, and this extension is unique.*

Proof. Let $x \in \overline{A}$. Then there exists $x_n \in A$ such that $x_n \to x$. Now, $f(x_n)$ is a bounded sequence in $\mathbb{R}$. Hence, by Bolzano-Weierstrass theorem, $f(x_n)$ has a convergent subsequence. Without loss of generality we can assume that $f(x_n)$ is convergent. Let $\tilde{f}(x) = \lim f(x_n)$ ($\because \lim f(x_n)$ exists). Notice that $\tilde{f}$ is well defined, because f is uniformly continuous on A. If $x_n, y_n \to x$, then $x_n - y_n \to 0 \implies f(x_n) - f(y_n) \to 0$ that is $\lim f(x_n) = \lim f(y_n)$ ($\because \lim f(x_n)$ and $\lim f(y_n)$) both exist). Hence $\tilde{f} : \overline{A} \to \mathbb{R}$ is well defined. Suppose $x, y \in \overline{A}$ and they are close enough to each other. Then there exist $x_n, y_n \in A$ such that $x_n \to x$ and $y_n \to y$. Hence,

$$\tilde{f}(x) - \tilde{f}(y) = \tilde{f}(x) - f(x_n) + f(x_n) - f(y_n) + f(y_n) - \tilde{f}(y)$$
$$\implies |\tilde{f}(x) - \tilde{f}(y)| \le |\tilde{f}(x) - f(x_n)| + |f(x_n) - f(y_n)| + |f(y_n) - \tilde{f}(y)|$$

Notice that $|\tilde{f}(x) - f(x_n)| < \varepsilon$ and $|\tilde{f}(y) - f(y_n)| < \varepsilon$ for $n \ge n_0$ (say). Let $|x - y| < \delta$ (small enough). Then there exists $n' \in \mathbb{N}$ such that $|x_n - y_n| < \delta$ for $n \ge n'$. Since f is uniformly continuous on A, it follows that $|f(x_n) - f(y_n)| < \epsilon$ for $n \ge n'$. Thus for sufficiently large $n \ge \max(n_0, n')$.

$$|\tilde{f}(x) - f(y)| \le 3\epsilon, \quad \text{where } |x - y| < \delta.$$

Hence, $\tilde{f}$ is uniformly continuous on $\overline{A}$.

This extension of f is unique: If there exists $\tilde{g} : \overline{A} \to \mathbb{R}$ which is uniformly continuous and $\tilde{g} = f$ on A, then for $x \in \overline{A}$, there is a sequence $x_n \in A$ such that $x_n \to x$. Hence,

$$\tilde{f}(x) = \lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} g(x_n) = \tilde{g}(x)$$

($\because g$ is uniformly continuous extension). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Next, we shall see that a uniformly continuous function grows at most linearly.

Theorem 2.94. *Let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous. Then there exist constants $A, B \ge 0$ such that $|f(x)| \le A|x| + B$ for all $x \in \mathbb{R}$.*

Proof. For any $\epsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < 1$. We divide the proof into two parts: one is near "0" and other is away from "0". Let $a > 0$. Then $|f(x)| \le A < \infty$ for $x \in [-a, a]$. Now, consider $f : [a, \infty) \to \mathbb{R}$. Then for $x \in [a, \infty)$, we can find $n \in \mathbb{N}$ such that $x \in [a + n\delta, a + (n+1)\delta]$. Then,

$$f(x) - f(a) = f(x) - f(a + n\delta) + f(a + n\delta) - f(a)$$

$$= f(x) - f(a + n\delta) + \sum_{j=1}^{n} [f(a + j\delta) - f(a + (j+1)\delta)]$$

$$\Rightarrow |f(x)| < 1 + n + |f(a)|$$

$$\Rightarrow \left| \frac{f(x)}{x} \right| < \frac{(n+1) + |f(a)|}{a + n\delta} < \frac{(n+1) + |f(a)|}{n\delta} < \left(1 + \frac{1}{n}\right)\frac{1}{\delta} + \frac{|f(a)|}{n\delta} \le B < \infty$$

Notice that B is independent of n, hence B is independent of x. That is, $|f(x)| \le B|x|$ if $x > a$. Hence, we can summarize that $|f(x)| \le B|x| + A \qquad$ for all $x \in \mathbb{R}$. $\qquad\qquad\square$

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Example 2.95. Notice that $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$, as it does not satisfy the conclusion of the above theorem.

Example 2.96. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and its derivative is bounded. Then f is uniformly continuous on $\mathbb{R}$. For any $x, y \in \mathbb{R}$, by the Mean Value Theorem,

$$|f(x) - f(y)| = |f'(t)(x - y)| \leq M|x - y|$$

where t is between x and y, and M is an upper bound for $|f'(t)|$. However, $f(x) = \sqrt{x}$ for $x \in (0, \infty)$ is uniformly continuous, but its derivative is $f'(x) = \frac{1}{2\sqrt{x}}$, is not bounded.

Exercises

Exercise 2.13. Verify that $d(x, y) = |x - y|$ defines a metric on $\mathbb{R}$, and that $d(x, y) = \|x - y\|_2$ defines a metric on $\mathbb{R}^n$.

Exercise 2.14. Let X be a set and define $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ if $x \neq y$. Show that d is a metric (the discrete metric). Describe the open sets in this metric.

Exercise 2.15. Let d be a metric on X. Prove that every open ball $B_r(x)$ is an open set and every closed ball $\overline{B}_r(x)$ is a closed set.

Exercise 2.16. Let (X, d) be a metric space. Show that a sequence (x_n) converges to x if and only if for every $\varepsilon > 0$ there exists N such that $x_n \in B_\varepsilon(x)$ for all $n \geq N$.

Exercise 2.17. Let $f : (X, d_X) \to (Y, d_Y)$ be Lipschitz: $d_Y(f(x), f(y)) \leq L\, d_X(x, y)$. Show that f is uniformly continuous.

Exercise 2.18. Let $\| \cdot \|$ be a norm on a vector space V and define $d(x, y) = \|x - y\|$. Prove that d is a metric. Conversely, show that if d is translation-invariant and homogeneous in the sense $d(x + z, y + z) = d(x, y)$ and $d(\lambda x, \lambda y) = |\lambda| d(x, y)$, then d comes from a norm.

Exercise 2.19. Prove Hölder's inequality and Minkowski's inequality for sequences, and use them to show that ℓ^p is a normed linear space for $1 \leq p \leq \infty$.

Exercise 2.20. Show that for $0 < p < 1$, $\|x\|_p = \left(\sum |x_n|^p\right)^{1/p}$ fails the triangle inequality, and thus is not a norm.

Exercise 2.21. Let (X, d) be a metric space and $A \subset X$. Define $\text{dist}(x, A) = \inf_{a \in A} d(x, a)$. Prove that $x \mapsto \text{dist}(x, A)$ is 1-Lipschitz.

Exercise 2.22. Show that two metrics d_1, d_2 on the same set X generate the same open sets if there exist constants $c, C > 0$ such that $c\, d_1 \leq d_2 \leq C\, d_1$.

Chapter 3

Completeness and Compactness in Metric and Normed Spaces

This chapter develops completeness in metric and normed spaces and its key consequences. We prove the Banach fixed-point theorem as a prototypical application, and we relate total boundedness and completeness to compactness and sequential compactness.

Learning objectives.

- Separate the notions of completeness, total boundedness, and compactness, and understand exactly where they interact.

- Apply the contraction mapping principle as a structural theorem rather than as an isolated trick.

- Use sequential arguments and covering arguments interchangeably when studying compactness in metric spaces.

3.1 Completeness and Fixed Points

Section overview.

- Completeness is the hypothesis that turns Cauchy control into genuine convergence.

- The contraction mapping principle is the flagship application: it transforms a quantitative estimate into existence and uniqueness.

- Many later local inversion arguments in several variables should be read as sophisticated descendants of this section.

3.1.1 Fixed Points

Fixed-point problems provide a unifying framework for many existence and approximation questions. Let (X, d) be a metric space and let $\varphi : X \to X$ be a self-map. A point $x \in X$ is called a *fixed point* of φ if $\varphi(x) = x$.

A standard method is *Picard iteration*: starting from $x_0 \in X$, define a sequence by $x_{n+1} := \varphi(x_n)$ for $n \geq 0$. If (x_n) converges to some $x \in X$ and φ is continuous, then $\varphi(x) = x$. In a complete metric space it is often enough to show that (x_n) is a Cauchy sequence, because completeness then guarantees convergence. The most important sufficient condition ensuring that (x_n) is Cauchy is that φ is a *contraction*.

Definition 3.1 (Contraction). A map $\varphi : (X, d) \to (X, d)$ is called a *contraction* if there exists a constant $\alpha \in (0, 1)$ such that

$$d(\varphi(x), \varphi(y)) \leq \alpha \, d(x, y), \qquad \forall x, y \in X. \tag{3.1}$$

Theorem 3.2. *Let (X, d) be a complete metric space. If $\varphi : (X, d) \to (X, d)$ is a contraction, then φ has a unique fixed point.*

Proof. Fix $x_0 \in X$ and define the Picard iterates by $x_n := \varphi^n(x_0)$ for $n \geq 0$. Since φ is a contraction with constant $\alpha \in (0, 1)$,

$$d(x_{n+1}, x_n) = d(\varphi(x_n), \varphi(x_{n-1})) \leq \alpha \, d(x_n, x_{n-1}) \leq \alpha^n d(x_1, x_0)$$

for every $n \geq 1$. Therefore, if $m > n$, then by the triangle inequality,

$$d(x_m, x_n) \leq \sum_{k=n}^{m-1} d(x_{k+1}, x_k) \leq \sum_{k=n}^{m-1} \alpha^k d(x_1, x_0) \leq \frac{\alpha^n}{1 - \alpha} \, d(x_1, x_0).$$

As $n \to \infty$, the right-hand side tends to 0, so (x_n) is a Cauchy sequence. Completeness of X now yields a point $x \in X$ such that $x_n \to x$.

Because every contraction is continuous, we may pass to the limit in $x_{n+1} = \varphi(x_n)$ to obtain

$$\varphi(x) = \varphi\Big(\lim_{n \to \infty} x_n\Big) = \lim_{n \to \infty} \varphi(x_n) = \lim_{n \to \infty} x_{n+1} = x.$$

Thus x is a fixed point of φ.

To prove uniqueness, let $y \in X$ be any other fixed point. Then

$$d(x, y) = d(\varphi(x), \varphi(y)) \leq \alpha \, d(x, y).$$

Since $0 < \alpha < 1$, this is possible only if $d(x, y) = 0$, and hence $x = y$. Therefore φ has a unique fixed point. $\qquad \square$

Notice that completeness property of the space is a sufficient condition for existence of fixed point. For example,

$$\varphi : (0, \infty) \to (0, \infty)$$

$$\varphi(x) = \frac{1}{2}\Big(x + \frac{a}{x}\Big), \qquad a > 0$$

satisfies $\varphi(\sqrt{a}) = \sqrt{a}$.

Also, contraction is a sufficient condition for existence of fixed point. Notice that φ above is not a contraction mapping, since

$$|\varphi(x) - \varphi(y)| = \frac{1}{2} \Big|1 - \frac{a}{xy}\Big| \, |x - y|$$

because the function $|1 - \frac{a}{xy}|$ is not bounded near zero.

Exercise 3.1. If (X, d) is a complete metric space and $f : X \to X$ is such that f^k is a contraction, then show that f has a unique fixed point. (*Hint.* do for $k = 2$, use the fact that f^k cannot have two fixed points. If $f^2(x_0) = x_0$ and $y_0 = f(x_0)$(say), implies that $f(y_0) = y_0 \Longrightarrow y_0 = x_0$)

Exercise 3.2. Let $T : C[0, 1] \to C[0, 1]$ be defined by

$$T(f)(x) = \int_0^x f(t)dt$$

Show that T^2 is a contraction but T is not a contraction.

The preceding examples also illustrate a useful principle: the tail of an orbit determines its convergence behaviour, so finitely many initial iterates do not affect the limit.

We next use the fixed point theorem to study existence and uniqueness for the initial value problem

$$\begin{cases} y' = f(x, y) \\ y(0) = y_0 \end{cases} \tag{3.2}$$

using the fixed point theorem.

Assume that f is continuous on a rectangle containing $(0, y_0)$ in its interior and is Lipschitz in the second variable; that is,

$$|f(x, y_1) - f(x, y_2)| \leq K|y_1 - y_2|,$$

where K is a fixed constant. Then equation (3.2) has a unique solution in some neighbourhood of $x = 0$. Observe that solving (3.2) is equivalent to solving

$$\int_0^x y'(t)dt = \int_0^x f(t, y(t))dt$$

that is,

$$y(x) = y_0 + \int_0^x f(t, y(t))dt \tag{3.3}$$

That is, we want $y(t)$ such that (3.3) holds. In other words, we want to get fixed point for the map $\varphi \mapsto F(\varphi)$, where

$$F(\varphi)(x) = y_0 + \int_0^x f(t, \varphi(t))dt,$$

with $\varphi \in C[-\delta, \delta]$ for some $\delta > 0$, which we get very soon. Now,

$$|F(\varphi)(x) - F(\psi)(x)| \leq \int_0^x |f(t, \varphi(t)) - f(t, \psi(t))|dt,$$

$$\leq K \int_0^x |\varphi(t) - \psi(t)|dt$$

$$\leq K \cdot 2\delta \cdot \|\varphi - \psi\|_\infty.$$

Thus, $F : C[-\delta, \delta] \to C[-\delta, \delta]$ is a contraction as long as $2K\delta < 1$, that is if $\delta < \frac{1}{2K}$. Hence F has a unique fixed point in $C[-\frac{1}{2K}, \frac{1}{2K}]$. That is, (3.2) has a unique solution in $|x| < \frac{1}{2K}$.

Example 3.3. Consider $y' = 2x(1 + y), \qquad y(0) = 0$. Then

$$\varphi(x) = \int_0^x 2t(1 + \varphi(t))dt.$$

With the initial guess $\varphi^0 \equiv 0$, we get

$$\varphi^1(x) = \int_0^x 2t(1 + 0)\, dt = x^2,$$

$$\varphi^2(x) = \int_0^x 2t(1 + t^2)\, dt = x^2 + \frac{x^4}{2},$$

$$\varphi^3(x) = x^2 + \frac{x^4}{2} + \frac{x^6}{6}.$$

Thus, by induction,

$$\varphi^n(x) = \sum_{k=1}^n \frac{x^{2k}}{k!} \longrightarrow e^{x^2} - 1, \tag{3.4}$$

and $\varphi(x) = e^{x^2} - 1$ is a solution, which is same as method of separation of variables. Notice that the series (3.4) converges uniformly on every interval $[-a, a]$, or on any interval $[a, b]$. On the other hand, $\varphi'(x) = 2x(1 + \varphi(x))$ has unique solution in neighborhood of any point x_0, that is, $[x_0 - \delta, x_0 + \delta]$ with $\delta < \frac{1}{4}$. (*Hint.* Lipschitz constant $= 2$.)

3.2 Total Boundedness and Compactness

3.2.1 Totally Bounded Set

Suppose A be a bounded set in $\mathbb{R}$ and, without loss of generality, $A \subset (0, 1)$. Then for $\epsilon = \frac{1}{n} > 0$:

$$A \subset \bigcup_{k=1}^n \left(\frac{k-1}{n}, \frac{k}{n} \right]$$

That is, A can be covered by finitely many intervals of arbitrarily small length. A similar argument can be produced for a bounded set $A \subset \mathbb{R}^m$ (or in finite dimensional spaces). Notice that if A is bounded in $\mathbb{R}$, then $A \subset [a, b]$ ($a = \inf A, b = \sup A, b - a < \infty$). Hence,

$$A \subseteq \bigcup_{k=1}^n \left[a + \frac{(k-1)(b-a)}{n}, \; a + \frac{k(b-a)}{n} \right]$$

Notice that, with small perturbation of the intervals, A can be covered by open intervals of arbitrarily small length $\epsilon > 0$. However, if the dimension of the space X is infinite, then the

above property need not be inherited for an arbitrary bounded set. For example, let $X = \ell^1$, $e_n = (0, 0, \ldots, 1, 0, \ldots)$ (with 1 in the nth position):

$$\|e_n - e_m\|_1 = 2, \quad \text{if } n \neq m$$

$$\Rightarrow A = \{e_n : n \in \mathbb{N}\} \subset B_1[0] \subset B_2[0]$$

This means A is bounded. Notice that for any ϵ, $0 < \epsilon < 1$, if $A \subset \bigcup_{n=1}^{\infty} B_\epsilon(e_n)$. But A cannot be covered by finitely many balls of arbitrarily small radius, that is if

$$A \subset \bigcup_{i=1}^{n} B_\epsilon(f_i) \quad \forall f_i \in \ell^1$$

Then, for $\epsilon < 1$, each ball $B_\epsilon(f_i)$ can contain exactly one point of A ($\because \|e_n - e_m\|_1 = 2$). Also, notice that A has no convergent subsequence. Since ℓ^1 is complete, it is equivalent to say that A has no Cauchy subsequence.

Definition 3.4. $A \subseteq (X, d)$ is said to be totally bounded if for every $\epsilon > 0$, there exist $x_1, x_2, \ldots, x_n \in X$ such that

$$A \subseteq \bigcup_{i=1}^{n} B_\epsilon(x_i)$$

We can show that centers of these balls can be taken from some points of A, since

$$A \subseteq \bigcup_{i=1}^{n} B_{\epsilon/2}(x_i)$$

Also, we can assume that $A \cap B_{\epsilon/2}(x_i) \neq \emptyset$, for all $i = 1, 2, \ldots, n$. Then there exists $y_i \in A \cap B_{\epsilon/2}(x_i)$. And it is easy to see that

$$A \subset \bigcup_{i=1}^{n} B_\epsilon(y_i)$$

(*Hint.* $x \in A \implies d(x, x_i) < \epsilon/2$ for some i and $y_i \in A \cap B_{\epsilon/2}(x_i) \implies d(x, y_i) < d(x, x_i) + d(x_i, y_i) < \epsilon$. Moreover, if A is totally bounded, then we can replace balls with sets in A with arbitrarily small diameter.
Result: A in (X, d) is totally bounded if and only if for every $\epsilon > 0$, there exist sets $A_1, \cdots, A_n \subset A$ with $\delta(A_i) < \delta$ such that $A \subseteq \bigcup_{i=1}^{n} A_i$.

Proof. Let A be totally bounded. Then for every $\epsilon > 0$, there exist points $x_1, \ldots, x_n \in A$ such that

$$A \subseteq \bigcup_{i=1}^{n} B_\epsilon(x_i)$$

Set $A_i = A \cap B_\epsilon(x_i) \subseteq A$ and $\delta(A_i) \leq 2\epsilon$. that is

$$A = \bigcup_{i=1}^{n} A_i, \quad \delta(A_i) \leq 2\epsilon.$$

Conversely, suppose for all $\epsilon > 0$, there exist $A_i \subset A$ such that $A \subseteq \bigcup_{i=1}^{n} A_i$, with $\delta(A_i) < \epsilon$. Let $x_i \in A$, then $A_i \subseteq B_{2\epsilon}(x_i)$. Since $\epsilon > 0$ is arbitrary, we get

$$A \subseteq \bigcup_{i=1}^{n} B_{2\epsilon}(x_i)$$

Notice that if $A \subseteq \bigcup_{i=1}^{n} B_i$, $B_i \subseteq X$, with $\delta(B_i) < \epsilon$, then for $A_i = A \cap B_i \subset A$,

$$A = \bigcup_{i=1}^{n} A_i; \qquad \delta(A_i) < \delta \tag{3.5}$$

It is easy to see that if A is totally bounded in (X, d), then A is bounded. Also, every finite set $A = \{x_1, x_2, \ldots, x_n\}$ is totally bounded because $A \subseteq \bigcup_{i=1}^{n} B_\epsilon(x_i)$. $\qquad\square$

Notice that total boundedness of a set solely depends upon the metric. In fact, in discrete metric space, (X, d_0), $A \subset X$ is totally bounded if and only if A is finite. (*Hint.* If $A \subseteq \bigcup_{i=1}^{n} B_\epsilon(x_i)$, $x_i \in A$, then for $0 < \epsilon < \frac{1}{2}$, each $B_\epsilon(x_i) = \{x_i\}$) However, if $X = \ell^1$, $\|e_n - e_m\|_1 = 2$ for $n \neq m$, $A = \{e_n : n \in \mathbb{N}\}$ cannot be covered by finitely many balls of radius < 2. In fact, $A = \{e_n : n \in \mathbb{N}\}$ with

$$d(e_n, e_m) = \begin{cases} 2 & n \neq m \\ 0 & \text{otherwise} \end{cases}$$

(in its own discrete metric) is not totally bounded.

Exercise 3.3. Every subset of a totally bounded set is totally bounded.

Exercise 3.4. $A \subset \mathbb{R}$ is totally bounded if and only if A is bounded.

Exercise 3.5. A is totally bounded if and only if A is covered by finitely many closed sets of arbitrarily small diameters. (*Hint.* $A \subset \bigcup_{i=1}^{n} A_i$; $\delta(A_i) < \epsilon$ but $\delta(\overline{A_i}) = \delta(A_i) < \epsilon$ and $A \subset \bigcup_{i=1}^{n} \overline{A_i}$)

Exercise 3.6. A is totally bounded if and only if $\overline{A}$ is totally bounded. If A is totally bounded, then $A \subseteq \bigcup_{i=1}^{n} A_i$, $\delta(A_i) < \epsilon$. So $\overline{A} \subseteq \bigcup_{i=1}^{n} \overline{A_i}$, $\delta(\overline{A_i}) < \epsilon$, so $\overline{A}$ is totally bounded. On the other hand, if $\overline{A}$ is totally bounded, then for $\epsilon > 0$, $\exists x_1, \ldots, x_n \in X$ such that

$$A \subseteq \overline{A} \subseteq \bigcup_{i=1}^{n} B_i, \qquad \delta(B_i) < \epsilon$$

Exercise 3.7. If $A \subset \mathbb{R}^n$ is bounded, then A is totally bounded.

Result: Let (x_n) be a sequence in (X, d) and let $A = \{x_n : n \in \mathbb{N}\}$ (range of (x_n)).

(i) If (x_n) is Cauchy sequence, then A is totally bounded.

(ii) If A is totally bounded, then (x_n) has a Cauchy subsequence.

Proof. **(i)** Since (x_n) is a Cauchy sequence, for $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that

$$d(x_n, x_N) < \epsilon \qquad \forall n \geq N$$

$$\implies \delta\{x_n : n \geq N\} \leq \epsilon$$

Let

$$A = \{x_i : i = 1, \ldots, N-1\} \cup \{x_n : n \geq N\}$$

$$A \subseteq \bigcup_{i=1}^{N-1} B_\epsilon(x_i) \cup B_\epsilon(x_N)$$

which shows A is totally bounded.

(ii) If A is finite, then trivial. Suppose A is an infinite set and totally bounded. Then A can be covered by finitely many sets of diameter < 1. And one of them, say A_1, will contain infinitely many points of A. But A_1 is also totally bounded, and hence covered by finitely many sets of diameter $< \frac{1}{2}$. Let A_2 be one of them having infinitely many points from A. Thus,

$$A_1 \supset A_2 \cdots \supset A_k \supset A_{k+1} \supset \cdots$$

where each A_k is an infinite set with $\delta(A_k) < \frac{1}{k}$. Choose $x_{n_k} \in A_k$. Then

$$\delta\{x_{n_k} : n \geq k\} \leq \delta(A_k) < \frac{1}{k}$$

($\because A_k$ are decreasing). Thus, x_{n_k} is a Cauchy sequence. $\qquad\square$

Example 3.5. Let $x_n = (-1)^n$ has Cauchy subsequences, as it is totally bounded.

Example 3.6. Let $e_n \in \ell^2$, $e_n = (0, 0, \ldots, 1, 0, \ldots)$. Then (e_n) has no Cauchy subsequence.

Theorem 3.7. *A set $A \subset (X, d)$ is totally bounded if and only if every sequence in A has a Cauchy subsequence.*

Proof. Let A be a totally bounded set in X, and (x_n) be a sequence in A. Then (x_n) is totally bounded and by the previous result (x_n) has a Cauchy subsequence. For the other implication, suppose A is not totally bounded. Then, there exists $\epsilon > 0$ such that

$$A \neq \bigcup_{i=1}^{n} B_\epsilon(x_i)$$

for every choice of finite set $\{x_1, \cdots, x_n\}$. Thus, for each $n \geq 1$, there exists $y_n \in A$ such that $d(y_n, x_i) \geq \epsilon \quad \forall i = 1, 2, \ldots, n$. Notice that the y_n's must be distinct (or an infinite set), else A will be covered by finitely many balls of radius ϵ. Also, notice that (y_n) cannot be a Cauchy sequence, else A would be covered by finitely many ϵ-balls and therefore A is covered by ϵ-balls.

This implies that (y_n) has no Cauchy subsequence (as y_n's are distinct). Therefore, A must be totally bounded. □

Corollary 3.8. *(The Bolzano-Weierstrass Theorem)* *Every bounded infinite subset of* $\mathbb{R}$ *has a limit point in* $\mathbb{R}$.

Proof. Let A be an infinite bounded set in $\mathbb{R}$. Then there exists a sequence of distinct points $x_n \in A$. Since A is totally bounded, (x_n) has a Cauchy subsequence, say (x_{n_k}). But $\mathbb{R}$ is complete, so $x_{n_k} \to x \in \mathbb{R}$. Thus, x is a limit point of A. □

We know that a metric space X is complete if and only if every Cauchy sequence in X converges to a point of X. Moreover, if X is complete, then a subset $A \subseteq X$ is complete if and only if A is closed. We can see that complete metric spaces have some common properties like $\mathbb{R}$:

Theorem 3.9. *Let (X, d) be a metric space. Then the following are equivalent:*

1. (X, d) *is complete.*

2. *(Nested Set Theorem:) Let F_n be a decreasing sequence of closed sets in X with $\delta(F_n) \to 0$, then $\bigcap_{n=1}^{\infty} F_n \neq \phi$ (exactly one point).*

3. *(Bolzano-Weierstrass Theorem:) Every infinite totally bounded subset of X has a limit point in X.*

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Proof. $(2.16) \Rightarrow (2.11)$: Let $F_n \supset F_{n+1} \supset \cdots$ and $\delta(F_n) \to 0$. Choose $x_n \in F_n$, then $\delta\{x_k : k \geq n\} \leq \delta(F_n) \to 0$. Hence, (x_n) is a Cauchy sequence in X, and by (2.16), $x_n \to x \in X$. Since F_n's are closed, $x \in F_n$ for each n,

$$\implies x \in \bigcap_{n=1}^{\infty} F_n \implies \bigcap_{n=1}^{\infty} F_n \neq \varnothing.$$

(In fact, $\bigcap_{n=1}^{\infty} F_n = \{x\}$, exactly one point.)

$(2.11) \Rightarrow (3)$: Let A be an infinite, totally bounded set in X. Notice that A contains a distinct Cauchy sequence x_n ($x_n \neq x_m$ for $n \neq m$), because A is totally bounded. Set $A_n = \{x_k : k \geq n\}$. Then $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots$, and $\delta(A_n) \to 0$ ($\because x_n$ is a Cauchy sequence). But then $\overline{A_n} \supseteq \overline{A_{n+1}} \cdots$ and $\delta(\overline{A_n}) = \delta(A_n) \to 0$. By (ii), there exists $x \in \bigcap_{n=1}^{\infty} \overline{A_n} \neq \varnothing$. Now, $x_n \in A$, and $d(x_n, x) \leq \delta(\overline{A_n}) \to 0$. Hence, $x_n \to x$. So x is a limit point of A.

$(3) \Rightarrow (2.16)$: Let x_n be a Cauchy sequence in X. We only need to show that (x_n) has a convergent subsequence. Note that $A = \{x_n : n \in \mathbb{N}\}$ is totally bounded, because (x_n) is a Cauchy sequence. If A is finite, the result is trivial. Otherwise (iii) implies A has a limit point. That is, there exists a subsequence $x_{n_k} \to x \in X$. Hence, $x_n \to x \in X$. □

Exercise 3.8. Suppose that every countable, closed subset in X is complete. Show that X is complete.

Exercise 3.9. Show that X is complete if and only if every closed ball in X is complete.

Remark 3.10. The total boundedness of a set is all about; an infinite set cannot be too scattered. That is, the substantial portion of the set can be put into (or lies in) a set of arbitrarily "small" size by a continuous dissection process by leaving finitely many.

3.2.2 Compact Metric Spaces

Definition 3.11. A metric space (X, d) is said to be compact if X is complete and totally bounded.

Theorem 3.12. *(X, d) is compact if and only if every sequence in X has a convergent subsequence.*

Proof. Suppose X is compact (complete and totally bounded). Let $x_n \in X$. Then $A = \{x_n : n \in \mathbb{N}\}$ is totally bounded and hence has Cauchy subsequence, say x_{n_k}. But X is complete, which implies $x_{n_k} \to x \in X$.

On the other hand, if every sequence (x_n) in X has a convergent subsequence (x_{n_k}), which is then Cauchy, and this implies that X is totally bounded. Also, let (x_n) be a Cauchy sequence in X. Then again $A = \{x_n : n \in \mathbb{N}\}$ is totally bounded and has convergent subsequence, say $x_{n_k} \to x \in X$. Thus, $x_n \to x$.

$$\begin{bmatrix} \text{Totally Bounded} \\ + \text{ Complete} \end{bmatrix} \iff \begin{bmatrix} \text{every sequence has a Cauchy subsequence} \\ + \text{ Cauchy sequence is convergent} \end{bmatrix}$$

$\square$

Corollary 3.13.

1. Let $A \subset X$. If A is compact, then A is closed.

2. If X is compact and A is closed, then A is compact.

    that is compact subsets of a compact metric space are closed sets.

(Hint.*(i) If A is compact, then for $x_n \in A$ and $x_n \to x \implies x_{n_k} \to y \in A \implies x_n \to x = y$.*
(ii) If A is closed and $x_n \in A$, then $x_n \in X \implies x_{n_k} \to x \in X \implies x \in A$, since A is closed.)

Exercise 3.10. If K is a compact subset of $(\mathbb{R}, u)$, then $\inf K$ and $\sup K \in K$. By definition of infimum, there exists $x_n \in K$ such that $x_n \to \inf K$. But, since K is compact, there exists $x_{n_k} \to x \in K$, which implies $\inf K = x$ etc.

Exercise 3.11. Let $E = \{x \in \mathbb{Q} : 2 < x^2 < 3\}$. Show that E is closed and bounded in $(\mathbb{Q}, u)$, but not compact. (*Hint.* $\mathbb{Q}$ is not complete.)

Exercise 3.12. Suppose $f : (X, d) \to (Y, \rho)$ is continuous, then for $K \subset X$ to be compact, $f(K)$ is compact in Y.

    Let $y_n \in f(K)$, then $y_n = f(x_n)$ for some $x_n \in K$. Therefore, there exists a subsequence $x_{n_k} \to x \in K$ such that $f(x_{n_k}) \to f(x) \in f(K)$.

Exercise 3.13. If $A \subset X$ is compact, then show that $\delta(A) < \infty$. If $A \neq \emptyset$, then there exist $x, y \in A$ such that $\delta(A) = d(x, y)$. Note that

$$\delta(A) = \sup\{d(x, y) : (x, y) \in A \times A\} \quad (\text{say} \quad S)$$

And $d : A \times A \to \mathbb{R}$ is (jointly) continuous. As $A \times A$ is compact, the set S is compact in $\mathbb{R}$. Hence, there exist $(x_0, y_0) \in A \times A$ such that $\delta(A) = d(x_0, y_0)$.

Exercise 3.14. Show that $S_1[0] = \{x \in \ell^2 : \|x\|_2 \leq 1\}$ is not compact. (*Hint.* The set $\{e_n : n \in \mathbb{N}\}$ is not totally bounded.)

Exercise 3.15. Show that $A = \left\{x \in \ell^2 : |x_n| \leq \frac{1}{n}, \ n = 1, 2, \ldots\right\}$ is compact. (*Hint.* A is closed, hence complete. A is totally bounded, since for $\varepsilon = 1$, okay. For $\varepsilon < 1$, only finitely many coordinates are left unpatched (uncovered), hence for each $\varepsilon < 1$, $A = A_\varepsilon \cup B_\varepsilon$, $A_\varepsilon \in \mathbb{R}^n$ for some n.

Corollary 3.14. *Let (X, d) be compact. Suppose $f : X \to \mathbb{R}$ is continuous, then f is bounded. Moreover, f attains its maximum and minimum.*

Proof. $f(X)$ is compact in $\mathbb{R}$, which implies $f(X)$ is closed and bounded. Hence,

$$\sup_{x \in X} f(x) \in \mathbb{R}, \qquad \inf_{x \in X} f(x) \in \mathbb{R}.$$

that is there exist $x_0, y_0 \in X$ such that $f(y_0) = \sup_{x \in X} f(x)$, $f(x_0) = \inf_{x \in X} f(x)$. Hence,

$$f(x_0) \leq f(x) \leq f(y_0) \quad \forall x \in X.$$

$\square$

Corollary 3.15. *If $f : [a, b] \to \mathbb{R}$ is continuous, then $f([a, b])$ is compact and $f([a, b]) = [c, d]$ for some $c, d \in \mathbb{R}$.*

Corollary 3.16. *If (X, d) is a compact metric space and*

$$C(X) = \{f : X \to \mathbb{R} \quad or \quad \mathbb{C} \mid f \text{ is continuous}\}.$$

Define

$$\|f\|_\infty = \sup_{x \in X} |f(x)| < \infty.$$

Then $(C(X), \| \cdot \|_\infty)$ is complete normed linear space.

Lemma 3.17. *Let (X, d) be a metric space. Then the following are equivalent:*

(a) If $\mathcal{G}$ is an arbitrary collection of open sets in X with $\bigcup_{G \in \mathcal{G}} G \supseteq X$, then there exist $G_1, \ldots, G_n$ (finitely many) such that $\bigcup_{i=1}^n G_i \supseteq X$. (In other words, every open cover has a finite subcover.).

(b) If $\mathcal{F}$ is a collection of closed sets in X with $\bigcap_{i=1}^{n} F_i \neq \emptyset$ for every choice of finitely many F_i's in $\mathcal{F}$, then

$$\bigcap_{F \in \mathcal{F}} F \neq \emptyset.$$

(This is called the finite intersection property.).

Notice that (a) $\implies X$ *is totally bounded, since*

$$X \subseteq \bigcup_{x \in X} B_\varepsilon(x) \implies X \subseteq \bigcup_{i=1}^{n} B_\varepsilon(x_i).$$

(b) $\implies X$ is complete, since every decreasing sequence of closed sets has non-empty intersection.

Proof.

(a) $\implies$ (b): Let $\mathcal{F}$ be a collection of closed sets in X such that $\bigcap_{i=1}^{n} F_i \neq \emptyset$ for every choice of finitely many F_i's in $\mathcal{F}$. On contrary, suppose $\bigcap_{F \in \mathcal{F}} F = \emptyset$. Then $X = \bigcup_{F \in \mathcal{F}} F^c$ is an open cover of X. Hence $X = \bigcup_{i=1}^{n} \{F_i^c \colon F_i \in \mathcal{F}\}$. This implies $\bigcap_{i=1}^{n} F_i = \emptyset$, a contradiction.

(b) $\implies$ (a): Suppose $X = \bigcup_{G \in \mathcal{G}} G$ but $X \neq \bigcup_{i=1}^{n} G_i$ for any choice of finitely many G_i's in $\mathcal{G}$. Then $X \setminus \bigcup_{i=1}^{n} G_i \neq \emptyset$ for every choice of finitely many G_i's in $\mathcal{G}$, which implies $\bigcap_{i=1}^{n} G_i^c \neq \emptyset$ for every choice of finitely many sets G_i's from $\mathcal{G}$.

$$\bigcap_{G \in \mathcal{G}} G^c \neq \emptyset \implies \bigcup \{G \colon G \in \mathcal{G}\} \neq X.$$

$\square$

Theorem 3.18. *X is compact if and only if either (a) or (b) (hence both) of the previous lemma is satisfied.*

Proof. If either condition (a) or condition (b) of the previous lemma holds, then X is both totally bounded and complete; hence X is compact.

Conversely, assume that X is compact, and let $\mathcal{G}$ be an open cover of X with no finite subcover. Since X is totally bounded, it can be covered by finitely many closed sets of diameter at most 1. At least one of these closed sets, call it A_1, cannot be covered by finitely many members of $\mathcal{G}$.

Proceed inductively. Once A_n has been chosen, it is compact and therefore totally bounded, so it can be covered by finitely many closed sets of diameter at most $1/(n+1)$. One of these closed sets, call it A_{n+1}, still cannot be covered by finitely many members of $\mathcal{G}$. In this way we obtain a decreasing sequence

$$A_1 \supset A_2 \supset \cdots \supset A_n \supset \cdots$$

of non-empty closed sets such that $\operatorname{diam}(A_n) \leq 1/n$ for every n and no A_n admits a finite subcover from $\mathcal{G}$.

Choose $x_n \in A_n$. Since X is compact, the sequence (x_n) has a convergent subsequence $x_{n_k} \to x \in X$. Because the sets are nested and closed, for each fixed m all sufficiently large terms of the subsequence lie in A_m, so $x \in A_m$. Thus $x \in \bigcap_{n=1}^{\infty} A_n$.

Now choose $G \in \mathcal{G}$ with $x \in G$. Since G is open, there exists $\epsilon > 0$ such that $B_\epsilon(x) \subset G$. Pick n so large that $1/n < \epsilon$. Because $x \in A_n$ and $\mathrm{diam}(A_n) \leq 1/n$, every point of A_n lies within distance $1/n < \epsilon$ of x. Hence $A_n \subset B_\epsilon(x) \subset G$, contradicting the choice of A_n. Therefore every open cover has a finite subcover, and X is compact. $\qquad\square$

Corollary 3.19. *X is compact if and only if every decreasing sequence of non-empty closed sets has non-empty intersection.*

$$that\,is \quad F_1 \supset F_2 \supset \cdots \supset F_n \supset F_{n+1} \cdots \implies \bigcap_{n=1}^{\infty} F_n \neq \emptyset$$

Proof. The forward implication follows from the previous theorem.

Conversely, assume that every decreasing sequence of non-empty closed subsets of X has non-empty intersection. To prove compactness, it is enough to show that every sequence in X has a convergent subsequence. Let (x_n) be a sequence in X, and for each n set

$$A_n := \{x_k : k \geq n\}.$$

Then the closed sets $\overline{A_n}$ are non-empty and decreasing, so by assumption there exists

$$x \in \bigcap_{n=1}^{\infty} \overline{A_n}.$$

Since $x \in \overline{A_n}$ for every n, each ball centred at x meets A_n. Choose n_1 such that $B_1(x) \cap A_{n_1} \neq \varnothing$, and then choose $k_1 \geq n_1$ with $x_{k_1} \in B_1(x)$. Proceed inductively: once k_j is chosen, pick $k_{j+1} > k_j$ such that $x_{k_{j+1}} \in B_{1/(j+1)}(x) \cap A_{k_j+1}$. Then

$$d(x_{k_j}, x) < \frac{1}{j}, \qquad j \in \mathbb{N},$$

so $x_{k_j} \to x$. Thus every sequence has a convergent subsequence, and hence X is compact. $\qquad\square$

Remark 3.20. Note that, as long as compactness is concerned, we do not require the diameter of F_n tends to zero. Hence $\bigcap_{n=1}^{\infty} F_n$ can contain more than one point. This is in sharp contrast with the condition for completeness.

Corollary 3.21. *X is compact if and only if every countable open cover admits a finite subcover.*

Proof. ($\implies$:) Compact $\implies$ lemma (a) holds $\implies$ countable cover has finite subcover. ($\impliedby$:) Suppose every countable open cover has a finite subcover. This is equivalent to every countable family of closed sets having the finite intersection property (can be proved similar to the previous lemma). Let $(x_n) \subset X$ be a sequence of distinct terms. Write $A_n = \overline{\{x_k : k \geq n\}}$. Then $x \in \bigcap_{n=1}^{\infty} A_n \neq \emptyset$, so there exists $x_{n_k} \in X$ such that $x_{n_k} \to x$. Hence, X is compact. $\qquad\square$

3.2.3 Separable Metric Spaces

If a space admits a countable dense set, we say that the space is *separable*. Eventually, it helps determine the size of the space, certainly not in terms of cardinality only, rather dimensions, or in a more general sense of size. Evidently, every totally bounded space is separable.

Definition 3.22. A metric space (X, d) is said to be *separable* if there exists a countable set $A \subset X$ such that $\overline{A} = X$.

For example, $\mathbb{Q}$ (the set of rationals) is a countable dense subset of $\mathbb{R}$. Likewise, $\mathbb{Q}^n$ and $\mathbb{Q}^n + i\mathbb{Q}^n$ are countable dense subsets of $\mathbb{R}^n$ and $\mathbb{C}^n$, respectively.

It is easy to see that $(\mathbb{R}^n, \|\cdot\|_p)$ is separable for $1 \le p \le \infty$. However, $(\ell^p, \|\cdot\|_p)$ is separable for $1 \le p < \infty$ and *not* separable for $p = \infty$. We know that $\overline{c_{00}} \subset \ell^p$, where c_{00} is the space of finite sequences. Let $x \in \ell^p$, $x = (x_1, x_2, \ldots, x_n, x_{n+1}, \ldots)$ Define $x_n = (x_1, \ldots, x_n, 0, 0, \ldots)$. Then

$$\|x - x_n\|_p \to 0 \quad \text{as } n \to \infty \tag{2.16}$$

Since $x_i \in \mathbb{C}$, there exists $x_i^k \in \mathbb{Q} + i\mathbb{Q}$ such that $|x_i^k - x_i|^p \to 0; \quad i = 1, 2, \ldots, n$. Thus,

$$\left(\sum_{i=1}^n |x_i^k - x_i|^p \right)^{1/p} \to 0$$

that is

$$\|x_n^k - x_n\|_p \to 0 \tag{2}$$

where $x_n^k = (x_1^k, \ldots, x_n^k) \in \mathbb{Q}^n + i\mathbb{Q}^n$. From (2.16) and (2.11),

$$\|x - x_n^k\|_p \le \|x_n^k - x_n\|_p + \|x_n - x\|_p \to 0$$

That is, $\overline{c_{00}(\mathbb{N}, \mathbb{Q} + i\mathbb{Q})} = \ell^p(\mathbb{N}, \mathbb{C})$.

Next, we shall show $\ell^\infty(\mathbb{N}, \mathbb{C})$ is not separable, by proving that ℓ^∞ cannot be the union of countably many balls of arbitrarily small radius. Let $A = \{\tilde{x}_1, \tilde{x}_2, \ldots\}$ be any countable set in ℓ^∞. Consider

$$S = \{x = (x_1, x_2, \ldots) \in \ell^\infty : x_i \in \{0, 1\} \}$$

Then S is an uncountable set. For this, $x \in S \implies y = \frac{x_1}{2} + \frac{x_2}{2^2} + \cdots, x_i \in \{0, 1\}$. Then the map from S to $[0, 1]$ is surjective, and hence S is uncountable. Let $x, y \in S$ be such that $x \ne y$. Then $\|x - y\|_\infty = 1$. Hence, $\{B_{\frac{1}{2}}(x) : x \in S\}$ is an uncountable, disjoint collection of open balls in ℓ^∞. Since A is countable, A can intersect only countably many balls $B_{\frac{1}{2}}$'s. Hence A cannot be dense.

Exercise 3.16. Show that $\overline{c_{00}} = c_0$ and hence deduce c_0 is separable.

Exercise 3.17. Let $B([0, 1])$ be the space of all bounded functions on $[0, 1]$. Show that $(B([0, 1]), \|\cdot\|_\infty)$ is not separable. For $t \in (0, 1)$, define $f_t = \chi_{[0,t)}$. Then for $s \ne t$, $s, t \in (0, 1)$, we get $\|f_s - f_t\|_\infty = 1$. Then $S = \{B_{1/2}(f_t) : t \in (0, 1)\}$ is an uncountable collection of disjoint open balls in $B([0, 1])$. If A is any countable set, say $A = \{g_1, g_2, \ldots\} \subset B([0, 1])$, then there

exists $t_0 \in (0,1)$ such that $B_{1/2}(f_{t_0}) \cap A = \varnothing$. That is, except countably many, all the balls in S are left un-intersected by A.

Exercise 3.18. The space $(C([0,1]), \|\cdot\|_\infty)$ is separable. (*Hint.* proof of this will be done by Weierstrass approximation theorem, which we do later.)

Exercise 3.19. Every totally bounded metric space is separable.

Let (X,d) be totally bounded. For $\epsilon = \frac{1}{n}$, there exist $x_{n_1}, \ldots, x_{n_k}$ such that

$$X = \bigcup_{j=1}^{n_k} B_{\frac{1}{n}}(x_{n_j}).$$

Let $D_{n_k} = \{x_{n_1}, \ldots, x_{n_k}\}$. Then $\mathcal{D} = \bigcup D_{n_k}$ is a countable dense set in X.

Next, we consider the compact subsets of the space of continuous functions $C(X)$, then X is a compact metric space. Notice that $\dim C(X) < \infty$ if and only if X is a finite set. Hence, closed and bounded subset of $C(X)$ are compact if X is finite. But the question of compact subsets of $C(X)$, X is compact, is same as when a subset of $C(X)$ is totally bounded? In terms of the Bolzano–Weierstrass theorem, we can rephrase, when (uniformly) bounded sequence in $C(X)$ have a uniformly convergent subsequence?

We will see later that this question is related to the earlier question of asking, When does a pointwise convergent sequence imply uniform convergence? That is, pointwise convergence + [something] $\implies$ uniform convergence.

Example 3.23. If $f_n \in C(X)$, X compact, $f_n \xrightarrow{\text{unif}} f$, then $\{f\} \cup \{f_n : n \in \mathbb{N}\}$ is compact. (that is, every Cauchy sequence is totally bounded).

Definition 3.24. A collection $\mathcal{F} \subset C(X)$ is said to be uniformly bounded if

$$\sup_{f \in \mathcal{F}} \sup_{x \in X} |f(x)| = \sup_{f \in \mathcal{F}} \|f\|_\infty < \infty.$$

Example 3.25. Any uniformly convergent sequence f_n in $B(X)$ (or $C(X)$) is uniformly bounded. (*Hint.* $\|f_n\|_\infty \le \|f\|_\infty + 1$ (for $\epsilon = 1$) for all $n \ge N$, $n \in \mathbb{N}$.)

Definition 3.26. A collection $\mathcal{F} \subset C(X)$ is said to be pointwise bounded if for each $x \in X$, $\sup_{f \in \mathcal{F}} |f(x)| < \infty$.

Example 3.27. If $f_n \to f$ pointwise, then f_n is pointwise bounded.

Theorem 3.28. *Let (X,d) be a compact metric space and $f : X \to \mathbb{R}$ (or $\mathbb{C}$) be continuous. Then f is uniformly continuous.*

Proof. Let $\epsilon > 0$. For each $x \in X$, continuity of f at x gives $\delta_x > 0$ such that

$$d(x,y) < \delta_x \implies |f(x) - f(y)| < \frac{\epsilon}{2}.$$

Then $\{B_{\delta_x/2}(x) : x \in X\}$ is an open cover of X. Since X is compact, there exist $x_1, \ldots, x_n \in X$ such that

$$X = \bigcup_{i=1}^{n} B_{\delta_{x_i}/2}(x_i).$$

Set

$$\delta = \frac{1}{2} \min_{1 \le i \le n} \delta_{x_i} > 0.$$

Now let $x, y \in X$ with $d(x, y) < \delta$. Choose i such that $x \in B_{\delta_{x_i}/2}(x_i)$. Then

$$d(y, x_i) \le d(y, x) + d(x, x_i) < \delta + \frac{\delta_{x_i}}{2} \le \delta_{x_i}.$$

Hence both x and y belong to $B_{\delta_{x_i}}(x_i)$, and therefore

$$|f(x) - f(y)| \le |f(x) - f(x_i)| + |f(x_i) - f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Thus f is uniformly continuous on X. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Next, we shall discuss the missing ingredient of pointwise convergence to the uniform convergence.

Exercises

Exercise 3.20. Prove that a metric space (X, d) is complete if and only if every Cauchy sequence converges.

Exercise 3.21. Let (X, d) be complete and $\varphi : X \to X$ a contraction with constant $0 < q < 1$. Prove that φ has a unique fixed point and that the Picard iterates $x_{n+1} = \varphi(x_n)$ converge to it.

Exercise 3.22. Show that every compact metric space is complete.

Exercise 3.23. Show that every totally bounded set in a metric space is bounded.

Exercise 3.24. Prove: a metric space is compact if and only if it is complete and totally bounded.

Exercise 3.25. Let (x_n) be a sequence in a compact metric space. Prove that (x_n) has a convergent subsequence (sequential compactness).

Exercise 3.26. Let X be a normed space. Prove that if X is finite-dimensional, then every closed and bounded subset of X is compact.

Exercise 3.27. Show that ℓ^p is complete for $1 \le p \le \infty$.

Exercise 3.28. Let $A \subset X$ and define $\operatorname{diam}(A) = \sup\{d(x, y) : x, y \in A\}$. Prove that $\operatorname{diam}(\overline{A}) = \operatorname{diam}(A)$.

Exercise 3.29. Give an example of a complete metric space that is not compact, and of a totally bounded metric space that is not complete.

Chapter 4

Connectedness in Metric and Normed Linear Spaces

Connectedness formalizes the intuition that a set cannot be split into two separated open pieces. We introduce connected and disconnected subsets of a metric space, and we develop basic criteria for connectedness. We then discuss path-connectedness and its relationship with connectedness. In $\mathbb{R}$, connectedness reduces to the familiar notion of an interval, which explains the central role played by the Intermediate Value Theorem.

Learning objectives.

- Detect connectedness and path-connectedness using both separation arguments and image arguments.

- Translate connectedness into analytic consequences such as interval-valued images and intermediate value principles.

- Build and analyze standard examples that distinguish connected, totally disconnected, and non-path-connected sets.

4.1 Connected and Disconnected Sets

Section overview.

- Connectedness is the inability to decompose a set into two separated nonempty open pieces.

- The central theme is that connected sets cannot support jump behavior under continuous maps.

- The examples matter: intervals, punctured spaces, and classical pathological subsets should be compared throughout.

The topology of the real line admits a particularly concrete description, because every open set is a countable disjoint union of open intervals. We have already seen that any open set $O \subset \mathbb{R}$ can be expressed as the disjoint union of countably many open intervals. That is,

$$O = \bigcup_{n=1}^{\infty} I_n, \text{ where } I_n = (a_n, b_n).$$

Hence, for any set $A \subset \mathbb{R}$, we get an open set $O \supset A$, and thus

$$A \subset O \subset \bigcup_{n=1}^{\infty} I_n.$$

Hence, any set can be embedded into countably many open intervals. The "connected set" has its natural meaning, and we can extract its definition from the intervals. We know that an interval cannot be broken into two relatively open parts. On the contrary, suppose that

$$[a, b] = A \cup B,$$

where A and B are non-empty, disjoint, and relatively open sets in $[a, b]$. This implies that A and B are disjoint closed sets too, as

$$A = [a, b] \setminus B, \qquad B = [a, b] \setminus A.$$

Thus A and B are disjoint, non-empty open and closed sets (called clopen sets). To start with, let $b \in B$. Since B is open, $(b - \varepsilon, b] \subset B$ for some $\varepsilon > 0$. Now, let $c = \sup A$. Then $a < c < b$. (if $a = c$, then $A = \{a\}$ (not open), and if $c = b$, then $A \cap B \neq \emptyset$.) By definition of supremum, $(c - \varepsilon, c) \cap A \neq \emptyset$ and $(c, c + \varepsilon) \cap B \neq \emptyset$ (since c is a boundary point of A). That is, $c \in \overline{A} = A$ and $c \in \overline{B} = B$, which is a contradiction that $A \cap B = \emptyset$. Motivated by this observation, we now introduce the general notions of connected and disconnected sets.

Definition 4.1. A metric space X is said to be *disconnected* (not connected) if there exist two non-empty open sets A and B such that $X = A \cup B$. The sets A and B are called a *disconnection* of X.

We say that X is *connected* if X cannot be expressed as a disjoint union of two non-empty open sets in X.

Thus, the interval $[a, b]$ is connected.

Note that, when $X = A \cup B$ where A and B are disjoint, non-empty open sets, it follows that A and B are closed sets too (as $A = B^c$, $B = A^c$). Thus, A and B are disjoint, non-empty clopen sets. Thus, X is connected if and only if X has no nontrivial clopen sets. (*Hint.* if A is clopen, then $X = A \cup A^c$, and A^c is also open.)

Definition 4.2. A subset E of a metric space X is called *disconnected in E* if there exist non-empty disjoint open sets U and V in E such that $E = U \cup V$.

Note that there exist open sets A and B in X such that

$$U = A \cap E, \qquad V = B \cap E$$

$$\implies E = (A \cap E) \cup (B \cap E) = (A \cup B) \cap E \implies E \subset A \cup B.$$

The sets A and B need not be disjoint. However, they can be refined to disjoint open sets of X whose intersections with E are still U and V, respectively.

Lemma 4.3. *Let $E \subset X$. If U and V are disjoint open sets in E, then there exist disjoint open sets A and B in X such that $U = A \cap E$ and $V = B \cap E$.*

Proof. For each $x \in U$, because U is open in the subspace topology on E, there exists $\varepsilon_x > 0$ such that

$$E \cap B_{\varepsilon_x}(x) \subset U.$$

Likewise, for each $y \in V$ there exists $\varepsilon_y > 0$ such that

$$E \cap B_{\varepsilon_y}(y) \subset V.$$

Set

$$A := \bigcup_{x \in U} B_{\varepsilon_x}(x), \qquad B := \bigcup_{y \in V} B_{\varepsilon_y}(y).$$

Then A and B are open in X, and clearly $U \subseteq A \cap E$ and $V \subseteq B \cap E$. The reverse inclusions also hold: if $z \in A \cap E$, then $z \in B_{\varepsilon_x}(x) \cap E$ for some $x \in U$, hence $z \in U$; similarly, $B \cap E = V$.

Finally, $A \cap B = \varnothing$. Indeed, if $z \in A \cap B$, then for some $x \in U$ and $y \in V$ we would have $z \in B_{\varepsilon_x}(x) \cap B_{\varepsilon_y}(y)$. Since $z \in E$ would then imply $z \in U \cap V$, which is impossible, we may shrink the radii if necessary so that the corresponding balls are disjoint; hence the unions remain disjoint. Therefore A and B are disjoint open subsets of X with $A \cap E = U$ and $B \cap E = V$. $\square$

Theorem 4.4. *A subset E of $\mathbb{R}$ (containing more than one point) is connected if and only if for every $x, y \in E$ with $x < y$ it follows that $[x, y] \subset E$.*

Proof. Suppose first that there exist $x, y \in E$ with $x < y$ and some z satisfying $x < z < y$ but $z \notin E$. Then

$$E \subseteq (-\infty, z) \cup (z, \infty),$$

and both sets on the right meet E because they contain x and y, respectively. Hence E is disconnected.

Conversely, assume that $[x, y] \subseteq E$ whenever $x, y \in E$ with $x < y$. We show that E is connected. If not, there exist disjoint non-empty open sets $A, B \subset \mathbb{R}$ such that $E \subseteq A \cup B$, with $A \cap E \neq \varnothing$ and $B \cap E \neq \varnothing$. Choose $a \in A \cap E$ and $b \in B \cap E$ with $a < b$. By hypothesis, $[a, b] \subseteq E \subseteq A \cup B$. Then $A \cap [a, b]$ and $B \cap [a, b]$ are disjoint non-empty open subsets of $[a, b]$ whose union is $[a, b]$, contradicting the connectedness of the interval $[a, b]$.

Thus E is connected exactly when it contains the interval between any two of its points. This property is equivalent to saying that E itself is an interval: indeed, if $\alpha = \inf E$ and $\beta = \sup E$ (allowing the values $\pm\infty$), then every point of (α, β) lies between two points of E and hence belongs to E. $\square$

Exercise 4.1. Show that the connected subsets of Cantor's set are only singletons (that is, Cantor set is totally disconnected).

Now, we simplify our study of connected sets with the help of continuous functions. Notice that a discrete metric space (containing more that one point) is always disconnected. We use this fact to identify disconnected sets through comparison via continuous maps.

Theorem 4.5. *A space X is disconnected if and only if there exists a continuous surjective map $f : X \to \{0, 1\}$ (two-point discrete space).*

Proof. If $f : X \to \{0, 1\}$ is continuous and surjective, then

$$A = f^{-1}(\{0\}), \quad B = f^{-1}(\{1\}). \tag{4.1}$$

are non-empty, disjoint open sets and $A \cup B = X$. Since f is continuous, A, B are closed. Thus, X has a disconnection.

Conversely, if $X = A \cup B$ where A and B are non-empty, disjoint open sets in X. Define

$$f(x) = \begin{cases} 0 & x \in A \\ 1 & x \in B \end{cases} \tag{4.2}$$

which is a continuous and surjective map. $\qquad\square$

This result gives a perfect replacement of definition of connected sets. Thus, we conclude that X is connected if and only if every continuous map from X into a discrete space is constant.

Theorem 4.6. *Let $f : (X, d) \to (Y, \rho)$ be continuous, and let $E \subseteq X$. If E is connected, then $f(E)$ is connected.*

Proof. Suppose $f(E)$ is not connected. Then there exists a continuous surjective map $g : f(E) \to \{0, 1\}$. Thus, $g \circ f : E \to \{0, 1\}$ is continuous and surjective, so E is disconnected, a contradiction. $\qquad\square$

Remark 4.7. A non-constant continuous image of an interval is again an interval. This is nothing but the intermediate value theorem.

Corollary 4.8. *Let I be an interval in $\mathbb{R}$, and $f : I \to \mathbb{R}$ be a non-constant continuous function, then $f(I)$ is an interval. In particular, if $a, b \in I$ and $f(a) \neq f(b)$, then f assumes all values between $f(a)$ and $f(b)$.*

Example 4.9. If A, B are connected subsets of a metric space X, then $A \times B$ is connected in $(X \times X, d \times d)$, where

$$(d \times d)\{(x_1, y_1), (x_2, y_2))\} = d(x_1, x_2) + d(y_1, y_2). \tag{4.3}$$

Suppose $f : A \times B \to \{0, 1\}$ is continuous. We claim f is constant. For $a \in A$ and $b \in B$, $f(a, \cdot)$ and $f(\cdot, b)$ are continuous function on A and B respectively . Since A and B are connected implies $f(a, \cdot)$ and $f(\cdot, b)$ both are constant. That is, f is constant on every vertical and horizontal lines. Hence, f is constant.

Exercise 4.2. Show that $(0, 1) \times (0, 1)$ cannot be written as disjoint union of countably many open balls. (*Hint:* $(0, 1) \times (0, 1)$ is connected)

Exercise 4.3. Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$ continuous. Show that D is connected if and only if the graph of f, $G_f = \{(x, f(x)) : x \in D\}$ is connected in $\mathbb{R}^2$. (*Hint:* $g : X \to X \times X$, $g(x) = (x, f(x))$ is continuous $\implies G_f$ is connected ($\because X$ is connected). On the other hand, projection $\rho_1 : G_f \to X \implies \rho_1(x, f(x)) = x$, is continuous $\implies X$ is connected)

Exercise 4.4. If $A \subset X$ is connected, then for $A \subseteq B \subseteq \overline{A}$, it implies that B is connected. In particular, $\overline{A}$ is connected.

Suppose $f : B \to \{0, 1\}$ is continuous and surjective, then $f|_A : A \to \{0, 1\}$ is continuous $\implies f$ is constant on B.

Exercise 4.5. Let $A \subset B \subset X$. If A and X are connected, does it imply B is connected? ($(0, 1) \subset (0, 1) \cup (1, 2) \subset \mathbb{R}$)

Exercise 4.6. *(Topologist's sine curve)* Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sin\left(\frac{\pi}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Show that f is not continuous, but G_f is connected.
(*Hint.* Consider $g : (0, 1] \to [-1, 1]$ by $g(x) = \sin\left(\frac{\pi}{x}\right)$. Then g is continuous, and hence $g((0, 1])$ is connected $\implies g$ is onto. Also, G_g is connected. Since $G_g \subset G_f \subset \overline{G_g} \implies G_f$ is connected.

Exercise 4.7. If $f : X \to Y$ is continuous and onto, Y not connected, then X is not connected. (*Hint.* $Y = C \cup D \implies X = f^{-1}(C) \cup f^{-1}(D)$)

Exercise 4.8. $\text{Ln}(\mathbb{R}) = \{\text{space of all } n \times n \text{ real matrices}\}$ and $\text{GL}_n(\mathbb{R}) = \{A = (x_{ij}) \in \text{Ln}(\mathbb{R}) : \det A \neq 0\}$. Then, $\text{GL}_n(\mathbb{R})$ is disconnected in the usual metric on $\text{Ln}(\mathbb{R})$. (*Hint.* $\det(A) = \sum_{i=1}^{n} x_{ii} \implies \det$ is continuous. $\implies \text{GL}_n(\mathbb{R}) = (\det)^{-1}(\mathbb{R} \setminus \{0\})$ is open. Now,

$$\det : \ \text{GL}_n(\mathbb{R}) \xrightarrow{\text{continuous, onto}} \mathbb{R} \setminus \{0\}$$

$\implies \text{GL}_n(\mathbb{R}) = \text{GL}_n^+(\mathbb{R}) \cup \text{GL}_n^-(\mathbb{R})$ is disconnected, where

$$(\det)^{-1}\{(-\infty, 0))\} = \text{GL}_n^-(\mathbb{R}), \quad (\det)^{-1}\{(0, \infty))\} = \text{GL}_n^+(\mathbb{R}).$$

(*Hint.* An easiest metric on $\text{Ln}(\mathbb{R})$ is $d(A, B) = \max_{ij} |a_{ij} - b_{ij}|$)

4.2 Path Connectedness

4.2.1 Path Connectedness

A set $E \subset X$ is said to be *path-connected* if for every $x, y \in E$, there exists a continuous function $\gamma : [0, 1] \to E$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Example 4.10. Show that the the continuous image of a path-connected set is path-connected.

Let $E \subset X$ be path-connected and let $f : E \to \mathbb{C}$ be continuous. Take any $f(x), f(y) \in f(E)$ with $x, y \in E$. Since E is path-connected, there exists a path $\gamma : [0, 1] \to E$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Therefore, $(f \circ \gamma)(0) = f(x)$ and $(f \circ \gamma)(1) = f(y)$, so $f \circ \gamma$ is a path joining $f(x)$ and $f(y)$.

Example 4.11. Let P be a polynomial in $\mathbb{C}^n$. Then $\mathbb{C}^n \setminus P^{-1}(0)$ is path-connected.

Let $z, w \in \mathbb{C}^n \setminus P^{-1}(0)$. Define $\gamma : \mathbb{C} \to \mathbb{C}^n$ by $\gamma(t) = (1 - t)z + tw$, $t \in \mathbb{C}$. Then $\{t \in \mathbb{C} : \gamma(t) \in P^{-1}(0)\} = (P \circ \gamma)^{-1}(0)$. Since $(P \circ \gamma)$ is a polynomial on $\mathbb{C}$, it implies that $(P \circ \gamma)^{-1}(0)$ is a finite set. Hence, $\mathbb{C} \setminus (P \circ \gamma)^{-1}(0)$ is path-connected in $\mathbb{C}$. Hence, $f(\mathbb{C} \setminus (P \circ \gamma)^{-1}(0))$ is path-connected in $\mathbb{C}^n \setminus P^{-1}(0)$ (since $\gamma(\mathbb{C} \setminus P^{-1}(0))$ is contained in $\mathbb{C}^n \setminus P^{-1}(0)$) containing z and w. Hence, $\mathbb{C}^n \setminus P^{-1}(0)$ is path-connected. (Note that γ is not onto unless $n = 1$, hence $\gamma(\mathbb{C} \setminus (P \circ \gamma)^{-1}(0)) \subsetneq \mathbb{C}^n \setminus P^{-1}(0)$.)

Topologist's Sine Curve: Let $f : [0, 1] \to [-1, 1]$ by

$$
f(x) = \begin{cases} \sin \frac{\pi}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}
$$

Then

$$
G_f = \{(x, \sin \frac{\pi}{x}) : x \in (0, 1]\} \cup \{(0, 0)\}
$$

is not open. G_f is not path-connected. (The hope comes from the fact that f is not continuous at 0.) Suppose, on the contrary, that there is a continuous path

$$
\gamma : [0, 1] \to G_f = \{(x, \sin \frac{1}{x}) : x \neq 0\} \cup \{(0, 0)\}
$$

where $\gamma(0) = (0, 0)$ and $\gamma(1) = (1, 0)$; write $\gamma = (\gamma_1, \gamma_2)$. Since γ is continuous, γ becomes uniformly continuous. For $\varepsilon = 1 > 0$, there exists $\delta > 0$ such that

$$
|s - t| < \delta \implies |\gamma_2(s) - \gamma_2(t)| < 1
$$

Since $0 \in \gamma^{-1}\{(0, 0)\}$, let $t^* = \sup \gamma^{-1}\{(0, 0)\} < 1$ $(\because \gamma(1) = (1, 0))$.
Choose $\delta_1 > 0$ such that $0 \leq t^* < t^* + \delta_1 < 1$ and $\delta_1 < \delta$. Note that

$$
t^* = \sup\{t : \gamma(t) = (\gamma_1(t), \gamma_2(t)) = (0, 0)\}
$$

So, there exists $t_n \to t^*$, with $\gamma_1(t_n) = 0 \implies \gamma_1(t^*) = 0$, but $\gamma_1(t^* + \delta_1) > 0$. Since $0 = \gamma_1(t^*) < \gamma_1(t^* + \delta_1) < 1$, by IVP, for large N, there exists $s, t \in (t^*, t^* + \delta_1)$ such that

$$
\gamma_1(t) = \frac{2}{N+1}, \quad \gamma_1(s) = \frac{2}{N}
$$

Therefore,

$$
\gamma_2(t) = \sin\left(\frac{N+1}{2}\right)\pi, \quad \gamma_2(s) = \delta_1 \sin\left(\frac{N\pi}{2}\right)
$$

So,
$$|\gamma_2(t) - \gamma_2(s)| = 1$$

This is a contradiction.

Example 4.12. $\mathbb{R}^n \setminus \{0\}, (\ n \geq 2)$ is connected.

Suppose not, let U be an open and closed set in $\mathbb{R}^n \setminus \{0\}$. For $x \in U$ and $y \in \mathbb{R}^n \setminus \{0\} \setminus U$, we get a line segment path connecting x and y, say L. Then $L \cap U$ is the finite union of open and closed sets in $\mathbb{R}$, but $\mathbb{R}$ is connected. Hence, our assumption is wrong, and $\mathbb{R}^n \setminus \{0\}$ is connected. In fact, path-connected.

Example 4.13. Let $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$. Then S^{n-1} is connected.

Define $\varphi : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ by
$$\varphi(x) = \frac{x}{\|x\|}$$

Then φ is continuous and onto, hence S^{n-1} is connected. In fact, S^{n-1} is the continuous image of the path-connected set $\mathbb{R}^n \setminus \{0\}$, and is therefore path-connected.

Example 4.14. Alternative: If $A \subset \mathbb{R}$ is connected, then A is an interval. Suppose there exist $x, y \in A$, $x < z < y$, but $z \notin A$. Define

$$f(s) = \begin{cases} 1 & \text{if } s < z \\ -1 & \text{if } s > z \end{cases}$$

So,
$$f : A \setminus \{z\} = A \to \{1, -1\}$$

is continuous and onto, so A is not connected.

Example 4.15. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $G_f = \{(x, f(x)) : x \in \mathbb{R}\}$ is closed and connected in $\mathbb{R}^2$. Then f is continuous.

Let $x_n \to x$. Assume $f(x_n) \to y$. Then $(x_n, f(x_n))$ is a Cauchy sequence in $\mathbb{R}^2$ and hence $(x_n, f(x_n)) \to (x, y)$. But G_f is closed, so $(x, y) \in G_f$, which implies $y = f(x)$. Hence, f is continuous.
Note that: $f(x_n) \to y$ was achieved by considering the boundedness of f where $x_n \to x$. If f is bounded, then $f(x_n)$ is bounded in $\mathbb{R}$, and by Bolzano-Weierstrass Theorem, there exists a subsequence $f(x_{n_k}) \to y \in \mathbb{R}$. Thus, $(x_{n_k}, f(x_{n_k}))$ is a Cauchy sequence in $\mathbb{R}^2$ and hence convergent, say $(x_{n_k}, f(x_{n_k})) \to (x, y)$. But G_f is closed, implies $y = f(x)$.
Notice that there is no other limit point for $(x_n, f(x_n))$ than $(x, f(x))$, else f will not be well-defined. Thus, $(x_n, f(x_n)) \to (x, f(x))$. Hence, f is continuous.
Notice that so far we have not used the fact that G_f is connected. Next case is when $|f(x_n)| \to \infty$, where $x_n \to x$. In this case, we reach to a contradiction that G_f is disconnected in a neighborhood of x. We claim that there exists $\delta > 0$ such that for $|x - y| < \delta \implies$ either

$|f(x) - f(y)| < 1$ or $|f(x) - f(y)| > 2$. (Bounded below and above in a neighbourhood of x) If it is false, then there is a sequence u_n with $|u_n - x| < \frac{1}{n}$ such that $1 \le |f(x) - f(u_n)| \le 2$ $\Rightarrow$ There is a subsequence $f(u_{n_k})$ of $f(u_n)$ such that $f(u_{n_k}) \to w$. Then $(u_{n_k}, f(u_{n_k})) \to (x, w)$, and the graph G_f is closed (by hypothesis), which implies $f(x) = w$. But $1 \le |f(x) - w| \le 2$ Thus, our claim is true.

Let $[a, b] = [x - \delta, x + \delta]$. We claim that $G_f \cap \{[a, b] \times \mathbb{R}\}$ is connected. On the other hand,

$$G_f \cap ([a, b] \times \mathbb{R})$$

$$= (G_f \cap \{[a, b] \times \mathbb{R}\}) \cap \{(t, s) : |f(x) - s| < 1\} \cup (G_f \cap ([a, b] \times \mathbb{R})) \cap \{(t, s) : |f(x) - s| > 1\}$$

that is $G_f \cap ([a, b] \times \mathbb{R}) = A \cup B$.   (3.5)
Thus, $G_f \cap ([a, b] \times \mathbb{R})$ is disconnected as $(x, f(x)) \in A$ and $(x_n, f(x_n)) \in B$ for large n. This implies $x_n \to x \implies f(x_n)$ is bounded. Hence, from the previous case, it follows that $f(x_n) \to f(x)$. To show G_f is connected, let

$$g : G_f \cap ([a, b] \times \mathbb{R}) \to \{0, 1\}$$

be continuous. Then g can be extended continuously outside $G_f \cap ([a, b] \times \mathbb{R})$ by constant. Hence $g : G \to \{0, 1\}$ is continuous. But G is connected, hence g is constant. Thus, $G_f \cap ([a, b] \times \mathbb{R})$ is connected.

Example 4.16. Let $K = \{\frac{1}{n} : n \ge 1\}$ and $E = ([0, 1] \times \{0\}) \cup (K \times [0, 1])$. Then E is path-connected (Why?)

Let $C = E \times (\{0\} \times [0, 1])$, known as the *comb space*, which is path-connected. The deleted comb space $C_0 = E \cup \{(0, 1)\}$ is connected, since $E \subset C_0 \subset \overline{E}$ and E is connected. But C_0 is *not* path-connected, because there is no path connecting $(0, 1)$ and $(1, 0)$. On the contrary, suppose

$$\gamma : [0, 1] \to C_0$$

Suppose there is a continuous path such that $\gamma(0) = (0, 1)$ and $\gamma(1) = (1, 0)$. Then $\gamma^{-1}((0, 1))$ is a closed set, and let $t_0 = \sup \gamma^{-1}((0, 1)) = \sup\{t \in [0, 1] : \gamma(t) = (0, 1)\}$. We claim that there exists $t_1 \in (t_0, 1]$ such that
$$(P_1 \circ \gamma)\{(t_0, t_1)\} \subseteq K,$$

where $P_1 : \mathbb{R}^2 \to \mathbb{R}$ is the projection onto the x-axis. Suppose the claim is false. Then $\exists t_n \in (t_0, 1]$ with $t_n \to t_0$. By assumption, $\exists s_n \in (t_0, t_n)$ such that $\gamma(s_n) = (x_n, 0)$ for some $x_n \in [0, 1] \setminus K$.

Note that $s_n \to t_0$. By continuity, $(x_n, 0) = \gamma(s_n) \to \gamma(t_0) = (0, 1)$, which is absurd. Thus, there exists $t_1 \in (t_0, 1]$ such that $(P_1 \circ \gamma)\{(t_0, t_1)\} \subseteq K \implies 1 \in (P_1 \circ \gamma)(t_0, t_1)$ is a connected subset of K. Hence $(P_1 \circ \gamma)(t_0, t_1) = \{1\}$ (by continuity), but $(P_1 \circ \gamma)(t_0) = 0$, an absurd.

Example 4.17. Let U be an open set in $\mathbb{R}^n$ (or $\mathbb{C}^n$). Then U is path-connected if and only if U is connected.

Fix a point $p \in U$, and let

$$\mathcal{A} = \{q \in U : \text{there exists a path in } U \text{ joining } p \text{ to } q\}.$$

We show that $\mathcal{A}$ is both open and closed in U.

If $q \in \mathcal{A}$, then $q \in U$, so there exists $r > 0$ such that $B_r(q) \subset U$. For any $s \in B_r(q)$, the line segment joining q to s lies in $B_r(q)$, hence in U. Since q is already joined to p by a path in U, concatenating the two paths shows that $s \in \mathcal{A}$. Therefore $B_r(q) \subset \mathcal{A}$, and $\mathcal{A}$ is open in U.

Now let $q \in U \setminus \mathcal{A}$. Again choose $r > 0$ such that $B_r(q) \subset U$. If some $s \in B_r(q)$ belonged to $\mathcal{A}$, then the line segment from s to q would lie in U, and concatenating it with a path from p to s would produce a path from p to q, contradicting $q \notin \mathcal{A}$. Hence $B_r(q) \subset U \setminus \mathcal{A}$, so $U \setminus \mathcal{A}$ is open in U.

Thus $\mathcal{A}$ is nonempty, open, and closed in the connected set U. Therefore $\mathcal{A} = U$, and every point of U can be joined to p by a path in U. Hence U is path-connected.

Exercises

Exercise 4.9. Prove that the image of a connected set under a continuous map is connected.

Exercise 4.10. Prove that an interval in $\mathbb{R}$ is connected.

Exercise 4.11. Show that if $E \subset \mathbb{R}$ is connected, then E is an interval (possibly a single point).

Exercise 4.12. Let X be a metric space. Prove that path-connectedness implies connectedness.

Exercise 4.13. Give an example of a connected set that is not path-connected.

Exercise 4.14. Let $A \subset X$. Define the connected component of $x \in A$ to be the union of all connected subsets of A that contain x. Prove that components form a partition of A and are closed in A.

Exercise 4.15. Prove the Intermediate Value Theorem using connectedness: if $f : [a, b] \to \mathbb{R}$ is continuous, then $f([a, b])$ is an interval.

Exercise 4.16. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and periodic. Prove that f is uniformly continuous.

Exercise 4.17. Show that $\mathbb{R} \setminus \{0\}$ is disconnected, but $\mathbb{R}^2 \setminus \{0\}$ is connected.

Exercise 4.18. Let U, V be disjoint nonempty open sets in X with $X = U \cup V$. Prove that every continuous map $f : X \to \mathbb{R}$ has disconnected range unless it is constant on one of U or V.

Chapter 5

Continuity and Function Theory on the Real Line

This chapter introduces the ε–δ definitions of limits and continuity for real-valued functions. We emphasize the logical structure of the quantifiers and the connection with sequential characterizations. We then study monotone functions as a rich class of examples: one-sided limits always exist, and discontinuities are necessarily jumps and form a countable set. Finally, we discuss inverses of strictly monotone maps and construct monotone functions with prescribed jump discontinuities on a given countable set.

Learning objectives.

- Compare pointwise continuity, monotonicity, and one-sided behavior on the real line.

- Understand how inverse maps of monotone functions inherit continuity and order properties.

- Use constructive examples to see how prescribed discontinuity patterns can coexist with monotonicity.

5.1 Limits and Continuity

5.1.1 Limits and one-sided limits

Let $I \subseteq \mathbb{R}$ be an interval and let a be a point such that $I \cap (a - \delta, a + \delta) \setminus \{a\} \neq \varnothing$ for every $\delta > 0$. Let $f : I \setminus \{a\} \to \mathbb{R}$ be a function.

Definition 5.1 (Limit at a point)**.** We say that $L \in \mathbb{R}$ is the *limit* of f at a, and we write $\lim_{x \to a} f(x) = L$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < |x - a| < \delta, \ x \in I \implies |f(x) - L| < \varepsilon. \tag{5.1}$$

Definition 5.2 (One-sided limits)**.** Assume that I contains points to the left of a. We say that $L \in \mathbb{R}$ is the *left-hand limit* of f at a, and we write $\lim_{x \to a^-} f(x) = L$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$a - \delta < x < a \implies |f(x) - L| < \varepsilon. \tag{5.2}$$

Assume that I contains points to the right of a. We say that $M \in \mathbb{R}$ is the *right-hand limit* of f at a, and we write $\lim_{x \to a^+} f(x) = M$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$a < x < a + \delta \implies |f(x) - M| < \varepsilon. \tag{5.3}$$

Remark 5.3. If both one-sided limits exist and are equal, then the (two-sided) limit exists and equals their common value. Conversely, if $\lim_{x \to a} f(x)$ exists, then both one-sided limits exist and coincide with it.

Theorem 5.4 (Sequential characterization of limits). *Let $f : I \setminus \{a\} \to \mathbb{R}$ and $L \in \mathbb{R}$. Then $\lim_{x \to a} f(x) = L$ if and only if for every sequence (x_n) in $I \setminus \{a\}$ with $x_n \to a$ we have $f(x_n) \to L$.*

Proof. Assume first that $\lim_{x \to a} f(x) = L$. Let (x_n) be any sequence in $I \setminus \{a\}$ with $x_n \to a$. Given $\varepsilon > 0$, choose $\delta > 0$ as in (5.1). Since $x_n \to a$, there exists N such that $|x_n - a| < \delta$ for all $n \geq N$, and hence $|f(x_n) - L| < \varepsilon$ for all $n \geq N$. Thus $f(x_n) \to L$.

Conversely, assume that the sequential condition holds but the ε–δ condition fails. Then there exists $\varepsilon_0 > 0$ such that for every $k \in \mathbb{N}$ there is a point $x_k \in I$ with $0 < |x_k - a| < 1/k$ and $|f(x_k) - L| \geq \varepsilon_0$. The sequence (x_k) satisfies $x_k \to a$, but $f(x_k) \not\to L$, which contradicts the assumption. $\square$

5.1.2   Continuity

Definition 5.5 (Continuity at a point). Let $f : I \to \mathbb{R}$ and let $a \in I$. We say that f is *continuous at a* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - a| < \delta \text{ and } x \in I \implies |f(x) - f(a)| < \varepsilon.$$

Equivalently, f is continuous at a if and only if $\lim_{x \to a} f(x) = f(a)$.

Remark 5.6. If both one-sided limits $f(a^-)$ and $f(a^+)$ exist and satisfy $f(a^-) = f(a) = f(a^+)$, then f is continuous at a. If $f(a^-)$ and $f(a^+)$ exist but are unequal, then f has a *jump discontinuity* at a.

5.2   Monotone Functions

5.2.1   One-sided limits and discontinuities

Definition 5.7 (Monotonicity). Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$. We say that f is *increasing* on I if $x < y$ implies $f(x) \leq f(y)$. We say that f is *decreasing* on I if $x < y$ implies $f(x) \geq f(y)$. A function is *monotone* if it is either increasing or decreasing.

Theorem 5.8 (Existence of one-sided limits). *Let $f : (a, b) \to \mathbb{R}$ be monotone and let $c \in (a, b)$. Then both one-sided limits $f(c^-)$ and $f(c^+)$ exist in $\mathbb{R}$.*

Proof. Assume that f is increasing. Set

$$L := \sup\{f(x) : a < x < c\}, \qquad M := \inf\{f(x) : c < x < b\}.$$

Then $L \leq f(c) \leq M$. Fix $\varepsilon > 0$. Choose $x_0 \in (a, c)$ such that $f(x_0) > L - \varepsilon$. If $c - \delta < x < c$ with $\delta := c - x_0$, then $x_0 \leq x < c$ and hence

$$L - \varepsilon < f(x_0) \leq f(x) \leq L,$$

so $|f(x) - L| < \varepsilon$. This shows that $\lim_{x \to c^-} f(x) = L$. The proof for $M = \lim_{x \to c^+} f(x)$ is analogous. $\qquad\square$

Corollary 5.9 (Discontinuities of a monotone function). *Let $f : (a, b) \to \mathbb{R}$ be monotone. Then every discontinuity of f is a jump discontinuity, and the set of discontinuities is at most countable.*

Proof. Assume that f is increasing. By the theorem, $f(c^-)$ and $f(c^+)$ exist for every $c \in (a, b)$ and satisfy $f(c^-) \leq f(c) \leq f(c^+)$. Thus f is discontinuous at c precisely when $f(c^-) < f(c^+)$, in which case the discontinuity is a jump.

For each discontinuity point c, choose a rational number $q_c \in \mathbb{Q}$ such that

$$f(c^-) < q_c < f(c^+).$$

If $c \neq d$ are two discontinuity points, then the open intervals $(f(c^-), f(c^+))$ and $(f(d^-), f(d^+))$ are disjoint, because monotonicity implies $c < d$ yields $f(c^+) \leq f(d^-)$. Hence the map $c \mapsto q_c$ is injective from the set of discontinuities into $\mathbb{Q}$. Since $\mathbb{Q}$ is countable, the set of discontinuities is at most countable. $\qquad\square$

5.3   Surjections, Inverses, and Continuity

Theorem 5.10 (Monotone surjections are continuous). *Let $f : [a, b] \to [c, d]$ be monotone and surjective. Then f is continuous on $[a, b]$.*

Proof. Assume that f is increasing. Surjectivity forces $f(a) = c$ and $f(b) = d$. If f were discontinuous at some $x_0 \in (a, b)$, then $f(x_0^-) < f(x_0^+)$ by the previous corollary. Choose y with $f(x_0^-) < y < f(x_0^+)$. For $x < x_0$ we have $f(x) \leq f(x_0^-)$, and for $x > x_0$ we have $f(x) \geq f(x_0^+)$. Thus $f(x) \neq y$ for every $x \in [a, b]$, contradicting surjectivity. Therefore f is continuous. $\qquad\square$

Proposition 5.11 (Inverse of a strictly monotone map). *Let $f : (a, b) \to (c, d)$ be strictly monotone and surjective. Then f is bijective, and its inverse $f^{-1} : (c, d) \to (a, b)$ is strictly monotone. If, in addition, f is continuous, then f^{-1} is continuous.*

Proof. Strict monotonicity implies injectivity, and surjectivity gives bijectivity. If f is strictly increasing, then $y_1 < y_2$ implies $f^{-1}(y_1) < f^{-1}(y_2)$, so f^{-1} is strictly increasing, and similarly for the decreasing case.

Assume that f is strictly increasing and continuous. Fix $y_0 \in (c, d)$ and set $x_0 = f^{-1}(y_0)$. Given $\varepsilon > 0$, continuity of f at x_0 yields $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - y_0| < \varepsilon$. Since f is increasing, this implies that $y_0 - \varepsilon < f(x_0 - \delta) \leq y \leq f(x_0 + \delta) < y_0 + \varepsilon$ forces $|f^{-1}(y) - x_0| < \delta$. Hence f^{-1} is continuous at y_0. $\qquad\square$

Example 5.12 (Cantor function). The Cantor (or "devil's staircase") function $\tilde{f} : [0, 1] \to [0, 1]$ is continuous, increasing, and surjective, but it is not injective.

5.4 Constructing Monotone Functions with Prescribed Jumps

Let D be a countable subset of $\mathbb{R}$. We now construct an increasing function whose discontinuities are contained in D, and whose jump at each point of D can be prescribed.

Proposition 5.13 (Construction). *Write $D = \{x_1, x_2, \dots\}$, and choose numbers $\varepsilon_n \in (0,1)$ such that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$. Define*

$$f(x) := \sum_{\substack{n \in \mathbb{N} \\ x_n \leq x}} \varepsilon_n,$$

with the convention that $f(x) = 0$ if the index set $\{n : x_n \leq x\}$ is empty. Then f is increasing on $\mathbb{R}$. Moreover, for each $k \in \mathbb{N}$,

$$f(x_k^-) = f(x_k) - \varepsilon_k, \qquad f(x_k^+) = f(x_k),$$

and hence the jump size at x_k equals ε_k. Finally, f is continuous at every point of $\mathbb{R} \setminus D$.

Proof. If $x < y$, then the index set $\{n : x_n \leq x\}$ is contained in $\{n : x_n \leq y\}$, so $f(y) \geq f(x)$. Fix k and let $y \downarrow x_k$. Then the set $\{n : x_k < x_n \leq y\}$ eventually becomes empty, so $\sum_{x_k < x_n \leq y} \varepsilon_n \to 0$, which gives $f(x_k^+) = f(x_k)$. Similarly, if $x \uparrow x_k$, then the only term that can "disappear" at the limit is ε_k, which yields $f(x_k^-) = f(x_k) - \varepsilon_k$.

If $x \notin D$, then for $y \to x$ the index set $\{n : \min\{x, y\} < x_n \leq \max\{x, y\}\}$ eventually becomes empty, so the corresponding sum tends to 0 and f is continuous at x. $\qquad\square$

Example 5.14. If $D = \mathbb{Z}$ and $\sum_{n \in \mathbb{Z}} \varepsilon_n < \infty$ (for instance by choosing $\varepsilon_n = 2^{-|n|-1}$), then the above construction yields a monotone function that is constant on each open interval $(n, n+1)$.

Example 5.15. Let D be the set of endpoints of the intervals removed in the construction of the Cantor set. By choosing a suitable summable sequence (ε_n) and applying the above construction to an enumeration of D, one obtains a monotone function whose discontinuities occur only at points of D. This viewpoint provides one route to the Cantor function.

Example 5.16. Define $f : [0, 1] \to \mathbb{R}$ by

$$f(x) = x + \sum_{n=0}^{n_x} 2^{-n}, \qquad n_x := \left\lfloor \frac{1}{1-x} \right\rfloor \text{ for } x < 1,$$

and set $f(1) = 3$. Show that f is strictly increasing and discontinuous at the countable set $\left\{ 1 - \frac{1}{k} : k \in \mathbb{N} \right\}$.

Exercises

Exercise 5.1. Give the ε–δ proof that $\lim_{x \to a}(x^2) = a^2$.

Exercise 5.2. Show that f is continuous at a if and only if $x_n \to a$ implies $f(x_n) \to f(a)$.

Exercise 5.3. Prove that sums, products, and compositions of continuous functions are continuous (with full ε–δ arguments).

Exercise 5.4. Show that a continuous function on a compact set is bounded and attains its maximum and minimum.

Exercise 5.5. Prove that a continuous function on $[a, b]$ is uniformly continuous.

Exercise 5.6. Let f be monotone on an interval. Prove that the one-sided limits $f(x^-)$ and $f(x^+)$ exist at every interior point.

Exercise 5.7. Show that a monotone function has at most countably many discontinuities.

Exercise 5.8. Let f be strictly increasing and continuous on an interval I. Prove that $f(I)$ is an interval and that f^{-1} is continuous on $f(I)$.

Exercise 5.9. Construct a monotone function with jump discontinuities exactly at the points of a given countable subset of $(0, 1)$.

Exercise 5.10. Prove that if f is continuous and injective on an interval, then f is strictly monotone.

Chapter 6

Sequences and Series of Functions: Compactness, Approximation, and Power Series

We study compactness phenomena in spaces of functions via equicontinuity and the Arzelà–Ascoli theorem. We then discuss semicontinuity and approximation results (including the Weierstrass approximation theorem), and we conclude with a systematic treatment of power series.

Learning objectives.

- Control families of functions by uniform boundedness, equicontinuity, and compactness in function spaces.

- Distinguish pointwise from uniform phenomena and understand which analytic properties pass to the limit.

- Use Arzelà–Ascoli, approximation arguments, and power-series methods as a coherent package.

6.1 Compactness in Spaces of Functions

Section overview.

- Compactness in $C(X)$ is subtler than compactness in Euclidean space because one must control entire families of functions at once.

- Uniform boundedness and equicontinuity are the correct replacements for ordinary boundedness.

- Arzelà–Ascoli should be read as a compactness criterion and as a blueprint for extracting convergent subsequences.

6.1.1 Equicontinuity

A collection $\mathcal{F} \subset C(X)$ is said to be (uniformly) equicontinuous if for every $\varepsilon > 0$, there exists $\delta > 0$ such that
$$d(x, y) < \delta \implies |f(x) - f(y)| < \varepsilon, \quad \forall f \in \mathcal{F}.$$

Example 6.1.

(i) Finite subset of $C(X)$ is (uniformly) equicontinuous and every sub-collection of a (uniformly) equicontinuous collection is equicontinuous.

(ii) Let $0 < \alpha \leq 1$ and $k > 0$. Define

$$\mathrm{Lip}_K^\alpha = \{f \in C([0,1]) : |f(x) - f(y)| \leq k|x - y|^\alpha\}$$

This collection is equicontinuous, but not totally bounded, since all constant functions are satisfying this condition.

Lemma 6.2. *If $\mathcal{F} \subset C(X)$ is totally bounded, then $\mathcal{F}$ is uniformly bounded and (uniformly) equicontinuous.*

Proof. Since a totally bounded set is (uniformly) bounded, we only need to show that $\mathcal{F}$ is equicontinuous. Since $\mathcal{F}$ is totally bounded, for $\varepsilon > 0$, there exists $f_1, \ldots, f_n \in \mathcal{F}$ such that for $f \in \mathcal{F}$, there exists f_i with $\|f - f_i\|_\infty < \varepsilon$. But $\{f_1, \cdots, f_n\}$ is equicontinuous, so for $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d(x, y) < \delta \implies |f_i(x) - f_i(y)| < \varepsilon, \quad \forall i = 1, \ldots, n.$$

Now, for any $f \in \mathcal{F}$,

$$|f(x) - f(y)| \leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)| \leq \varepsilon + \varepsilon + \varepsilon = 3\varepsilon.$$

$\square$

Corollary 6.3. *If $f_n \xrightarrow{unif} f$ in $C(X)$, then $\{f_n\}$ is uniformly bounded and (uniformly) equicontinuous.*

Proof. Notice that $\{f\} \cup \{f_n : n \in \mathbb{N}\}$ is compact, hence $\{f_n\}$ is totally bounded, so (uniformly) equicontinuous. $\square$

6.1.2 Arzelà-Ascoli Theorem

Theorem 6.4 (Arzelà–Ascoli). *Let X be a compact metric space, and let $\mathcal{F} \subset C(X)$. Then $\mathcal{F}$ is compact in $(C(X), \|\cdot\|_\infty)$ if and only if $\mathcal{F}$ is closed, uniformly bounded, and uniformly equicontinuous.*

Proof strategy. The forward direction packages compactness into total boundedness and hence into uniform boundedness and equicontinuity. For the converse, one proves that every sequence in $\mathcal{F}$ admits a uniformly Cauchy subsequence by combining finitely many point evaluations with equicontinuity on a finite δ-net of the compact space.

Proof. The forward implication follows from the previous lemma.

Conversely, let $(f_n) \subset \mathcal{F}$ be a sequence. *Claim:* (f_n) has a (uniformly) convergent subsequence. Note that (f_n) is equicontinuous. For $\varepsilon > 0$, there exists $\delta > 0$ such that

$$d(x, y) < \delta \implies |f_n(x) - f_n(y)| < \varepsilon, \quad \forall n \geq 1.$$

Since X is totally bounded, there exists a finite set $x_1, \ldots, x_k \in X$ such that

$$X = \bigcup_{i=1}^{k} B_\delta(x_i).$$

Let $x \in X$, then there exist x_i such that $d(x, x_i) < \delta$. Also, (f_n) is uniformly bounded, so for each i, $\{f_n(x_i)\}_{n=1}^{\infty}$ is bounded in $\mathbb{R}$. So Without loss of generality, we may assume that $\{f_n(x_i)\}_{n=1}^{\infty}$ is convergent for each $i = 1, \ldots, k$. In particular, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$|f_m(x_i) - f_n(x_i)| < \varepsilon$$

for all $m, n \geq N$, for each $i = 1, 2, \ldots, k$. Now, for $x \in X$, there exists x_i such that $d(x, x_i) < \delta$. Hence,

$$|f_m(x) - f_n(x)| \leq |f_m(x) - f_m(x_i)| + |f_m(x_i) - f_n(x_i)| + |f_n(x_i) - f_n(x)|$$
$$< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon.$$

So $\|f_m - f_n\|_\infty \leq 3\varepsilon$ for all $m, n \geq N$. Therefore, (f_n) is a (uniformly) Cauchy sequence, hence convergent (because $C(X)$ is complete). $\qquad\square$

Takeaway. Arzelà–Ascoli is the function-space analogue of Bolzano–Weierstrass: compactness is recovered not from pointwise boundedness alone, but from a simultaneous control of size and oscillation.

Corollary 6.5. *Let X be compact. If (f_n) is uniformly bounded and (uniformly) equicontinuous in $C(X)$, then (f_n) has a uniformly convergent subsequence. (Hint: $A = \overline{\{f_n : n \in \mathbb{N}\}}$ is closed.)*

Example 6.6. Let $X = (0, 1)$ and define

$$f_n(t) = \begin{cases} 1 - nt & \text{if } t < \frac{1}{n} \\ 0 & \text{if } t \geq \frac{1}{n} \end{cases}$$

Show that $(f_n)_{n=1}^{\infty}$ is pointwise equicontinuous but not uniformly equicontinuous on $(0, 1)$.

Notice that for any point $t \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that for each $n \geq n - 0$, $f_n(t) = 0$ in a small neighborhood of t. Hence, $(f_n)_{n \geq 1}$ is pointwise equicontinuous on $(0, 1)$. However,

$$\left| f_n\left(\frac{1}{2n}\right) - f_n\left(\frac{1}{n}\right) \right| = |1 - n \cdot \frac{1}{2n} - 0| = \frac{1}{2}$$

where $|\frac{1}{2n} - \frac{1}{n}| = \frac{1}{2n} \to 0$. Hence, $(f_n)_{n \geq 1}$ is not uniformly equicontinuous on $(0, 1)$.

Example 6.7. For $X = [0, 1]$, define

$$f_n(t) = \max\left\{1 - 2(n+1)^2 \left|t - \frac{1}{n}\right|, 0\right\}.$$

Then $(f_n)_{n \geq 1}$ is equicontinuous at each point $t > 0$, but not at $t = 0$.

For $t_0 > 0$, it follows from the fact that

$$1 - 2(n+1)^2\left|t_0 - \frac{1}{n}\right| \leq 0 \quad \text{if and only if} \quad \frac{1}{n} + \frac{1}{2(n+1)^2} \leq t_0.$$

And hence, $f_n(t) = 0$ for $n \geq n_0$ in a small neighborhood of $t_0 > 0$. Notice that the above means that $(f_n)_{n=n_0}^{\infty}$ is pointwise equicontinuous at $t_0 > 0$. Since $\{f_1, \cdots, f_{n_0-1}\}$ (finitely many) is always equicontinuous. Thus $(f_n)_{n \geq 1}$ is pointwise equicontinuous for $t > 0$. However, for $t = 0$, $f_n(0) = 0$, $f_n\left(\frac{1}{n}\right) = 1$, but $|0 - \frac{1}{n}| \to 0$ and $|f_n(0) - f_n(\frac{1}{n})| = 1$. Thus, $(f_n)_{n \geq 1}$ is not pointwise equicontinuous at $t = 0$.

Remark 6.8. We end this section with a remark on a structural property of subsets of the real line. Any open set can be written as a countable union of pairwise disjoint open intervals; moreover, a bounded (equivalently, totally bounded) set can be covered by finitely many intervals of arbitrarily small length.

Remark 6.9. A closer look at totally bounded sets reveals that many properties verified on finitely many centers can be propagated to the whole space, since every point lies in a ball of arbitrarily small radius.

6.1.3   Dini's Theorem

Theorem 6.10. *Let X be a compact metric space, and $f, f_n \in C(X)$ such that $f_n \downarrow f$ pointwise on X. Then $f_n \downarrow f$ uniformly on X.*

Proof. Let $g_n = f_n - f$. Then $g_n \downarrow 0$ pointwise on X. Notice that for each $\epsilon > 0$, $|g_n(x)| < \epsilon$ for large n (depending upon x). Let

$$E_n = \{x \in X : g_n(x) < \epsilon\}.$$

Then $E_n = g_n^{-1}(-\infty, \epsilon)$, hence open. Also, $E_n \subset E_{n+1} \subset \cdots$. Since $g_n \downarrow 0$ at each point, it follows that $X = \bigcup_{n=1}^{\infty} E_n$. (If $x \in X$ and $x \notin E_n$ for all $n \in \mathbb{N}$, then $g_n(x) \geq \epsilon$ for all $n \in \mathbb{N}$, which is a contradiction.)

But X is compact, hence there exists $N \in \mathbb{N}$ such that $X = \bigcup_{n=1}^{N} E_n = E_N$. Thus, for $x \in X$ and $n \geq N$, we have $g_n(x) \leq g_N(x) < \epsilon$, that is, $|g_n(x)| < \epsilon$ for all $n \geq N$ and all $x \in X$. Hence $g_n \downarrow 0$ uniformly on X. $\square$

Corollary 6.11. *Suppose $f_n, f \in C(X)$ and $f_n \uparrow f$ pointwise, then $f_n \uparrow f$ uniformly. (Hint. $g_n = f - f_n \downarrow 0$ pointwise, so use the above argument.)*

1. Notice that the limit function f must be continuous, else $f_n(x) = x^n$ will contradict the above theorem.

2. If X is not compact, then the conclusion of the theorem might not be true.

   For $X = \mathbb{R}$,

   $$f_n(t) = \begin{cases} 0 & \text{if } -\infty < t \leq n \\ \frac{t}{n} - 1 & \text{if } n < t \leq 2n \\ 1 & \text{if } t > 2n \end{cases}$$

   $f_n \downarrow 0$ pointwise, but $\|f_n\|_\infty = 1..$

Remark 6.12. However, a pointwise convergent sequence can differ with uniform convergence on an arbitrarily small set (*Egoroff's Theorem*).

6.2   Semicontinuity and Approximation

6.2.1   Upper Semi-Continuity

Let $f : (X, d) \to \mathbb{R}$. Then f is said to be *upper semi-continuous* on X if for each $\alpha \in \mathbb{R}$, the set $\{x \in X : f(x) < \alpha\}$ is open.

Result: $f : X \to \mathbb{R}$ is upper semi-continuous if and only if for any $x \in X$, and each sequence $x_n \to x \implies \limsup_{n \to \infty} f(x_n) \leq f(x)$.

Proof. Let $x_0 \in X$ and $\epsilon > 0$. Then $x_0 \in \{x : f(x) < f(x_0) + \epsilon\}$ is open. This implies, there exists a neighbourhood $B_\delta(x_0)$ such that $f(x) < f(x_0) + \epsilon$ for all $x \in B_\delta(x_0)$. Let $\frac{1}{n} < \delta$ and $x_n \to x$ then $x_n \in B_{\frac{1}{n}}(x_0)$ such that $f(x_n) < f(x_0) + \epsilon$. Hence,

$$x_n \to x_0 \implies \limsup_{n \to \infty} f(x_n) \leq f(x_0) + \epsilon \quad \text{for all} \quad \epsilon > 0$$

So $\limsup_{n \to \infty} f(x_n) \leq f(x_0)$.

Conversely, suppose, on the contrary, that f is not upper semi-continuous on X. Then there exists $\alpha \in \mathbb{R}$ such that $A_\alpha = \{x \in X : f(x) < \alpha\}$ is not open. That is, there exists $x_0 \in A_\alpha$ such that for every neighbourhood $B_\delta(x_0)$, there exists $x_\delta \in B_\delta(x_0)$ with $x_\delta \notin A_\alpha$, hence $f(x_\delta) \geq \alpha$. For $\delta = \frac{1}{n}$, choose $x_n \in B_{\frac{1}{n}}(x_0) \implies x_n \to x_0$, but $f(x_n) > \alpha > f(x_0)$. Thus,

$$\limsup_{n \to \infty} f(x_n) \geq \alpha > f(x_0)$$

which is a contradiction. $\square$

Example 6.13. If X is compact and $f : X \to \mathbb{R}$ is upper semi-continuous, then f attains its maximum.

Note that $X = \bigcup_{\alpha \in \mathbb{R}} \{x \in X : f(x) < \alpha\}$, but X is compact, hence $X = \bigcup_{i=1}^{k} \{x \in X : f(x) < \alpha_i\}$. For any $x \in X$, $f(x) < \alpha_i < \max\{\alpha_i\} = \alpha < \infty$. Hence f is bounded above. Next, f attains

its supremum on X. If not, then $f(x) < \sup f$ for all $x \in X$. For $n \in \mathbb{N}$, there exists $x_n \in X$ such that $\sup f - \frac{1}{n} < f(x_n)$ Now, $x_n \in X$, X is compact, hence $\exists$ subsequence $x_{n_k} \to x \in X$. But, then

$$\sup f \leq \limsup_{k \to \infty} f(x_{n_k}) \leq f(x)$$

Thus $\sup f \leq f(x)$, which is not possible, as it contradicts our assumption.

Similarly, one defines *lower semicontinuity*: a function $f \colon X \to \mathbb{R}$ is lower semicontinuous if $\{x \in X : f(x) > \alpha\}$ is open for every $\alpha \in \mathbb{R}$. Equivalently,

$$f(x) \leq \liminf_{n \to \infty} f(x_n)$$

whenever $x_n \to x$. Thus, f is continuous if and only if it is both lower semicontinuous and upper semicontinuous.

Remark 6.14. Note that if $f : X \to \mathbb{R}$ is *upper semi-continuous* (USC), then $f^{-1}\{(-\infty, \alpha)\}$ is open, and hence $f^{-1}\{[\beta, \alpha)\}$ is open if $\beta < \alpha$, but it does *not* imply that $f^{-1}\{(\beta, \alpha)\}$ is open for each $\alpha, B \in \mathbb{R}$, else f is continuous. (However, f is *Lebesgue measurable*!) But if f is both lower semi-continuous (LSC) and upper semi-continuous (USC), then

$$f^{-1}\{(\alpha, \beta)\} = f^{-1}\{(-\infty, \beta) \cap (\alpha, \infty)\}$$

is open, hence f is continuous.

Remark 6.15. There is no relation between lower semi-continuity and upper semi-continuity with left limit and right limit.

Example 6.16.
$$f(x) = \begin{cases} \sin \frac{1}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

is upper semi-continuous, but none of left limit and right limit exists at $x = 0$.

Exercise 6.1. Check for lower semi-continuity and upper semi-continuity for $f(x) = \lfloor x \rfloor$, the greatest integer function.

6.2.2   Weierstrass Approximation Theorem

We shall see that polynomials are dense in $(C[a,b], \|\cdot\|_\infty)$ if $b - a < \infty$. As a consequence, $C[a,b]$ is a separable space. The question of density of polynomials in $C[a,b]$ can be transferred to $C[0,1]$ with the help of the map:
$$f(t) = \frac{t-a}{b-a}$$

For $f \in C[0,1]$ and $n = 0, 1, 2, \ldots$, define (Bernstein polynomial):

$$B_n(f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}$$

Then $B_n(f)$ is a polynomial of degree at most n. Here, $B_n(f)$ is known as *Bernstein polynomial.* In fact, we have

$$B_n(f)(0) = f(0), \qquad B_n(f)(1) = f(1)$$

Let us denote $f_n(x) = x^n$ for $n = 0, 1, 2, \ldots.$ The following lemma, which is involved with combinatorics, is crucial in proving the density of $B_n(f)$ in $C[0, 1]$.

Lemma 6.17.

(i) $B_n(f_0) = f_0$ and $B_n(f_1) = f_1.$

(ii) $B_n(f_2) = \left(1 - \frac{1}{n}\right) f_2 + \frac{1}{n} f_1,$ hence $B_n(f_2) \to f_2$ uniformly.

(iii) $\sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^2 \binom{n}{k} x^k (1 - x)^{n-k} = \frac{x(1-x)}{n} \leq \frac{1}{4n}.$

(iv) Given $\delta > 0$, $0 \leq x \leq 1$, let F denote the set of $F = \left\{k \in \{0, 1, \ldots, n\} : \left|\frac{k}{n} - x\right| \geq \delta\right\}.$ Then

$$\sum_{k \in F} \binom{n}{k} x^k (1 - x)^{n-k} \leq \frac{1}{4n\delta^2}$$

Proof. (i) is trivial, as it follows from simple binomial expansions. *Hint.*

$$\sum_{k=0}^{n} \frac{k}{n} \binom{n}{k} x^k (1 - x)^{n-k} = x \sum_{k=1}^{n} \binom{n-1}{k-1} x^{k-1} (1 - x)^{n-k}$$

$$= x \sum_{j=0}^{n-1} \binom{n-1}{j} x^j (1 - x)^{(n-1)-j} = x[x + (1 - x)]^{n-1} = x$$

So $B_n(f_1) = f_1.$
(ii) To compute $B_n(f_2)$, we break the sum into two parts:

$$\left(\frac{k}{n}\right)^2 \binom{n}{k} = \frac{k}{n} \binom{n-1}{k-1} = \left(1 - \frac{1}{n}\right) \binom{n-2}{k-2} + \frac{1}{n} \binom{n-1}{k-1} \quad \text{for } k \geq 2$$

Thus,

$$B_n(f_2) = \left(1 - \frac{1}{n}\right) \sum_{k=2}^{n} \binom{n-2}{k-2} x^k (1 - x)^{n-k} + \frac{1}{n} \sum_{k=1}^{n} \binom{n-1}{k-1} x^k (1 - x)^{n-k}$$

$$= \left(1 - \frac{1}{n}\right) x^2 + \frac{1}{n} x \to f_2 \text{ uniformly}$$

(iii) Note that

$$\left(\frac{k}{n} - x\right)^2 = \left(\frac{k}{n}\right)^2 - 2x\frac{k}{n} + x^2$$

Hence,

$$\sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^2 \binom{n}{k} x^k (1 - x)^{n-k} = (1 - \frac{1}{n})x^2 + \frac{1}{n}x - 2x^2 + x^2 = \frac{x(1 - x)}{n} \leq \frac{1}{4n} \quad (\text{ by (ii)})$$

(iv) For $k \in F$, $1 \leq \frac{(\frac{k}{n} - x)^2}{\delta^2}$. Hence,

$$\sum_{k \in F} \binom{n}{k} x^k (1 - x)^{n-k} \leq \frac{1}{\delta^2} \sum_{k \in F} \left(\frac{k}{n} - x \right)^2 \binom{n}{k} x^k (1 - x)^{n-k}$$

$$\leq \frac{1}{\delta^2} \sum_{k=0}^{n} \left(\frac{k}{n} - x \right)^2 \binom{n}{k} x^k (1 - x)^{n-k} \leq \frac{1}{4n\delta^2}.$$

$\square$

Theorem 6.18 (Bernstein). *Let $f \in C[0,1]$. Then $B_n(f) \to f$ uniformly on $[0,1]$.*

Proof. Since f is uniformly continuous, for $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{2}$. Now,

$$|f(x) - B_n(f)(x)| = \left| \sum_{k=0}^{n} \left(f(x) - f\left(\frac{k}{n} \right) \right) \binom{n}{k} x^k (1 - x)^{n-k} \right|$$

$$\leq \sum_{k=0}^{n} \left| f(x) - f\left(\frac{k}{n} \right) \right| \binom{n}{k} x^k (1 - x)^{n-k},$$

Let us fix a n (to be specified soon). Let F denote the set of $k \in \{0, 1, \ldots, n\}$ such that $\left| \frac{k}{n} - x \right| \geq \delta$. Then

$$\left| f(x) - f\left(\frac{k}{n} \right) \right| < \frac{\varepsilon}{2} \quad \text{for } k \notin F,$$

and

$$\left| f(x) - f\left(\frac{k}{n} \right) \right| \leq 2\|f\|_\infty \quad \text{for } k \in F.$$

Thus,

$$|f(x) - B_n(f)(x)| \leq \frac{\varepsilon}{2} \sum_{k \notin F} \binom{n}{k} x^k (1 - x)^{n-k} + 2\|f\|_\infty \sum_{k \in F} \binom{n}{k} x^k (1 - x)^{n-k}$$

$$\leq \frac{\varepsilon}{2} \cdot 1 + 2\|f\|_\infty \left(\frac{1}{4n\delta^2} \right)$$

$$< \varepsilon$$

provided $n > \frac{\|f\|_\infty}{\varepsilon\delta^2}$. Therefore,

$$\|B_n(f) - f\|_\infty < \varepsilon \quad \text{whenever} \quad n > \frac{\|f\|_\infty}{\varepsilon\delta^2}.$$

$\square$

Exercise 6.2. If $f \in C[0,1]$ and $\int_0^1 x^n f(x)\, dx = 0$ for all $n \geq 0$, then $f = 0$.

6.3   Power series

Power series provide a canonical source of uniformly convergent series of functions and connect real analysis with elementary complex-analytic ideas. We record the basic convergence theory and the standard termwise operations.

Definition 6.19. A *power series* centered at $x_0 \in \mathbb{R}$ is a series of the form

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n, \tag{6.1}$$

where $(a_n)_{n \geq 0} \subset \mathbb{R}$ (or $\mathbb{C}$).

Theorem 6.20 (Radius of convergence). *For the power series (6.1) there exists $R \in [0, \infty]$ such that:*

(i) the series converges absolutely for all x with $|x - x_0| < R$;

(ii) the series diverges for all x with $|x - x_0| > R$.

The number R is called the radius of convergence. *Moreover,*

$$\frac{1}{R} = \limsup_{n \to \infty} |a_n|^{1/n} \tag{6.2}$$

(with the convention $1/\infty = 0$ and $1/0 = \infty$).

Proof. Set $\rho := \limsup_{n \to \infty} |a_n|^{1/n} \in [0, \infty]$ and define R by $1/R = \rho$. Fix x with $|x - x_0| < R$. Choose r with $|x - x_0| < r < R$. Then $\rho < 1/r$, hence there exists N such that $|a_n|^{1/n} \leq 1/r$ for all $n \geq N$. Thus $|a_n(x - x_0)^n| \leq (|x - x_0|/r)^n$ for $n \geq N$, and the tail is dominated by a geometric series; hence convergence is absolute.

If $|x - x_0| > R$, choose r with $R < r < |x - x_0|$. Then $\rho > 1/r$, hence for infinitely many n we have $|a_n|^{1/n} \geq 1/r$, that is $|a_n(x - x_0)^n| \geq (|x - x_0|/r)^n$, which does not tend to 0. Therefore the series diverges. $\qquad\square$

Theorem 6.21 (Uniform convergence on compact subintervals). *Let $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ have radius of convergence $R > 0$. Then for every $0 < r < R$ the series converges uniformly on the closed interval*

$$[x_0 - r, \ x_0 + r].$$

Proof. Fix $r \in (0, R)$. By the definition of R there exists $M > 0$ such that $|a_n|r^n \leq M$ for all n large enough. For $x \in [x_0 - r, x_0 + r]$ we have $|x - x_0| \leq r$, hence $|a_n(x - x_0)^n| \leq |a_n|r^n$. Since $\sum |a_n|r^n$ converges (as $r < R$), the Weierstrass M-test implies uniform convergence on $[x_0 - r, x_0 + r]$. $\qquad\square$

Theorem 6.22 (Termwise differentiation and integration). *Let $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ have radius of convergence $R > 0$, and define*

$$f(x) := \sum_{n=0}^{\infty} a_n(x - x_0)^n \qquad (|x - x_0| < R).$$

Then f is differentiable on $(x_0 - R, x_0 + R)$ and

$$f'(x) = \sum_{n=1}^{\infty} na_n(x - x_0)^{n-1}, \qquad |x - x_0| < R. \tag{6.3}$$

Moreover, for every x with $|x - x_0| < R$ we have

$$\int_{x_0}^{x} f(t)\,dt = \sum_{n=0}^{\infty} a_n \frac{(x - x_0)^{n+1}}{n + 1}. \tag{6.4}$$

Both derived series have the same radius of convergence R.

Proof. Fix $r \in (0, R)$ and work on the closed interval $I = [x_0 - r, x_0 + r]$. By theorem 6.21, the original series converges uniformly on I. The differentiated series in (6.3) also converges uniformly on I: indeed, since $\sum |a_n| r^n$ converges, the series $\sum n|a_n| r^{n-1}$ converges by a standard comparison (e.g. ratio test), and hence the M-test applies to $\sum na_n(x - x_0)^{n-1}$ on I. Let $S_N(x) = \sum_{n=0}^{N} a_n(x - x_0)^n$. Then S_N is differentiable and $S_N'(x) = \sum_{n=1}^{N} na_n(x - x_0)^{n-1}$. Uniform convergence of S_N' on I together with pointwise convergence of $S_N(x_0)$ implies that (S_N) converges uniformly to a differentiable limit f on I and that $f' = \lim S_N'$ on I. Since $r < R$ is arbitrary, the formula holds for all $|x - x_0| < R$. The integral formula (6.4) follows by integrating the partial sums and passing to the limit using uniform convergence. $\qquad \square$

Example 6.23. The exponential series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ has infinite radius of convergence and defines a C^{∞} function on $\mathbb{R}$. Moreover, it satisfies $f' = f$ and $f(0) = 1$, hence it coincides with the exponential function e^x.

Exercises

Exercise 6.3. Let (f_n) be a sequence of continuous functions on $[a, b]$ that converges uniformly to f. Prove that f is continuous.

Exercise 6.4. Give an example of pointwise convergence $f_n \to f$ on $[0, 1]$ where f is discontinuous.

Exercise 6.5. Prove that if $f_n \to f$ uniformly and each f_n is bounded, then $\sup |f_n - f| \to 0$.

Exercise 6.6. State and prove the Weierstrass M-test, and use it to prove uniform convergence of a power series inside its radius of convergence.

Exercise 6.7. Prove the Arzelà–Ascoli theorem for a uniformly bounded, equicontinuous family in $C([a, b])$.

Exercise 6.8. Let f_n be differentiable on $[a, b]$, assume $f_n(x_0)$ converges for some x_0 and f_n' converges uniformly. Prove that f_n converges uniformly and that $(\lim f_n)' = \lim f_n'$.

Exercise 6.9. Prove that every continuous function on $[0, 1]$ can be uniformly approximated by polynomials (Weierstrass approximation theorem).

Exercise 6.10. For a power series $\sum a_n(x - a)^n$ with radius $R > 0$, prove that it converges uniformly on $[a - r, a + r]$ for each $0 < r < R$.

Exercise 6.11. Compute the radius of convergence of $\sum_{n \geq 1} \frac{n!}{n^n}(x - 2)^n$.

Exercise 6.12. Let $\sum a_n x^n$ and $\sum b_n x^n$ have radii R_a and R_b. Prove that the Cauchy product has radius at least $\min(R_a, R_b)$.

Chapter 7

Differential Calculus in One Variable

This chapter develops the basic differential calculus of functions $f : \mathbb{R} \to \mathbb{R}$ (or $f : (a, b) \to \mathbb{R}$). After reviewing differentiability and the standard algebraic rules, we prove Rolle's theorem and the Mean Value Theorem and derive their classical consequences (monotonicity criteria and estimates). We then establish Taylor's theorem with remainder, which provides local polynomial approximation and is the main engine behind many asymptotic arguments in analysis.

Learning objectives.

- Treat differentiability as first-order approximation and use it to derive qualitative global information.

- Apply Rolle's theorem, the mean value theorems, and Taylor's theorem with full control of their hypotheses.

- Convert derivative information into monotonicity, Lipschitz estimates, and approximation results.

Learning objectives.

- Understand differentiability as a local linear approximation and relate it to continuity.

- Prove and use Rolle's theorem, the Mean Value Theorem, and Cauchy's Mean Value Theorem.

- Derive standard consequences: monotonicity, Lipschitz-type estimates, and convexity tests.

- State and prove Taylor's theorem with Lagrange's remainder and use it for approximation.

7.1 Differentiability

Section overview.

- Differentiability is stronger than continuity because it asserts the existence of a linear model at small scales.

- The goal of this section is to identify the exact approximation statement hidden behind the derivative.

- Later results in the chapter should be read as consequences of this linear approximation viewpoint.

Definition 7.1. Let $I \subseteq \mathbb{R}$ be an interval and let $f : I \to \mathbb{R}$. We say that f is *differentiable at* $x_0 \in I$ if the limit

$$f'(x_0) := \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \tag{7.1}$$

exists (finite). If f is differentiable at every $x \in I$, then f is *differentiable on I*.

Proposition 7.2. *If f is differentiable at x_0, then f is continuous at x_0.*

Proof. Assume (7.1) holds. Write

$$f(x_0 + h) - f(x_0) = h \cdot \frac{f(x_0 + h) - f(x_0)}{h}.$$

As $h \to 0$, the quotient tends to $f'(x_0)$, hence is bounded near 0. Therefore the product tends to 0, that is $f(x_0 + h) \to f(x_0)$. $\qquad\square$

Remark 7.3. Differentiability is precisely the existence of a first-order approximation: f is differentiable at x_0 if and only if there exists $m \in \mathbb{R}$ and a function $\eta(h)$ with $\eta(h) \to 0$ as $h \to 0$ such that

$$f(x_0 + h) = f(x_0) + mh + h\eta(h). \tag{7.2}$$

In that case $m = f'(x_0)$.

7.2 Mean value theorems

Theorem 7.4 (Rolle)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.*

Proof. By the extreme value theorem, f attains a maximum and a minimum on $[a, b]$. If both extrema occur at the endpoints, then f is constant and $f' \equiv 0$ on (a, b). Otherwise, at least one of the extrema occurs at some interior point $c \in (a, b)$. At an interior maximum or minimum, the difference quotients from the left and right have opposite signs, which forces $f'(c) = 0$. $\quad\square$

Theorem 7.5 (Mean Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{7.3}$$

Idea. Subtract the secant line joining $(a, f(a))$ and $(b, f(b))$. The resulting function has equal endpoint values, so Rolle's theorem turns the mean slope statement into a zero-derivative statement.

Proof. Consider the auxiliary function

$$g(x) := f(x) - \ell(x), \qquad \ell(x) := f(a) + \frac{f(b) - f(a)}{b - a}(x - a).$$

Figure 7.1. Geometric content of the Mean Value Theorem: some tangent slope equals the secant slope.

Then g is continuous on $[a, b]$, differentiable on (a, b), and $g(a) = g(b) = 0$. By Rolle's theorem theorem 7.4, there exists $c \in (a, b)$ with $g'(c) = 0$, that is $f'(c) = \ell'(c)$, which is exactly (7.3). $\square$
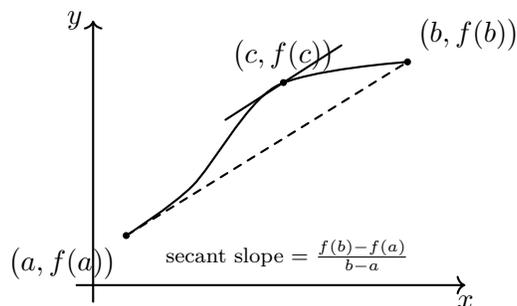
Theorem 7.6 (Cauchy Mean Value Theorem). *Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that*

$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c). \tag{7.4}$$

If moreover $g'(x) \neq 0$ on (a, b), then

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}. \tag{7.5}$$

Proof. Define
$$h(x) := (f(b) - f(a))\, g(x) - (g(b) - g(a))\, f(x).$$

Then h is continuous on $[a, b]$, differentiable on (a, b), and $h(a) = h(b)$. By Rolle's theorem theorem 7.4, there exists $c \in (a, b)$ such that $h'(c) = 0$, which is precisely (7.4). $\square$

7.3 Consequences of the MVT

Proposition 7.7 (Monotonicity test). *Let $f : (a, b) \to \mathbb{R}$ be differentiable.*

(i) *If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is nondecreasing on (a, b).*

(ii) *If $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing on (a, b).*

Proof. Fix $x < y$ in (a, b). Apply the Mean Value Theorem theorem 7.5 to f on $[x, y]$ to find $c \in (x, y)$ such that
$$f(y) - f(x) = f'(c)(y - x).$$

If $f' \geq 0$, then the right-hand side is ≥ 0, hence $f(y) \geq f(x)$. If $f' > 0$, then $f(y) > f(x)$. $\square$

Proposition 7.8 (Lipschitz estimate)**.** *Let* $f : [a, b] \to \mathbb{R}$ *be differentiable and suppose* $|f'(x)| \leq M$ *on* (a, b)*. Then for all* $x, y \in [a, b]$,

$$|f(x) - f(y)| \leq M|x - y|. \tag{7.6}$$

Proof. Assume $x < y$. By theorem 7.5, there exists $c \in (x, y)$ with $f(y) - f(x) = f'(c)(y - x)$, hence $|f(y) - f(x)| \leq M|y - x|$. $\qquad\square$

7.4 Taylor's theorem

Theorem 7.9 (Taylor with Lagrange remainder)**.** *Let* $n \in \mathbb{N}$ *and let* $f : [a, b] \to \mathbb{R}$ *be* $(n + 1)$ *times continuously differentiable on* $[a, b]$*. Fix* $x_0 \in [a, b]$*. Then for every* $x \in [a, b]$ *there exists* ξ *between* x *and* x_0 *such that*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(\xi)}{(n + 1)!}(x - x_0)^{n+1}. \tag{7.7}$$

Proof. Fix $x \neq x_0$ and define the degree-n Taylor polynomial at x_0:

$$P_n(t) := \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(t - x_0)^k.$$

Consider the function

$$\Phi(t) := f(t) - P_n(t) - \lambda(t - x_0)^{n+1},$$

where λ is chosen so that $\Phi(x) = 0$. Since $\Phi(x_0) = 0$ as well, repeated application of Rolle's theorem yields a point ξ between x and x_0 with $\Phi^{(n+1)}(\xi) = 0$. But $P_n^{(n+1)} \equiv 0$ and $\frac{d^{n+1}}{dt^{n+1}}(t - x_0)^{n+1} = (n + 1)!$, hence

$$0 = \Phi^{(n+1)}(\xi) = f^{(n+1)}(\xi) - \lambda(n + 1)!,$$

so $\lambda = \frac{f^{(n+1)}(\xi)}{(n+1)!}$. Substituting the definition of λ (from $\Phi(x) = 0$) gives (7.7). $\qquad\square$

Corollary 7.10. *Under the assumptions of theorem 7.9, if* $|f^{(n+1)}(t)| \leq M$ *on* $[a, b]$*, then*

$$\left| f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \right| \leq \frac{M}{(n + 1)!}|x - x_0|^{n+1}.$$

Exercises

Exercise 7.1. Prove Rolle's theorem and deduce the Mean Value Theorem.

Exercise 7.2. Let f be differentiable on (a, b) and continuous on $[a, b]$. Show that if $f' = 0$ on (a, b) then f is constant on $[a, b]$.

Exercise 7.3. Show that if f' exists and is bounded on (a, b), then f is Lipschitz on $[a, b]$.

Exercise 7.4. Prove that if $f' > 0$ on an interval then f is strictly increasing there.

Exercise 7.5. Let f be twice differentiable on (a, b). Prove that $f'' \geq 0$ implies f is convex.

Exercise 7.6. Prove Taylor's theorem with Lagrange remainder and use it to derive $\sin x < x$ for $x > 0$.

Exercise 7.7. Show that if $\sum a_n$ converges absolutely, then the series $\sum a_n x^n$ defines a differentiable function on $(-1, 1)$ with termwise differentiation.

Exercise 7.8. Give an example of a function differentiable everywhere whose derivative is unbounded on every interval.

Exercise 7.9. Prove that if f is differentiable at a, then f is continuous at a.

Exercise 7.10. Suppose f is differentiable on (a, b) and f' is increasing. Prove that f is convex.

Chapter 8

Differential Calculus in Several Variables

This chapter develops the basic analytic framework for functions on $\mathbb{R}^n$. We begin with limits and continuity, phrased in the language of the Euclidean metric. We then introduce partial and directional derivatives and, more importantly, the correct notion of differentiability as a linear approximation (Fréchet differentiability). The chain rule is stated and proved in forms suited to compositions and coordinate changes, and we discuss higher derivatives and mixed partials. We next establish Taylor's theorem as a systematic higher-order approximation scheme. The chapter culminates in the inverse and implicit function theorems, which provide precise criteria for local invertibility and for representing level sets as graphs.

Learning objectives.

- Understand differentiability in several variables as a linear approximation, not merely as the existence of partial derivatives.

- Use the Jacobian, Hessian, and Taylor expansion to analyze local geometry and local extrema.

- Master the conceptual content of the inverse and implicit function theorems as local structure theorems.

8.1 Syllabus map

This chapter develops the theory of functions of several variables in a self-contained manner. The central viewpoint is that differentiability is *linear approximation with a remainder that is small relative to $\|h\|$*. The chapter develops this idea in a sequence that mirrors the syllabus:

(1) **Limits and continuity in $\mathbb{R}^n$:** the ε–δ definition and its sequential characterization, together with basic permanence properties.

(2) **Differentiability and the Fréchet derivative:** partial and directional derivatives, differentiability as the existence of a best linear approximation, and the chain rule.

(3) **Taylor's theorem and quantitative estimates:** higher-order expansions and remainder bounds that control approximation errors.

(4) **Local structure of smooth maps:** the inverse function theorem and implicit function theorem, with applications to local parametrizations and constraint manifolds.

8.2 Limits and continuity

8.2.1 Notation and basic definitions in Euclidean space

Throughout this chapter we work in $\mathbb{R}^n$ with the standard inner product

$$\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{j=1}^{n} x_j y_j, \qquad \mathbf{x} = (x_1, \ldots, x_n),\ \mathbf{y} = (y_1, \ldots, y_n),$$

and the associated Euclidean norm

$$\|\mathbf{x}\| := \langle \mathbf{x}, \mathbf{x} \rangle^{1/2} = (x_1^2 + \cdots + x_n^2)^{1/2}. \tag{8.1}$$

We will use (8.1) as the default norm on $\mathbb{R}^n$ (and on $\mathbb{R}^m$ when needed). The Euclidean distance is

$$d(\mathbf{x}, \mathbf{y}) := \|\mathbf{x} - \mathbf{y}\|.$$

For $\mathbf{x} \in \mathbb{R}^n$ and $r > 0$, the open ball of radius r centered at $\mathbf{x}$ is

$$B_r(\mathbf{x}) := \{\mathbf{y} \in \mathbb{R}^n : \|\mathbf{y} - \mathbf{x}\| < r\}.$$

Conventions. Vectors are written in boldface, and $\mathbf{x}_k \to \mathbf{x}$ always means $\|\mathbf{x}_k - \mathbf{x}\| \to 0$. When the meaning is clear from context we may suppress boldface.

We record two inequalities that will be used repeatedly.

Theorem 8.1 (Cauchy–Schwarz inequality)**.** *For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,*

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\|\,\|\mathbf{y}\|.$$

Moreover, equality holds if and only if $\mathbf{x}$ and $\mathbf{y}$ are linearly dependent.

Proof. If $\mathbf{y} = \mathbf{0}$ the claim is trivial. Otherwise, consider the quadratic polynomial

$$\begin{aligned}
p(t) :&= \|\mathbf{x} - t\mathbf{y}\|^2 \\
&= \langle \mathbf{x} - t\mathbf{y}, \mathbf{x} - t\mathbf{y} \rangle \\
&= \|\mathbf{x}\|^2 - 2t\langle \mathbf{x}, \mathbf{y} \rangle + t^2 \|\mathbf{y}\|^2.
\end{aligned}$$

Since $p(t) \geq 0$ for all $t \in \mathbb{R}$, its discriminant is non-positive:

$$4\langle \mathbf{x}, \mathbf{y} \rangle^2 - 4\|\mathbf{x}\|^2 \|\mathbf{y}\|^2 \leq 0.$$

This is exactly $|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\|\,\|\mathbf{y}\|$. Equality holds if and only if $p(t)$ has a real root, i.e. $\mathbf{x} - t\mathbf{y} = \mathbf{0}$ for some t. $\qquad\square$

Proposition 8.2 (Triangle inequality)**.** *For all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|.$$

Proof. By expanding the square and applying Cauchy–Schwarz,

$$\|\mathbf{x} + \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\langle \mathbf{x}, \mathbf{y} \rangle$$
$$\leq (\|\mathbf{x}\| + \|\mathbf{y}\|)^2.$$

Taking square roots gives the result. □

Theorem 8.3 (Bolzano–Weierstrass in $\mathbb{R}^n$)**.** *Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*

Proof. Let $\mathbf{x}_k = (x_1^k, \ldots, x_n^k)$ be bounded. Then each coordinate sequence $(x_j^k)_{k \geq 1}$ is bounded in $\mathbb{R}$. Apply the one-dimensional Bolzano–Weierstrass theorem to (x_1^k) to obtain a subsequence along which the first coordinate converges. Restrict to that subsequence and repeat for the second coordinate, and so on. After n steps we obtain a subsequence $(\mathbf{x}_{k_\ell})$ such that each coordinate converges; hence $\mathbf{x}_{k_\ell} \to \mathbf{x} \in \mathbb{R}^n$. □

8.2.2 Limits in Euclidean space

The definition of a limit in $\mathbb{R}^n$ uses the Euclidean distance.

Definition. Let $D \subseteq \mathbb{R}^n$ and let $f : D \to \mathbb{R}^m$. Given $x_0 \in \mathbb{R}^n$ that is a limit point of D, we write

$$\lim_{x \to x_0} f(x) = L$$

if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$0 < \|x - x_0\| < \delta \ \text{ and } \ x \in D \quad \Longrightarrow \quad \|f(x) - L\| < \varepsilon.$$

Sequential criterion. The limit equals L if and only if for every sequence $(x_k) \subset D$ with $x_k \to x_0$ and $x_k \neq x_0$, one has $f(x_k) \to L$. We will freely use either formulation, depending on convenience.

Paths and polar coordinates in $\mathbb{R}^2$. When $n = 2$ and $x_0 = (0,0)$, a common way to *disprove* the existence of a limit is to find two paths approaching $(0,0)$ along which f approaches different values. A common way to *compute* limits is to use polar coordinates $x = r\cos\theta$, $y = r\sin\theta$; then $(x,y) \to (0,0)$ is equivalent to $r \to 0^+$. In many examples, showing that $\lim_{r \to 0^+} f(r\cos\theta, r\sin\theta)$ exists and is independent of θ suffices.

A caution about the path test. Agreement along a few special curves never *proves* the existence of a multivariable limit; it only rules out some obvious obstructions. To establish existence, one needs an estimate that works for *all* nearby points—for example, a bound in terms

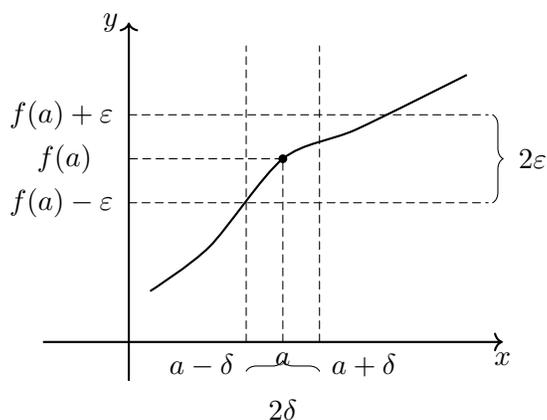of $\sqrt{x^2 + y^2}$, or a representation in polar coordinates whose absolute value tends to 0 uniformly in θ.

8.2.3 Continuity in Euclidean space

Let $D \subseteq \mathbb{R}^n$ and $f : D \to \mathbb{R}^m$.

Definition 8.4. The function f is *continuous at $x_0 \in D$* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\|x - x_0\| < \delta \ \text{ and } \ x \in D \quad \Longrightarrow \quad \|f(x) - f(x_0)\| < \varepsilon.$$

Equivalently, f is continuous at x_0 if and only if $\lim_{x \to x_0} f(x) = f(x_0)$.

Figure 8.1. Geometric interpretation of the ε–δ definition of continuity at a.

Sequential criterion. The function f is continuous at x_0 if and only if for every sequence $(x_k) \subset D$ with $x_k \to x_0$ one has $f(x_k) \to f(x_0)$.

Componentwise continuity. If $f = (f_1, \ldots, f_m) : D \to \mathbb{R}^m$, then f is continuous at x_0 if and only if each component f_i is continuous at x_0. Indeed, $\|f(x) - f(x_0)\| < \varepsilon$ implies $|f_i(x) - f_i(x_0)| \le \|f(x) - f(x_0)\| < \varepsilon$, while conversely one can estimate

$$\|f(x) - f(x_0)\|^2 = \sum_{i=1}^{m} |f_i(x) - f_i(x_0)|^2.$$

Negation of continuity. The function f fails to be continuous at x_0 if and only if there exists $\varepsilon_0 > 0$ such that for every $\delta > 0$ there exists $x \in D$ with $\|x - x_0\| < \delta$ but $\|f(x) - f(x_0)\| \ge \varepsilon_0$.

Example 8.5. Define

$$f(x, y) = \begin{cases} 1, & xy \ne 0, \\ 0, & xy = 0. \end{cases}$$

Then $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist, because along the path $y = 0$ the function is identically 0, whereas along the path $y = x$ it is identically 1 for $x \neq 0$. By contrast, if one defines

$$g(x,y) = \begin{cases} 1, & xy = 1, \\ 0, & xy \neq 1, \end{cases}$$

then $\lim_{(x,y)\to(0,0)} g(x,y) = 0$, since the condition $xy = 1$ cannot hold near $(0,0)$.

Exercise 8.1. Let $f : \mathbb{R}^2 \to \mathbb{R}$, check the continuity of f at $(0,0)$.

1. $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}} & \text{if } x^2 + y^2 \neq 0 \\ 0 & \text{otherwise} \end{cases}$

2. $f(x,y) = \dfrac{\sin^2(x - y)}{\sqrt{x^2 + y^2}}, \qquad f(0,0) = 0.$

3. $f(x,y) = \begin{cases} \dfrac{x^2 y}{x^2 + y} & \text{if } x^2 + y \neq 0 \\ 0 & \text{otherwise} \end{cases}$

4. $f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & \text{if } x^4 + y^2 \neq 0 \\ 0 & \text{otherwise} \end{cases}$

5. $f(x,y) = \begin{cases} \dfrac{\sin xy}{xy} & \text{if } xy \neq 0 \\ 0 & \text{otherwise} \end{cases}$

Remark 8.6 (A typical ε–δ negation). Consider the function

$$f(x,y) := \begin{cases} \dfrac{xy}{x^2 + y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

Along the diagonal $y = x$ we have $f(x,x) = \frac{1}{2}$ for all $x \neq 0$. Hence, for $\varepsilon_0 := \frac{1}{4}$ and for every
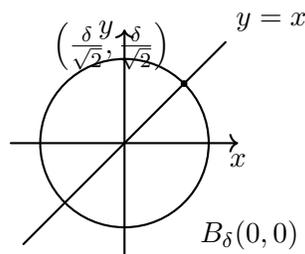
Figure 8.2. The diagonal path $y = x$ intersects every ball $B_\delta(0,0)$, which forces the negation estimate in Remark 8.6.

$\delta > 0$, the point

$$(x_\delta, y_\delta) := \left(\frac{\delta}{\sqrt{2}}, \frac{\delta}{\sqrt{2}} \right)$$

satisfies $\sqrt{x_\delta^2 + y_\delta^2} = \delta$ but

$$|f(x_\delta, y_\delta) - f(0,0)| = \frac{1}{2} \geq \varepsilon_0.$$

Therefore f is not continuous at $(0,0)$, and consequently $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Proposition 8.7 (Composition preserves continuity). *Let $D \subseteq \mathbb{R}^n$ and let $f\colon D \to \mathbb{R}^m$ be continuous at $x_0 \in D$. Let $I \subseteq \mathbb{R}^m$ and let $g\colon I \to \mathbb{R}^k$ be continuous at $f(x_0)$, and assume that $f(D) \subseteq I$. Then the composition $g \circ f\colon D \to \mathbb{R}^k$ is continuous at x_0.*

Proof. Fix $\varepsilon > 0$. Since g is continuous at $f(x_0)$, there exists $\eta > 0$ such that

$$\|u - f(x_0)\| < \eta \implies \|g(u) - g(f(x_0))\| < \varepsilon, \qquad u \in I. \tag{8.2}$$

Since f is continuous at x_0, there exists $\delta > 0$ such that

$$\|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \eta, \qquad x \in D. \tag{8.3}$$

Combining (8.3) with (8.2) yields

$$\|x - x_0\| < \delta \implies \|g(f(x)) - g(f(x_0))\| < \varepsilon.$$

Hence $g \circ f$ is continuous at x_0. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Remark 8.8 (Sequential proof). Alternatively, if $x_n \to x_0$ in D, then $f(x_n) \to f(x_0)$ by continuity of f, and hence $g(f(x_n)) \to g(f(x_0))$ by continuity of g.

Example 8.9.

$$f(x,y) = \begin{cases} \dfrac{\sin xy}{xy} & \text{if } xy \neq 0 \\ 1 & \text{otherwise} \end{cases}$$

$$f(x,y) = p \circ g(x,y), \quad \text{where } p(t) = \begin{cases} \dfrac{\sin t}{t} & t \neq 0 \\ 1 & t = 0 \end{cases}$$

8.3   Differentiation in $\mathbb{R}^n$

Section overview.

- Partial derivatives are useful coordinates, but differentiability is fundamentally the existence of a single linear map approximating the function.

- This section emphasizes how the Jacobian packages the first-order behavior of a multivariable map.

- Watch carefully which statements require only directional information and which require a genuinely uniform linear approximation.

8.3.1 Partial derivatives

Let $D \subseteq \mathbb{R}^2$ be open and $f : D \to \mathbb{R}$. Fix $(x_0, y_0) \in D$.

$$\frac{\partial f}{\partial x}(x_0, y_0) := \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}, \tag{8.4}$$

provided the limit exists. Equation (8.4) is the defining limit for the partial derivative in the x-direction. Similarly,

$$\frac{\partial f}{\partial y}(x_0, y_0) := \lim_{k \to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k},$$

when the limit exists. We also use the notation $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$.

First-order expansion along coordinate lines. If $f_x(x_0, y_0)$ exists, then

$$f(x_0 + h, y_0) = f(x_0, y_0) + h \, f_x(x_0, y_0) + o(h) \qquad (h \to 0),$$

and analogously for f_y. The important point is that the existence of $\partial f / \partial x$ depends only on the values of f along the line segment $\{(x_0 + h, y_0) : |h| \text{ small}\}$, not on the behavior of f in two dimensions.
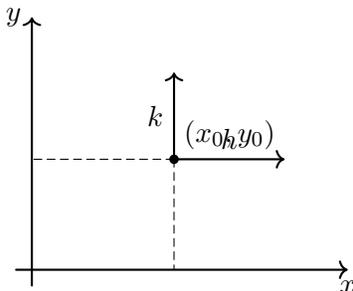
Figure 8.3. Partial derivatives measure change along coordinate directions: f_x varies x while holding $y = y_0$ fixed (horizontal arrow), and f_y varies y while holding $x = x_0$ fixed (vertical arrow).

Warning. The existence of partial derivatives at a point does *not* imply continuity or differentiability at that point.

Example 8.10. $f(x, y) = \dfrac{xy}{x^2 + y^2}$, $f(0, 0) = 0$. Then $f_x(0, 0) = 0 = f_y(0, 0)$ but f is not continuous at $(0, 0)$.

8.3.2 Directional derivatives

The directional derivative measures the instantaneous rate of change of f along a prescribed direction.

Let $D \subseteq \mathbb{R}^n$ be open and $f : D \to \mathbb{R}$. Fix $x_0 \in D$ and a unit vector $\mathbf{v} \in \mathbb{R}^n$ (i.e. $\|\mathbf{v}\| = 1$). The *directional derivative* of f at x_0 in the direction $\mathbf{v}$ is

$$D_{\mathbf{v}}f(x_0) := \lim_{t \to 0} \frac{f(x_0 + t\mathbf{v}) - f(x_0)}{t}, \tag{8.5}$$

provided the limit exists. As in (8.5), only the values of f along the line segment $\{x_0 + t\mathbf{v} : |t| \text{ small}\}$ matter.
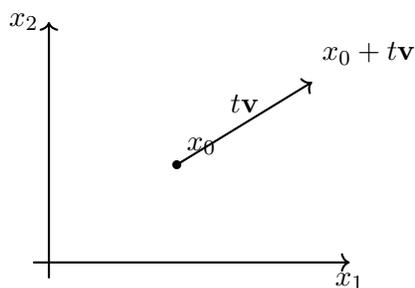
Figure 8.4. Directional derivatives probe f along the line $t \mapsto x_0 + t\mathbf{v}$. Only the behavior on this one-dimensional slice is relevant.

Example 8.11. Define

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Then every directional derivative $D_{\mathbf{v}}f(0,0)$ exists at $(0,0)$. Indeed, for $\mathbf{v} = (v_1, v_2)$ with $\|\mathbf{v}\| = 1$,

$$\frac{f(t\mathbf{v}) - f(0,0)}{t} = \frac{v_1^2 v_2}{t^2 v_1^4 + v_2^2} \xrightarrow{t \to 0} \begin{cases} 0, & v_2 = 0, \\ \dfrac{v_1^2}{v_2}, & v_2 \neq 0. \end{cases}$$
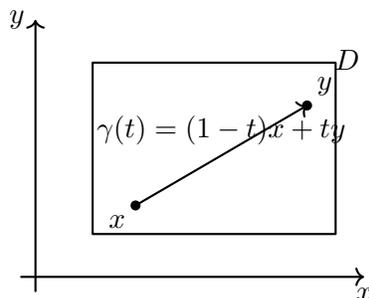
Figure 8.5. On a convex set D, the straight-line path from x to y stays inside D, so the one-variable mean value theorem applied to $t \mapsto f(\gamma(t))$ yields constancy when $\nabla f \equiv 0$.

8.3.3 Differentiability

Partial and directional derivatives capture *one-dimensional* behavior of f along lines. The correct multivariable notion is differentiability as *linear approximation.*

Let $D \subseteq \mathbb{R}^n$ be open and let $f : D \to \mathbb{R}^m$. Fix $x_0 \in D$ and write $H \in \mathbb{R}^n$ for an increment.

Definition 8.12 (Fréchet differentiability)**.** We say that f is *differentiable at x_0* if there exists a linear map $A : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$\frac{\|f(x_0 + H) - f(x_0) - AH\|}{\|H\|} \longrightarrow 0 \qquad \text{as } \|H\| \to 0. \tag{8.6}$$

The linear map A is unique; it is called the *derivative* of f at x_0 and is denoted by $Df(x_0)$ or $f'(x_0)$.

Uniqueness of the derivative. Suppose A and B both satisfy (8.6). Then

$$\frac{\|(A - B)H\|}{\|H\|} \leq \frac{\|f(x_0 + H) - f(x_0) - AH\|}{\|H\|} + \frac{\|f(x_0 + H) - f(x_0) - BH\|}{\|H\|} \longrightarrow 0.$$

Taking $H = t\mathbf{v}$ with $\mathbf{v} \neq \mathbf{0}$ and letting $t \to 0$ shows $\|(A - B)\mathbf{v}\| = 0$ for every $\mathbf{v}$, hence $A = B$. $\square$

Consequences. If f is differentiable at x_0, then f is continuous at x_0, and every directional derivative exists with

$$D_{\mathbf{v}} f(x_0) = f'(x_0)\mathbf{v} \qquad (\mathbf{v} \in \mathbb{R}^n, \ \|\mathbf{v}\| = 1).$$

In particular, for scalar-valued $f : D \to \mathbb{R}$ the derivative $f'(x_0)$ can be identified with the gradient row vector $\nabla f(x_0)$, and $f'(x_0)\mathbf{v} = \nabla f(x_0) \cdot \mathbf{v}$.

Example 8.13 (Bounded partial derivatives imply continuity)**.** Let $D \subseteq \mathbb{R}^2$ be open and let $f : D \to \mathbb{R}$ have bounded partial derivatives f_x and f_y on D. Then f is (locally) Lipschitz on D, hence continuous on D.

Proof. Fix $(x_0, y_0) \in D$ and consider $(x_0 + h, y_0 + k)$ close enough to (x_0, y_0) so that the rectangle with corners (x_0, y_0) and $(x_0 + h, y_0 + k)$ lies in D. Using the one-variable mean value theorem along the two coordinate segments,

$$f(x_0 + h, y_0 + k) - f(x_0, y_0) = (f(x_0 + h, y_0 + k) - f(x_0, y_0 + k)) + (f(x_0, y_0 + k) - f(x_0, y_0)),$$

and hence

$$|f(x_0 + h, y_0 + k) - f(x_0, y_0)| \leq |h| \sup_D |f_x| + |k| \sup_D |f_y|$$

$$\leq \sqrt{h^2 + k^2} \sqrt{\left(\sup_D |f_x|\right)^2 + \left(\sup_D |f_y|\right)^2}.$$

Letting $(h, k) \to (0, 0)$ gives continuity at (x_0, y_0). $\square$

Exercise 8.2. Even if $f_x(x_0)$ and $f_y(x_0)$ exist, f need not be differentiable at x_0. Find an explicit example (or verify one of the examples below).

Example 8.14 (Directional derivatives may all exist without differentiability)**.** Define

$$f(x, y) = \begin{cases} \dfrac{y}{|y|} \sqrt{x^2 + y^2}, & y \neq 0, \\ 0, & y = 0. \end{cases}$$

Then f is continuous at $(0,0)$ and every directional derivative $D_{\mathbf{v}}f(0,0)$ exists. However, f is not differentiable at $(0,0)$ since the map $\mathbf{v} \mapsto D_{\mathbf{v}}f(0,0)$ is not linear.

Proof. For $\mathbf{v} = (v_1, v_2)$ with $\|\mathbf{v}\| = 1$,

$$\frac{f(t\mathbf{v}) - f(0,0)}{t} = \begin{cases} 1, & v_2 > 0, \\ -1, & v_2 < 0, \\ 0, & v_2 = 0, \end{cases} \qquad (t \downarrow 0).$$

Thus $D_{\mathbf{v}}f(0,0)$ exists for every $\mathbf{v}$. If f were differentiable at $(0,0)$, then $D_{\mathbf{v}}f(0,0) = Df(0,0) \cdot \mathbf{v}$ would depend linearly on $\mathbf{v}$, which is impossible. $\qquad\square$

Theorem 8.15 (A sufficient condition for differentiability)**.** *Let $D \subseteq \mathbb{R}^2$ be an open set, and let $(x_0, y_0) \in D$. Suppose that the partial derivatives f_x and f_y exist in a neighborhood of (x_0, y_0) and are continuous at (x_0, y_0). Then f is differentiable at (x_0, y_0). Moreover,*

$$f'(x_0, y_0) = \big(f_x(x_0, y_0),\ f_y(x_0, y_0)\big).$$

Proof strategy. Decompose the increment of f into a horizontal step and a vertical step, then apply the one-variable mean value theorem to each leg. Continuity of the partial derivatives lets the coefficients in this decomposition converge to the values of the Jacobian at the base point.

Proof. Write $X_0 = (x_0, y_0)$ and let $H = (h, k)$. Since D is open, there exists $r > 0$ such that the closed ball $\overline{B(X_0, r)}$ is contained in D. We may therefore assume that $\|H\| < r$, so that all points used below lie in D.

Decompose the increment as

$$f(x_0 + h, y_0 + k) - f(x_0, y_0) = \big[f(x_0 + h, y_0 + k) - f(x_0, y_0 + k)\big]$$
$$+ \big[f(x_0, y_0 + k) - f(x_0, y_0)\big].$$

By the one-variable mean value theorem, there exist $\theta_i \in (0, 1)$; $i = 1, 2$, such that

$$f(x_0 + h, y_0 + k) - f(x_0, y_0) = f_x(x_0 + \theta_1 h, y_0 + k)\, h + f_y(x_0, y_0 + \theta_2 k)\, k.$$

Subtracting $f_x(x_0, y_0)h + f_y(x_0, y_0)k$, we get

$$f(x_0 + h, y_0 + k) - f(x_0, y_0) - f_x(x_0, y_0)h - f_y(x_0, y_0)k$$
$$= [f_x(x_0 + \theta_1 h, y_0 + k) - f_x(x_0, y_0)]h$$
$$+ [f_y(x_0, y_0 + \theta_2 k) - f_y(x_0, y_0)]k.$$

Now let $\epsilon > 0$. Since f_x and f_y are continuous at (x_0, y_0), there exists $\delta > 0$ such that

$$\|(u, v) - (x_0, y_0)\| < \delta \quad \Longrightarrow \quad |f_x(u, v) - f_x(x_0, y_0)| < \epsilon, \quad |f_y(u, v) - f_y(x_0, y_0)| < \epsilon.$$

If $\|H\| < \delta$, then both points $(x_0 + \theta_1 h, y_0 + k)$ and $(x_0, y_0 + \theta_2 k)$ lie within distance at most $\|H\|$ of (x_0, y_0). Therefore

$$\left| f(x_0 + h, y_0 + k) - f(x_0, y_0) - f_x(x_0, y_0)h - f_y(x_0, y_0)k \right|$$
$$\leq \epsilon|h| + \epsilon|k| \leq \sqrt{2}\, \epsilon\, \|(h, k)\|.$$

Thus

$$\frac{\left| f(X_0 + H) - f(X_0) - \nabla f(X_0) \cdot H \right|}{\|H\|} \leq \sqrt{2}\, \epsilon$$

whenever $0 < \|H\| < \delta$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Exercise 8.3. Prove that

$$f(x, y) = \begin{cases} (x^2 + y^2) \sin\left(\dfrac{1}{x^2 + y^2}\right), & x^2 + y^2 \neq 0, \\ 0, & (x, y) = (0, 0), \end{cases}$$

is differentiable at $(0, 0)$ and $f'(0, 0) = \mathbf{0}$, but neither f_x nor f_y is continuous at $(0, 0)$.

Geometric interpretation (scalar case). If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at X_0, then the graph of f has a tangent hyperplane at $(X_0, f(X_0))$ given by

$$z = f(X_0) + f'(X_0)(X - X_0) = f(X_0) + \nabla f(X_0) \cdot (X - X_0).$$

For $n = 1$ this is the familiar tangent line; for $n = 2$ it is a tangent plane. In general, $f'(X_0)$ is the best linear approximation to f near X_0.

Proposition 8.16 (Tangent line, tangent plane, and tangent hyperplane)**.** *If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at X_0, then*

$$f(X_0 + H) = f(X_0) + f'(X_0)H + o(\|H\|),$$

so the graph of f near $(X_0, f(X_0))$ is approximated by the affine hyperplane

$$z = f(X_0) + f'(X_0)(X - X_0).$$

For $n = 1$ this reduces to the tangent line

$$z = f(x_0) + f'(x_0)(x - x_0),$$

and for $n = 2$ it becomes the tangent plane

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Proof. The differentiability relation

$$f(X_0 + H) - f(X_0) - f'(X_0)\, H = o(\|H\|)$$

says precisely that the affine function

$$L(X) := f(X_0) + f'(X_0)\, (X - X_0)$$

approximates $f(X)$ with an error negligible compared to $\|X - X_0\|$. Thus L is the tangent hyperplane to the graph at $(X_0, f(X_0))$. The formulas for $n = 1$ and $n = 2$ are the corresponding coordinate expressions. $\qquad\square$

8.3.4 Chain rule

One-variable chain rule. Let $I, J \subseteq \mathbb{R}$ be intervals, $g : I \to J$, and $f : J \to \mathbb{R}$. If g is differentiable at $x \in I$ and f is differentiable at $g(x) \in J$, then $f \circ g$ is differentiable at x and

$$(f \circ g)'(x) = f'(g(x))\, g'(x).$$
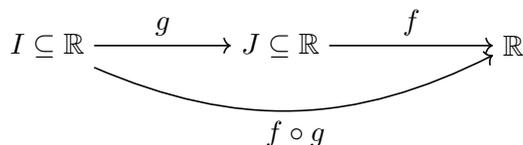
Figure 8.6. The chain rule differentiates the composition $f \circ g$ by combining the derivatives of g and f at the appropriate points.

Proof. Set $y = g(x)$ and $k = g(x + h) - g(x)$. Since f is differentiable at y, there exists a function η with $\eta(k) \to 0$ as $k \to 0$ such that

$$f(y + k) - f(y) - f'(y)k = k\,\eta(k).$$

Since g is differentiable at x, we may write $k = hg'(x) + h\mu(h)$ with $\mu(h) \to 0$ as $h \to 0$. Consider the error term

$$\varepsilon(h) := \frac{f(g(x + h)) - f(g(x)) - f'(g(x))\, g'(x)\, h}{h}. \tag{8.7}$$

We will show that $\varepsilon(h) \to 0$ as $h \to 0$; this is exactly the statement that the difference quotient in (8.7) tends to $f'(g(x))g'(x)$. Substituting the expansions above and simplifying gives

$$\varepsilon(h) = \eta(k)(g'(x) + \mu(h)) + f'(y)\mu(h).$$

As $h \to 0$ we have $k \to 0$, hence $\eta(k) \to 0$; also $\mu(h) \to 0$. Therefore $\varepsilon(h) \to 0$, which means that the difference quotient tends to $f'(g(x))g'(x)$. $\qquad\square$

Notation 8.17 (Matrix spaces and the operator norm). Let $L_n(\mathbb{R})$ denote the space of linear maps $\mathbb{R}^n \to \mathbb{R}^n$ (equivalently, real $n \times n$ matrices), and let $GL_n(\mathbb{R}) \subset L_n(\mathbb{R})$ be the group of invertible matrices.

For a linear map $A : \mathbb{R}^n \to \mathbb{R}^m$ we use the (subordinate) *operator norm*

$$\|A\| := \sup_{\|x\|=1} \|Ax\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}.$$

It satisfies the basic estimates

$$\|Ax\| \leq \|A\|\,\|x\|, \qquad \|AB\| \leq \|A\|\,\|B\|.$$

Examples.

(1) If $A : \mathbb{R}^2 \to \mathbb{R}$ is given by $A(x,y) = 4x + 3y$, then by Cauchy–Schwarz, $|A(x,y)| \leq \sqrt{4^2 + 3^2}\,\sqrt{x^2 + y^2} = 5\|(x,y)\|$, and equality is attained at $(x,y) = (4,3)/5$. Hence $\|A\| = 5$.

(2) If $A : \mathbb{R}^2 \to \mathbb{R}^2$ is given by $A(x,y) = (3x, 4y)$, then $\|A(x,y)\|^2 = 9x^2 + 16y^2 \leq 16(x^2 + y^2)$, with equality when $x = 0$. Hence $\|A\| = 4$.

Since $L_n(\mathbb{R})$ is finite-dimensional, all norms on $L_n(\mathbb{R})$ are equivalent, and the choice affects only constants, not topological statements.

Proposition 8.18 (Existence of the operator norm)**.** *Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be linear. Then the quantity*

$$\|A\| = \sup_{\|x\|=1} \|Ax\|$$

is finite, and in fact the supremum is attained:

$$\|A\| = \max_{\|x\|=1} \|Ax\|.$$

Equivalently,

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|},$$

and therefore

$$\|Ax\| \leq \|A\|\,\|x\| \qquad \text{for all } x \in \mathbb{R}^n.$$

Proof. Write $A = (R_1, \ldots, R_m)^t$, where $R_i \in \mathbb{R}^n$ is the ith row of the matrix of A. For any $x \in \mathbb{R}^n$, Cauchy–Schwarz gives

$$|(Ax)_i| = |R_i \cdot x| \leq \|R_i\| \, \|x\|, \qquad i = 1, \ldots, m.$$

Hence

$$\|Ax\|^2 = \sum_{i=1}^m |(Ax)_i|^2 \leq \left(\sum_{i=1}^m \|R_i\|^2 \right) \|x\|^2,$$

so $\|Ax\|$ is bounded on the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$. The map $x \mapsto \|Ax\|$ is continuous, and S^{n-1} is compact; therefore a maximum is attained on S^{n-1}.

If $x \neq 0$, write $x = \|x\| u$ with $\|u\| = 1$. Then

$$\frac{\|Ax\|}{\|x\|} = \|Au\| \leq \max_{\|v\|=1} \|Av\| = \|A\|.$$

Taking the supremum over $x \neq 0$ gives

$$\sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} \leq \|A\|.$$

The reverse inequality follows by restricting the supremum to vectors with $\|x\| = 1$, so the two formulas agree. The estimate $\|Ax\| \leq \|A\| \, \|x\|$ is immediate. $\qquad \square$

Chain rule in $\mathbb{R}^n$. Let $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ be open. If $g : U \to V$ is differentiable at $x \in U$ and $f : V \to \mathbb{R}^k$ is differentiable at $g(x)$, then $f \circ g$ is differentiable at x and

$$(f \circ g)'(x) = f'(g(x)) \, g'(x),$$

where the right-hand side is ordinary matrix multiplication of the Jacobians.

Proof. Set $y = g(x)$. Since f is differentiable at y, there exists a function $\eta : \mathbb{R}^m \to \mathbb{R}^k$ such that

$$f(y + k) - f(y) - f'(y)k = \|k\| \, \eta(k), \qquad \eta(k) \to 0 \quad (k \to 0).$$

Since g is differentiable at x, there exists a function $\varepsilon : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$g(x + h) - g(x) = g'(x)h + \|h\| \, \varepsilon(h), \qquad \varepsilon(h) \to 0 \quad (h \to 0).$$

Now set

$$k := g(x + h) - g(x).$$

Because differentiability implies continuity, g is continuous at x, hence $\|h\| \to 0$ implies $\|k\| \to 0$. Moreover,

$$\|k\| = \|g'(x)h + \|h\| \, \varepsilon(h)\| \leq \|g'(x)\| \, \|h\| + \|h\| \, \|\varepsilon(h)\|.$$

Consider

$$\mu(h) := \frac{(f \circ g)(x + h) - (f \circ g)(x) - f'(g(x))\, g'(x)\, h}{\|h\|}.$$

Using $y = g(x)$ and $k - g'(x)h = \|h\|\, \varepsilon(h)$, we obtain

$$\begin{aligned}
\mu(h) &= \frac{f(y + k) - f(y) - f'(y)\, g'(x)h}{\|h\|} \\
&= \frac{f(y + k) - f(y) - f'(y)k + f'(y)(k - g'(x)h)}{\|h\|} \\
&= \frac{\|k\|\, \eta(k) + \|h\|\, f'(y)\varepsilon(h)}{\|h\|}.
\end{aligned}$$

Hence

$$\|\mu(h)\| \le \frac{\|k\|}{\|h\|}\, \|\eta(k)\| + \|f'(y)\|\, \|\varepsilon(h)\| \le \|\eta(k)\|(\|g'(x)\| + \|\varepsilon(h)\|) + \|f'(y)\|\, \|\varepsilon(h)\|.$$

As $h \to 0$, we have $k \to 0$, so $\eta(k) \to 0$, and also $\varepsilon(h) \to 0$. Therefore $\mu(h) \to 0$. This proves that $f \circ g$ is differentiable at x and

$$(f \circ g)'(x) = f'(g(x))\, g'(x).$$

$\square$

Proposition 8.19 (Vanishing gradient implies constancy on convex sets). *Let $D \subseteq \mathbb{R}^n$ be open and convex, and let $f \in C^1(D)$ satisfy $\nabla f \equiv \mathbf{0}$ on D. Then f is constant on D.*

Proof. Fix $x, y \in D$. Since D is convex, the line segment $\gamma(t) = (1 - t)x + ty$ lies in D for $t \in [0, 1]$. Define $\varphi(t) = f(\gamma(t))$. By the chain rule,

$$\varphi'(t) = \nabla f(\gamma(t)) \cdot (y - x) = 0 \quad \text{for all } t \in [0, 1].$$

Hence φ is constant, so $f(y) = \varphi(1) = \varphi(0) = f(x)$. $\square$

Worked examples and a useful identity.

Example 8.20 (Radial compositions). Let $\phi : \mathbb{R} \to \mathbb{R}$ be differentiable and define $F : \mathbb{R}^n \to \mathbb{R}$ by $F(x) = \phi(\|x\|^2)$. Then F is differentiable and

$$DF(x) = \phi'(\|x\|^2)\, D(\|x\|^2) = 2\phi'(\|x\|^2)\, x^t, \qquad \text{so} \qquad \nabla F(x) = 2\phi'(\|x\|^2)\, x.$$

More generally, for an integer $k \ge 1$ and $G(x) = \phi(\|x\|^{2k})$ one has (for $x \ne 0$)

$$\nabla G(x) = 2k\, \|x\|^{2k-2}\, \phi'(\|x\|^{2k})\, x.$$

Theorem 8.21 (Euler's homogeneous function theorem)**.** *Let $U \subseteq \mathbb{R}^n \setminus \{0\}$ be open and let $f : U \to \mathbb{R}$ be differentiable. Assume that f is* homogeneous of degree $\alpha \in \mathbb{R}$*, meaning that*

$$f(tx) = t^\alpha f(x) \qquad \text{for all } x \in U \text{ and all } t > 0 \text{ with } tx \in U.$$

Then for every $x \in U$,

$$x \cdot \nabla f(x) = \alpha f(x).$$

Proof. Fix $x \in U$ and consider the one-variable function $\psi(t) = f(tx)$ for t near 1. By the chain rule, $\psi'(t) = f'(tx)\, x = \nabla f(tx) \cdot x$. The homogeneity hypothesis gives $\psi(t) = t^\alpha f(x)$, so $\psi'(t) = \alpha t^{\alpha-1} f(x)$. Evaluating at $t = 1$ yields $\nabla f(x) \cdot x = \alpha f(x)$. $\qquad \square$

Corollary 8.22 (Homogeneous maps near the origin)**.** *Suppose $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ is homogeneous of degree $\alpha > 1$ and extends continuously to 0 by setting $f(0) := 0$. Then f is differentiable at 0 and $f'(0) = 0$.*

Proof. Write $h = \|h\|v$ with $\|v\| = 1$. By homogeneity, $|f(h)| = \|h\|^\alpha |f(v)|$. Continuity on the compact unit sphere implies $M := \sup_{\|v\|=1} |f(v)| < \infty$, hence

$$\frac{|f(h) - f(0)|}{\|h\|} = \frac{|f(h)|}{\|h\|} \leq M \, \|h\|^{\alpha-1} \xrightarrow[h \to 0]{} 0.$$

This is exactly differentiability at 0 with derivative 0. $\qquad \square$

Mean value theorem on convex domains. Let $\mathcal{D} \subseteq \mathbb{R}^n$ be open and convex, and let $f : \mathcal{D} \to \mathbb{R}$ be differentiable. Then for any $x, y \in \mathcal{D}$ there exists a point c on the open line segment (x, y) such that

$$f(y) - f(x) = f'(c)\, (y - x) = \nabla f(c) \cdot (y - x).$$

Proof. Define $\varphi(t) = f((1 - t)x + ty)$ for $t \in [0, 1]$. By the chain rule, φ is differentiable on $(0, 1)$ and

$$\varphi'(t) = f'((1 - t)x + ty)\, (y - x).$$

Applying the one-variable mean value theorem to φ yields $\lambda \in (0, 1)$ such that

$$\begin{aligned}
f(y) - f(x) &= \varphi(1) - \varphi(0) \\
&= \varphi'(\lambda) \\
&= f'((1 - \lambda)x + \lambda y)\, (y - x).
\end{aligned}$$

Set $c = (1 - \lambda)x + \lambda y$. $\qquad \square$

Exercise 8.4. Let U be an open and connected set in $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ be such that $f'(x) = 0$ for all $x \in U$. Show that $f \equiv 0$.

Jacobian matrix for vector-valued maps. Let $U \subseteq \mathbb{R}^n$ be open and $f = (f_1, \ldots, f_m) : U \to \mathbb{R}^m$ be differentiable at $x_0 \in U$. Then each partial derivative $\partial f_i / \partial x_j(x_0)$ exists and

$$f'(x_0) = \left(\frac{\partial f_i}{\partial x_j}(x_0) \right)_{1 \le i \le m, \; 1 \le j \le n}.$$

Proof. Let e_j be the jth standard basis vector of $\mathbb{R}^n$. Differentiability gives

$$\frac{\|f(x_0 + he_j) - f(x_0) - f'(x_0)(he_j)\|}{|h|} \longrightarrow 0 \qquad (h \to 0).$$

Since $f'(x_0)(he_j) = h\, f'(x_0)e_j$, dividing by h shows that

$$\lim_{h \to 0} \frac{f(x_0 + he_j) - f(x_0)}{h} = f'(x_0)e_j \in \mathbb{R}^m,$$

so every component limit exists; equivalently, each $\partial f_i / \partial x_j(x_0)$ exists. Moreover, the vector $f'(x_0)e_j$ is exactly the jth column of the Jacobian matrix, hence $f'(x_0)$ is the matrix of partial derivatives. $\qquad \square$

Example 8.23 (Jacobian at a point does not guarantee differentiability). Define $F : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(x, y) := \begin{cases} \left(\dfrac{x^2 y}{x^2 + y^2}, \; \dfrac{xy^2}{x^2 + y^2} \right), & (x, y) \ne (0, 0), \\ (0, 0), & (x, y) = (0, 0). \end{cases}$$

Then all first partial derivatives of the component functions exist at $(0, 0)$ and the Jacobian matrix $F'(0, 0)$ is the zero matrix, but F is *not* differentiable at $(0, 0)$.

Proof. Along the coordinate axes we have $F(h, 0) = (0, 0)$ and $F(0, k) = (0, 0)$, so each partial derivative at $(0, 0)$ exists and equals 0, hence $F'(0, 0) = 0$. If F were differentiable at $(0, 0)$ with derivative 0, we would have $\|F(h, k)\|/\sqrt{h^2 + k^2} \to 0$ as $(h, k) \to (0, 0)$. However, along the diagonal $y = x$,

$$F(t, t) = \left(\frac{t}{2}, \frac{t}{2} \right), \qquad t \ne 0.$$

Thus

$$\frac{\|F(t, t)\|}{\|(t, t)\|} = \frac{\sqrt{(t/2)^2 + (t/2)^2}}{\sqrt{t^2 + t^2}} = \frac{1}{2}$$

for all $t \ne 0$, which does not tend to 0 as $t \to 0$. Therefore F is not differentiable at $(0, 0)$. $\quad \square$

Higher derivatives and the Hessian (scalar case). If $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ is differentiable on U, then each component of ∇f may itself be differentiated. The matrix of second partial derivatives, when it exists, is called the *Hessian*:

$$f''(x) = \left(\frac{\partial^2 f}{\partial x_j \, \partial x_k}(x) \right)_{1 \le j, k \le n}.$$

The next theorem justifies that under mild regularity, mixed second partials agree.

Example 8.24 (Mixed partials can disagree without continuity). Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$
f(x, y) := \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & (x, y) \neq (0, 0), \\[2mm] 0, & (x, y) = (0, 0). \end{cases}
$$

Then the mixed partials at $(0, 0)$ exist but are different: $f_{yx}(0, 0) = 1$ whereas $f_{xy}(0, 0) = -1$. Indeed, for $h \neq 0$ one computes

$$
f_y(h, 0) = \lim_{k \to 0} \frac{f(h, k) - f(h, 0)}{k} = \lim_{k \to 0} h \frac{h^2 - k^2}{h^2 + k^2} = h,
$$

so $f_{yx}(0, 0) = \lim_{h \to 0} \frac{f_y(h,0) - f_y(0,0)}{h} = 1$. Similarly, for $k \neq 0$,

$$
f_x(0, k) = \lim_{h \to 0} \frac{f(h, k) - f(0, k)}{h} = \lim_{h \to 0} k \frac{h^2 - k^2}{h^2 + k^2} = -k,
$$

so $f_{xy}(0, 0) = \lim_{k \to 0} \frac{f_x(0,k) - f_x(0,0)}{k} = -1$.

Notation 8.25 (Smoothness classes). Let $\mathcal{D} \subset \mathbb{R}^n$ be open. For $k \in \mathbb{N}$, we write $C^k(\mathcal{D})$ for the space of functions $f : \mathcal{D} \to \mathbb{R}$ whose partial derivatives of order $\leq k$ exist and are continuous on $\mathcal{D}$. In particular, $C^1(\mathcal{D})$ consists of continuously differentiable functions and $C^2(\mathcal{D})$ consists of twice continuously differentiable functions.

Theorem 8.26 (Clairaut–Schwarz). *Let $\mathcal{D} \subset \mathbb{R}^2$ be open and let $f \in C^2(\mathcal{D})$. Then for every $(x_0, y_0) \in \mathcal{D}$,*

$$
\frac{\partial^2 f}{\partial x \, \partial y}(x_0, y_0) = \frac{\partial^2 f}{\partial y \, \partial x}(x_0, y_0).
$$

More generally, if $\mathcal{D} \subset \mathbb{R}^n$ is open and $f \in C^2(\mathcal{D})$, then

$$
\frac{\partial^2 f}{\partial x_j \, \partial x_k} = \frac{\partial^2 f}{\partial x_k \, \partial x_j}, \qquad 1 \leq j, k \leq n.
$$

Proof. Fix $(x_0, y_0) \in \mathcal{D}$. Since $\mathcal{D}$ is open, there exists $r > 0$ such that the rectangle $[x_0 - r, x_0 + r] \times [y_0 - r, y_0 + r]$ is contained in $\mathcal{D}$. For (x, y) in this rectangle define

$$
F(x, y) := f(x, y) - f(x_0, y) - f(x, y_0) + f(x_0, y_0).
$$

Note that $F(x_0, y) = 0$ and $F(x, y_0) = 0$. Fix $x \neq x_0$ and apply the one-variable mean value theorem to the map $y \mapsto f(x, y) - f(x_0, y)$ to obtain a point η between y and y_0 such that

$$
F(x, y) = (y - y_0)\big(f_y(x, \eta) - f_y(x_0, \eta)\big).
$$

Now fix $y \neq y_0$ and apply the mean value theorem to the map $x \mapsto f_y(x, \eta)$ to obtain a point ξ between x and x_0 such that
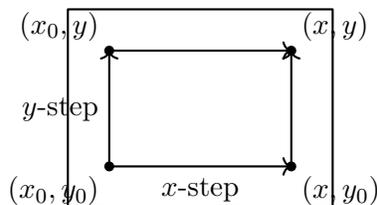
Figure 8.7. The rectangle used in the proof of Clairaut–Schwarz: compare increments by moving in the x-direction then y-direction, versus y then x.

$$f_y(x, \eta) - f_y(x_0, \eta) = (x - x_0) f_{xy}(\xi, \eta).$$

Consequently,

$$\frac{F(x, y)}{(x - x_0)(y - y_0)} = f_{xy}(\xi, \eta).$$

Letting $(x, y) \to (x_0, y_0)$ forces $(\xi, \eta) \to (x_0, y_0)$, hence by continuity of f_{xy} we obtain

$$\lim_{(x,y)\to(x_0,y_0)} \frac{F(x, y)}{(x - x_0)(y - y_0)} = f_{xy}(x_0, y_0).$$

Repeating the argument with the roles of x and y interchanged yields the same limit equal to $f_{yx}(x_0, y_0)$. Therefore $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$. $\qquad\square$

8.3.5   Taylor's theorem

Theorem 8.27 (Second-order Taylor formula in $\mathbb{R}^n$). *Let $\mathcal{D} \subseteq \mathbb{R}^n$ be open and let $f \in C^2(\mathcal{D})$. Fix $X \in \mathcal{D}$. Then there exists $\delta > 0$ such that for every $H \in \mathbb{R}^n$ with $\|H\| < \delta$ there exists $\lambda \in (0, 1)$ for which*

$$f(X + H) = f(X) + f'(X) H + \frac{1}{2} H^t f''(C) H, \qquad C = X + \lambda H,$$

where $f'(X)$ is the derivative (gradient row vector) and $f''(C)$ is the Hessian matrix at C.

How to read the formula. For scalar-valued f, the derivative $f'(X)$ can be identified with the gradient row vector $\nabla f(X)$, so the first-order term is $f'(X) H = \nabla f(X) \cdot H$. The matrix $f''(C)$ is the Hessian at C, and $H^t f''(C)H$ is the associated quadratic form.

Proof. Fix $X \in \mathcal{D}$ and choose $\delta > 0$ so that $X + tH \in \mathcal{D}$ for all $t \in [0, 1]$ whenever $\|H\| < \delta$. For such H, define $g : [0, 1] \to \mathbb{R}$ by $g(t) = f(X + tH)$. Then $g \in C^2([0, 1])$, and by the chain rule,

$$g'(t) = f'(X + tH) H, \qquad g''(t) = H^t f''(X + tH) H.$$

Apply the one-variable Taylor theorem (with Lagrange remainder) at $t = 0$ to obtain $\lambda \in (0, 1)$ such that

$$g(1) = g(0) + g'(0) + \frac{1}{2}\, g''(\lambda).$$

Substituting the expressions for g, g', g'' and writing $C = X + \lambda H$ gives the stated formula. $\quad\square$

8.3.6 Multi-index notation and the general Taylor formula

For higher-order expansions in $\mathbb{R}^n$, it is convenient to use *multi-indices*. A multi-index is a vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$. We write $|\alpha| := \alpha_1 + \cdots + \alpha_n$, $\alpha! := \alpha_1! \cdots \alpha_n!$, and for $h = (h_1, \ldots, h_n) \in \mathbb{R}^n$,

$$h^\alpha := h_1^{\alpha_1} \cdots h_n^{\alpha_n}.$$

For a sufficiently differentiable function f, define

$$D^\alpha f := \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}.$$

Theorem 8.28 (Taylor's theorem in $\mathbb{R}^n$ (multi-index form)). *Let $U \subset \mathbb{R}^n$ be open, let $k \in \mathbb{N}$, and let $f \in C^{k+1}(U)$. Fix $a \in U$. Then there exists a neighborhood V of 0 such that $a + h \in U$ for all $h \in V$, and for each such h,*

$$f(a + h) = \sum_{|\alpha| \le k} \frac{D^\alpha f(a)}{\alpha!}\, h^\alpha + R_k(a, h),$$

where the remainder satisfies

$$\lim_{h \to 0} \frac{R_k(a, h)}{\|h\|_2^k} = 0.$$

In particular, the polynomial part gives the best k-th order approximation of f near a in the sense of little-o.

Remark 8.29. There are several equivalent remainder formulations (integral form, mean-value form) under additional smoothness. For many applications in analysis, the little-*o* remainder above is the most robust.

A mean value inequality for vector-valued functions.

Theorem 8.30. *Let $f : [a, b] \to \mathbb{R}^n$ be differentiable on (a, b) and continuous on $[a, b]$. Then there exists $\lambda \in (a, b)$ such that*

$$\|f(b) - f(a)\| \le \|f'(\lambda)\|\, (b - a).$$

Proof. Consider the scalar function $g(t) = (f(b) - f(a)) \cdot f(t)$ on $[a, b]$. Then g is continuous on $[a, b]$ and differentiable on (a, b), with

$$g'(t) = (f(b) - f(a)) \cdot f'(t).$$

By the one-variable mean value theorem, there exists $\lambda \in (a, b)$ such that

$$g(b) - g(a) = g'(\lambda)(b - a).$$

Since $g(b) - g(a) = (f(b) - f(a)) \cdot (f(b) - f(a)) = \|f(b) - f(a)\|^2$, we obtain

$$\|f(b) - f(a)\|^2 = (b - a)(f(b) - f(a)) \cdot f'(\lambda)$$
$$\leq (b - a)\|f(b) - f(a)\| \|f'(\lambda)\|,$$

and the conclusion follows (trivially if $f(b) = f(a)$). $\qquad \square$

Theorem 8.31 (Mean value inequality along line segments). *Let $\mathcal{D} \subseteq \mathbb{R}^n$ be open and let $f : \mathcal{D} \to \mathbb{R}^m$ be differentiable. Fix $X \in \mathcal{D}$. Then there exists $\varepsilon > 0$ such that whenever $\|H\| < \varepsilon$ we can find $\lambda \in (0, 1)$ with*

$$\|f(X + H) - f(X)\| \leq \|f'(X + \lambda H)\| \|H\|.$$

Proof. Choose $\varepsilon > 0$ so that $X + tH \in \mathcal{D}$ for all $t \in [0, 1]$ whenever $\|H\| < \varepsilon$. Fix such H and define $g(t) = f(X + tH)$ for $t \in [0, 1]$. Then $g : [0, 1] \to \mathbb{R}^m$ is differentiable and $g'(t) = f'(X + tH)H$. Apply Theorem 8.30 to g on $[0, 1]$ to obtain $\lambda \in (0, 1)$ such that

$$\|f(X + H) - f(X)\| = \|g(1) - g(0)\| \leq \|g'(\lambda)\| \leq \|f'(X + \lambda H)\| \|H\|.$$

$\qquad \square$

Remark 8.32 (Equality need not hold in the vector mean value inequality). The scalar mean value theorem produces an exact identity, but for vector-valued functions the correct statement is generally only an inequality. For example, define $g : [-1, 1] \to \mathbb{R}^2$ by

$$g(t) = (t^3, 1 - t^2).$$

Then

$$g(1) - g(-1) = (2, 0), \qquad \|g(1) - g(-1)\| = 2.$$

Also

$$g'(t) = (3t^2, -2t), \qquad \|g'(t)\| = \sqrt{9t^4 + 4t^2} = |t|\sqrt{9t^2 + 4}.$$

If one had an exact vector-valued mean value formula

$$\|g(1) - g(-1)\| = \|g'(c)\| (1 - (-1))$$

for some $c \in (-1, 1)$, then $\|g'(c)\| = 1$, hence $9c^4 + 4c^2 - 1 = 0$, so $c^2 = \frac{1}{3}$. At such a point,

$$g'(c) = (1, -2c),$$

which is not parallel to $g(1) - g(-1) = (2, 0)$.

Geometric picture. The image of g is the planar curve

$$g(t) = (t^3, 1 - t^2), \qquad -1 \le t \le 1,$$

which joins $g(-1) = (-1, 0)$ to $g(1) = (1, 0)$. The secant vector $g(1) - g(-1)$ is horizontal, whereas at the distinguished point $c = \frac{1}{\sqrt{3}}$ the tangent direction $g'(c) = (1, -\frac{2}{\sqrt{3}})$ is not horizontal. The figure below makes this mismatch visible.
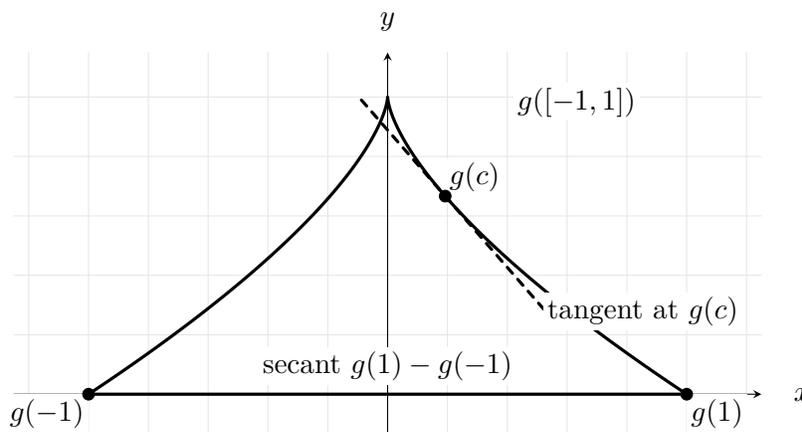
Figure for Remark 8.32: the secant direction determined by the endpoints is horizontal, but the tangent direction at the candidate mean-value point is not.

Thus the one-dimensional equality has no direct vector-valued analogue in general; the norm inequality is the correct replacement.

Proposition 8.33 (Stability of invertibility and continuity of inversion). *Let $A \in GL_n(\mathbb{R})$ and $B \in L_n(\mathbb{R})$ satisfy*

$$\|B - A\| < \frac{1}{\|A^{-1}\|}.$$

Then $B \in GL_n(\mathbb{R})$. Moreover, the map $A \mapsto A^{-1}$ is continuous on $GL_n(\mathbb{R})$.

Proof. Set $\alpha = \frac{1}{\|A^{-1}\|}$ and $\beta = \|B - A\|$, so $\beta < \alpha$. For any $x \in \mathbb{R}^n$,

$$\begin{aligned}
\alpha\|x\| &= \alpha\|A^{-1}Ax\| \\
&\le \alpha\|A^{-1}\|\,\|Ax\| \\
&= \|Ax\| \\
&\le \|(A - B)x\| + \|Bx\| \\
&\le \beta\|x\| + \|Bx\|.
\end{aligned}$$

Hence $(\alpha - \beta)\|x\| \le \|Bx\|$ for all x. If $Bx = 0$, this forces $x = 0$, so B is injective and therefore invertible (finite dimensions). For continuity, note that for $y \ne 0$ and $x = B^{-1}y$,

$$(\alpha - \beta)\|B^{-1}y\| \le \|y\|$$
$$\Rightarrow \quad \|B^{-1}\|$$
$$\le \frac{1}{\alpha - \beta}.$$

Then

$$\|B^{-1} - A^{-1}\| = \|B^{-1}(A - B)A^{-1}\|$$
$$\le \|B^{-1}\|\,\|A - B\|\,\|A^{-1}\|$$
$$\le \frac{\|A - B\|}{\alpha(\alpha - \beta)} \xrightarrow[B \to A]{} 0.$$

Thus inversion is continuous on $GL_n(\mathbb{R})$. □

Example 8.34 (Derivative of the inverse in one dimension)**.** Let $f : \mathbb{R} \to \mathbb{R}$ be bijective and continuously differentiable on a neighborhood of x_0, with $f'(x_0) \ne 0$. Then f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}.$$

Proof. Let $k \to 0$ and set $h = f^{-1}(y_0 + k) - f^{-1}(y_0)$, so that $y_0 + k = f(x_0 + h)$ and $k = f(x_0 + h) - f(x_0)$. By the mean value theorem applied to f on the interval between x_0 and $x_0 + h$, there exists $\theta \in (0, 1)$ such that

$$k = h\, f'(x_0 + \theta h).$$

Since $f'(x_0) \ne 0$ and f' is continuous, for h small we have $|f'(x_0 + \theta h)| \ge m > 0$, hence $k \to 0$ implies $h \to 0$. Now compute

$$\frac{f^{-1}(y_0 + k) - f^{-1}(y_0) - \dfrac{k}{f'(x_0)}}{|k|} = \frac{h - \dfrac{f(x_0 + h) - f(x_0)}{f'(x_0)}}{|f(x_0 + h) - f(x_0)|}$$
$$= \frac{|f'(x_0) - f'(x_0 + \theta h)|}{|f'(x_0 + \theta h)|\,|f'(x_0)|} \xrightarrow[h \to 0]{} 0.$$

Since f^{-1} is differentiable at $y_0 = f(x_0)$, by differentiating $f^{-1}(f(x)) = x$ at $x = x_0$ yields $(f^{-1})'(f(x_0))\, f'(x_0) = 1$. □

Example 8.35 (Invertible Jacobian does not imply global one-to-one behavior)**.** Define $F : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(x, y) = (e^x \cos y,\ e^x \sin y).$$

Then

$$DF(x, y) = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}, \qquad \det DF(x, y) = e^{2x} > 0$$

for every $(x, y) \in \mathbb{R}^2$. Thus the Jacobian matrix is invertible at every point. Nevertheless F is not injective on $\mathbb{R}^2$, because

$$F(x, y + 2\pi) = F(x, y) \qquad \text{for all } (x, y) \in \mathbb{R}^2.$$

This example is a useful warning before the inverse function theorem: invertibility of the derivative guarantees *local* invertibility, not a global inverse on the whole domain.

8.4 Inverse and implicit function theorems

We now turn to two fundamental results describing the local structure of C^1 maps: when a map is locally invertible (inverse function theorem) and when a level set can be solved locally as a graph (implicit function theorem).

8.4.1 Inverse function theorem

Theorem 8.36 (Inverse Function Theorem). *Let Ω be an open set in $\mathbb{R}^n$. Suppose $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 map such that $\det f'(x_0) \neq 0$. Then*

(i) There exist open sets U and $V \subset \mathbb{R}^n$ such that $f : U \to V$ $(= f(U))$ is bijective.

(ii) f^{-1} is a C^1 map on V, and

$$(f^{-1})'(f(x_0)) = (f'(x_0))^{-1}.$$

Idea. Near x_0, the map f is well-approximated by its linearization $A = f'(x_0)$. The hypothesis $\det A \neq 0$ means this linear approximation is invertible. The proof turns the equation $f(x) = y$ into a fixed-point problem whose solution follows from the contraction mapping principle, uniformly for y in a small neighborhood of $f(x_0)$.

Proof. Let $A = f'(x_0)$. Since $\det A \neq 0$, the matrix A is invertible. **Step 0 (fixed-point reformulation).** For $y \in \mathbb{R}^n$, define

$$\varphi_y(x) := x + A^{-1}(y - f(x)), \qquad x \in \Omega. \tag{8.8}$$

Then $\varphi_y(x) = x$ if and only if $f(x) = y$. In other words, solving $f(x) = y$ is equivalent to finding a fixed point of φ_y. **Step 1 (a uniform contraction estimate).** Because f' is continuous at x_0, for

$$\varepsilon = \frac{1}{2\|A^{-1}\|} > 0$$

there exists $\delta > 0$ such that

$$\|x - x_0\| < \delta \implies \|f'(x) - A\| < \frac{1}{2\|A^{-1}\|}.$$

Set $U := B_\delta(x_0) \subset \Omega$ and $V := f(U)$. For $x \in U$ we have

$$\begin{aligned}
\|\varphi_y'(x)\| &= \|I - A^{-1}f'(x)\| \\
&= \|A^{-1}(A - f'(x))\| \\
&\leq \|A^{-1}\| \, \|A - f'(x)\| < \frac{1}{2}.
\end{aligned}$$

By the mean value estimate for C^1 maps on convex sets (applied to φ_y on U), it follows that

$$\|\varphi_y(x_1) - \varphi_y(x_2)\| \leq \frac{1}{2}\|x_1 - x_2\| \qquad \text{for all } x_1, x_2 \in U. \tag{8.9}$$

Thus φ_y is a contraction on U (with contraction constant $1/2$), uniformly in y. **(i)** f **is one-to-one on** U**.** If $f(x_1) = f(x_2) = y$ with $x_1, x_2 \in U$, then both x_1 and x_2 are fixed points of φ_y. By (8.9), a contraction has at most one fixed point, hence $x_1 = x_2$. **(ii)** V **is open (and** $f : U \to V$ **is bijective).** Fix $y^* \in V$. Choose $x^* \in U$ with $y^* = f(x^*)$ and pick $r > 0$ such that the closed ball $\overline{B}_r(x^*) \subset U$. We claim that

$$\|y - y^*\| < \frac{r}{2\|A^{-1}\|} \quad \implies \quad y \in V. \tag{8.10}$$

Indeed, for such y we have

$$\|\varphi_y(x^*) - x^*\| = \|A^{-1}(y - y^*)\| \leq \|A^{-1}\| \, \|y - y^*\| < \frac{r}{2}.$$

If $x \in \overline{B}_r(x^*)$, then using (8.9) we obtain

$$\begin{aligned}
\|\varphi_y(x) - x^*\| &\leq \|\varphi_y(x) - \varphi_y(x^*)\| + \|\varphi_y(x^*) - x^*\| < \frac{1}{2}\|x - x^*\| + \frac{r}{2} \\
&\leq r.
\end{aligned}$$

Hence $\varphi_y(\overline{B}_r(x^*)) \subset \overline{B}_r(x^*)$. Since $\overline{B}_r(x^*)$ is complete and φ_y is a contraction there, the contraction mapping principle gives a unique fixed point $x \in \overline{B}_r(x^*)$. This fixed point satisfies $f(x) = y$, so $y \in f(U) = V$, proving (8.10). Thus every point of V is an interior point, so V is open. Since $V = f(U)$ and f is injective on U, the restriction $f : U \to V$ is bijective. **(iii) Differentiability of** f^{-1} **at** $f(x_0)$**.** Let $y_0 = f(x_0)$ and let k be small enough that $y_0 + k \in V$. Set

$$h := f^{-1}(y_0 + k) - x_0, \qquad \text{so that} \qquad k = f(x_0 + h) - f(x_0).$$

Using (8.8) with $y = y_0 + k$, the fixed-point property $\varphi_{y_0+k}(x_0 + h) = x_0 + h$ and $\varphi_{y_0+k}(x_0) = x_0 + A^{-1}k$ yield

$$h - A^{-1}k = \varphi_{y_0+k}(x_0 + h) - \varphi_{y_0+k}(x_0).$$

By the contraction estimate (8.9),

$$\|h - A^{-1}k\| \leq \frac{1}{2}\|h\|.$$

Consequently,

$$\frac{1}{2}\|h\| \leq \|A^{-1}k\| \leq \|A^{-1}\|\,\|k\|, \qquad \text{so} \qquad \|h\| \leq 2\|A^{-1}\|\,\|k\|. \tag{8.11}$$

Since f is differentiable at x_0, we can write

$$f(x_0 + h) - f(x_0) - Ah = \|h\|\,\rho(h), \qquad \rho(h) \to 0 \text{ as } \|h\| \to 0.$$

But $k = f(x_0 + h) - f(x_0)$, hence

$$k - Ah = \|h\|\,\rho(h), \qquad \text{and therefore} \qquad h - A^{-1}k$$
$$= -A^{-1}(k - Ah)$$
$$= -\|h\|\,A^{-1}\rho(h).$$

Divide by $\|k\|$ and use (8.11):

$$\frac{\|h - A^{-1}k\|}{\|k\|} \leq \frac{\|h\|\,\|A^{-1}\|\,\|\rho(h)\|}{\|k\|}$$
$$\leq 2\|A^{-1}\|^2\,\|\rho(h)\| \longrightarrow 0.$$

Thus f^{-1} is differentiable at y_0 with

$$(f^{-1})'(y_0) = A^{-1} = (f'(x_0))^{-1}.$$

(iv) f^{-1} **is C^1 on V.** The above argument applies at any $y \in V$ with $x = f^{-1}(y)$ and $A = f'(x)$ (which is invertible on U by continuity of $\det f'$). Hence

$$(f^{-1})'(y) = \left(f'(f^{-1}(y))\right)^{-1}, \qquad y \in V.$$

The map $y \mapsto f^{-1}(y)$ is continuous, the derivative f' is continuous on U, and the inversion map $A \mapsto A^{-1}$ is continuous on $GL_n(\mathbb{R})$. Therefore $y \mapsto (f^{-1})'(y)$ is continuous on V, i.e., $f^{-1} \in C^1(V)$. $\qquad\square$

Example 8.37 (A concrete Jacobian computation). Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$f(x, y) = (x - e^{-y},\ y - e^x).$$

Then

$$f'(0,0) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \qquad \det f'(0,0) = 2 \neq 0. \tag{8.12}$$

The non-vanishing determinant in (8.12) is precisely the inverse-function hypothesis. Therefore f is locally invertible near $(0,0)$, and

$$(f^{-1})'(f(0,0)) = (f'(0,0))^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

This example is best read as a computational template: first verify invertibility of the Jacobian, then obtain the derivative of the local inverse for free.

Example 8.38 (Local vs. global invertibility). Define $F : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(x,y) = (e^x \cos y, \ e^x \sin y).$$

Then F is C^1 and

$$\det F'(x,y) = e^{2x} \neq 0 \qquad \text{for all } (x,y) \in \mathbb{R}^2.$$

Hence, by the inverse function theorem, F is locally one-to-one (and locally onto its image) at every point. Nevertheless, F is not injective on all of $\mathbb{R}^2$, since

$$F(x, y + 2\pi) = F(x,y) \qquad \text{for all } (x,y) \in \mathbb{R}^2.$$

This illustrates that the inverse function theorem is inherently a *local* result.

8.4.2 Implicit function theorem

A motivating example. Consider $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = x^2 + y^2 - 1.$$

The level set $\{(x,y) : f(x,y) = 0\}$ is the unit circle.
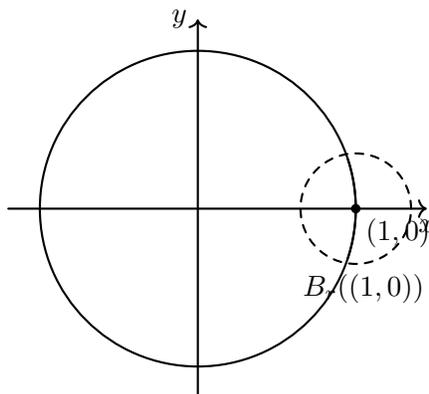
Figure 8.8. Near $(1,0)$, the circle $x^2 + y^2 = 1$ is the graph $x = \sqrt{1 - y^2}$, but it cannot be written as a single-valued graph $y = \psi(x)$ near $x = 1$.

We have $f'(x, y) = (2x, 2y)$, so at $(1, 0)$

$$\left.\frac{\partial f}{\partial x}\right|_{(1,0)} = 2 \neq 0, \qquad \left.\frac{\partial f}{\partial y}\right|_{(1,0)} = 0.$$

Because $\partial f/\partial x \neq 0$ at $(1, 0)$, we can solve $f(x, y) = 0$ locally for x as a C^1 function of y:

$$x = \varphi(y) = \sqrt{1 - y^2}, \qquad |y| < r,$$

for some small $r > 0$. On the other hand, we *cannot* solve locally for y as a single-valued function of x near $x = 1$, since for x slightly less than 1 the equation $x^2 + y^2 = 1$ has two solutions $y = \pm\sqrt{1 - x^2}$. This illustrates that which variable can be solved for depends on a non-degeneracy condition on the relevant partial derivative.

A linear-algebra model. Let

$$A : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$$

be a linear map. Writing $(h, k) = (h, 0) + (0, k)$, we may decompose

$$A(h, k) = A(h, 0) + A(0, k) = A_x h + A_y k,$$

where $A_x : \mathbb{R}^n \to \mathbb{R}^n$ and $A_y : \mathbb{R}^m \to \mathbb{R}^n$ are linear.

Lemma 8.39. *If A_x is invertible, then for each $k \in \mathbb{R}^m$ there exists a unique $h \in \mathbb{R}^n$ such that $A(h, k) = 0$, namely*

$$h = -A_x^{-1} A_y k.$$

Proof. The equation $A(h, k) = 0$ is equivalent to $A_x h + A_y k = 0$. If A_x is invertible, we can solve uniquely for h: $h = -A_x^{-1} A_y k$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Block form of the derivative. If $\Omega \subset \mathbb{R}^n \times \mathbb{R}^m$ is open and $f : \Omega \to \mathbb{R}^n$ is differentiable, then for $(x, y) \in \Omega$ the derivative $f'(x, y)$ is an $n \times (n + m)$ matrix. It is convenient to write it in block form

$$f'(x, y) = \begin{pmatrix} A_x & A_y \end{pmatrix},$$

where

$$A_x = \left(\frac{\partial f_i}{\partial x_j}(x, y)\right)_{1 \leq i \leq n,\ 1 \leq j \leq n}, \qquad A_y = \left(\frac{\partial f_i}{\partial y_k}(x, y)\right)_{1 \leq i \leq n,\ 1 \leq k \leq m}.$$

Theorem 8.40 (Implicit Function Theorem). *Let Ω be an open subset of $\mathbb{R}^n \times \mathbb{R}^m$ and let $f : \Omega \to \mathbb{R}^n$ be a C^1 map. Assume that $f(x_0, y_0) = 0$ for some $(x_0, y_0) \in \Omega$ and that the $n \times n$ block*

$$A_x := \left[f'(x_0, y_0)\right]_x$$

is invertible (equivalently, $\det A_x \neq 0$). Then:

(i) There exist open sets $U \subset \mathbb{R}^n \times \mathbb{R}^m$ with $(x_0, y_0) \in U$ and $W \subset \mathbb{R}^m$ with $y_0 \in W$ such that for every $y \in W$ there exists a unique $x \in \mathbb{R}^n$ with $(x, y) \in U$ and $f(x, y) = 0$.

(ii) Writing this unique solution as $x = g(y)$, we obtain a C^1 map $g : W \to \mathbb{R}^n$ with $g(y_0) = x_0$ and $f(g(y), y) = 0$ for all $y \in W$. Moreover,

$$g'(y_0) = -A_x^{-1} A_y, \qquad where \qquad A_y := \left[f'(x_0, y_0) \right]_y.$$

In particular, near (x_0, y_0) the zero set $\{(x, y) \in \Omega : f(x, y) = 0\}$ is the graph of the function $x = g(y)$.

Geometric meaning. The equation $f(x, y) = 0$ describes a level set in $\mathbb{R}^{n+m}$. The invertibility of the x-block of the Jacobian says that, near (x_0, y_0), the equation can be solved uniquely for x as a C^1 function of y. In other words, the level set is locally the graph of a function $x = \varphi(y)$.

Proof. (i) Define $F : \Omega \to \mathbb{R}^n \times \mathbb{R}^m$ by

$$F(x, y) := (f(x, y), y).$$

Then F is a C^1 map and its derivative at (x_0, y_0) has block form

$$F'(x_0, y_0) = \begin{bmatrix} A_x & A_y \\ 0 & I \end{bmatrix}.$$

Hence $\det F'(x_0, y_0) = \det(A_x) \neq 0$. By the inverse function theorem, there exist open sets $U \subset \mathbb{R}^n \times \mathbb{R}^m$ and $V \subset \mathbb{R}^n \times \mathbb{R}^m$ such that

$$F : U \to V$$

is a C^1 bijection and $F^{-1} : V \to U$ is C^1. Let

$$W := \{\, y \in \mathbb{R}^m : \ (0, y) \in V \,\}.$$

Since V is open, W is open in $\mathbb{R}^m$. For any $y \in W$, we have $(0, y) \in V$, and since F is onto V there exists $(x, y) \in U$ such that

$$(0, y) = F(x, y) = (f(x, y), y).$$

Thus $f(x, y) = 0$. If also $(x', y) \in U$ satisfies $f(x', y) = 0$, then $F(x', y) = (0, y) = F(x, y)$, and injectivity of F on U forces $x' = x$. This proves existence and uniqueness of x for each $y \in W$.

(ii) Define $g : W \to \mathbb{R}^n$ by declaring $g(y)$ to be the unique x with $(x, y) \in U$ and $f(x, y) = 0$. Then $(g(y), y) \in U$ and

$$F(g(y), y) = (0, y), \qquad y \in W.$$

Equivalently, $F^{-1}(0, y) = (g(y), y)$ for $y \in W$. Since F^{-1} is C^1, it follows that g is C^1. To compute $g'(y_0)$, differentiate the identity $f(g(y), y) = 0$ at $y = y_0$ and use the chain rule:

$$0 = \frac{d}{dy} f(g(y), y)\Big|_{y=y_0} = A_x \, g'(y_0) + A_y.$$

Since A_x is invertible, this gives $g'(y_0) = -A_x^{-1} A_y$.

<div align="right">□</div>

Exercise 8.5. Prove that $x^2 + ye^x - \sin(xy) = 0$ can be solved for y in a neighborhood of $(0, 0)$, but cannot be solved for x in any neighborhood of $(0, 0)$.

$$F(x, y) = x^2 + ye^x - \sin(xy)$$

(i) $F(0, 0) = 0$, $\frac{\partial F}{\partial y}\big|_{(0,0)} = 1 \neq 0$. By the implicit function theorem, there exists a neighborhood of $(0, 0)$ and a function g such that

$$F(x, g(x)) = 0$$

for $|x| < r$, equivalently $y = g(x)$ near 0.

(ii) $\frac{\partial F}{\partial x}\big|_{(0,0)} = 0$. Hence, the implicit function theorem *cannot* be applied.

On the contrary, suppose that $x = \phi(y)$ near $y = 0$. Then $\phi(0) = 0$, and

$$(\phi(y))^2 + ye^{\phi(y)} - \sin(\phi(y)y) = 0$$

for $|y| < r$ for some $r > 0$. Then

$$2\phi(0)\phi'(0) + 1 \cdot e^{\phi(0)} + 0 \cdot e^{\phi(0)}\phi'(0) - \cos(\phi(0)0)\left(\phi'(0)0 + \phi(0) \cdot 1\right) = 0$$

$$\implies 1 = 0 \quad \text{(contradiction)}$$

Example 8.41. Let $f : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$

$$f(x, y, z) = (xe^y + ye^z, \ xe^z + ze^y)$$

Then f is a C^1-map.

$$f'(x, y, z) = \begin{pmatrix} e^y & xe^y + e^z & ye^z \\ e^z & ze^y & xe^z + e^y \end{pmatrix}$$

$$f(-1, 1, 1) = (0, 0)$$

Let $f = (f_1, f_2)$. Then

$$\begin{pmatrix} \dfrac{\partial f_1}{\partial y} & \dfrac{\partial f_1}{\partial z} \\ \dfrac{\partial f_2}{\partial y} & \dfrac{\partial f_2}{\partial z} \end{pmatrix}(-1, 1, 1) = \begin{pmatrix} 0 & e \\ e & 0 \end{pmatrix}$$

By the implicit function theorem, there exists an open ball U in $\mathbb{R}^3$ and open ball V in $\mathbb{R}^2$, such that
$$(y, z) = (\phi(x), \psi(x)), \quad |x| < r \quad \text{for some } r > 0.$$

Exercise 8.6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^1-map such that $f(0,0) = 0$, $f_x(0,0) = 1$. Let $F(x, y) = (f(x, y), y)$. Prove that F is injective in some neighborhood of $(0,0)$. Does F remain injective in any neighborhood of $(0,0)$?

Remark: Condition in implicit function theorem or inverse mapping theorem on derivatives are sufficient.

Example 8.42. $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^2 - y^3$.

$$f(0,0) = 0,$$
$$\frac{\partial f}{\partial y}(0,0) = 0,$$

but $y = x^{2/3}$ is a solution of $f(x, y) = 0$ near $(0,0)$.

Example 8.43. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(x, y) = (x^3, y^3)$. Then $\det f'(0,0) = 0$ but f is one-to-one, onto.

8.5   Extrema, the Hessian test, and Lagrange multipliers

Many applications of multivariable calculus involve optimizing a function under no constraints (unconstrained extrema) or under one or more constraints (constrained extrema). The correct language is geometric: level sets and gradients.

8.5.1   Unconstrained extrema and critical points

Let $f : \mathbb{R}^n \to \mathbb{R}$ and let $a \in \mathbb{R}^n$. We say that a is a *local maximizer* of f if there exists $r > 0$ such that $f(a) \geq f(x)$ for all $x \in B_r(a)$. Local minimizers are defined analogously.

Theorem 8.44 (Fermat's theorem for C^1 functions). *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}^n$. If a is a local maximizer or local minimizer of f, then $\nabla f(a) = 0$.*

Proof. Fix any unit vector $v \in \mathbb{R}^n$ and consider the one-variable function $\varphi(t) = f(a + tv)$ for t near 0. If a is a local extremum of f, then $t = 0$ is a local extremum of φ. Since φ is differentiable at 0, we have $\varphi'(0) = 0$. By the chain rule, $\varphi'(0) = \nabla f(a) \cdot v$. Because v is arbitrary, it follows that $\nabla f(a) = 0$. $\qquad\square$

Points a with $\nabla f(a) = 0$ are called *critical points*. Fermat's theorem says that local extrema can occur only at critical points (or on the boundary of the domain).

8.5.2 The Hessian and the second derivative test

Assume $f \in C^2$ in a neighborhood of $a \in \mathbb{R}^n$. The *Hessian matrix* at a is

$$H_f(a) := \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(a) \right]_{i,j=1}^{n}.$$

It is symmetric when $f \in C^2$, and thus determines a quadratic form $q(h) = \frac{1}{2} h^{\mathsf{T}} H_f(a) h$.

Theorem 8.45 (Second derivative test)**.** *Let $f \in C^2$ in a neighborhood of a and assume $\nabla f(a) = 0$.*

  (i) If $H_f(a)$ is positive definite, then a is a strict local minimizer.

  (ii) If $H_f(a)$ is negative definite, then a is a strict local maximizer.

  (iii) If $H_f(a)$ is indefinite, then a is a saddle point.

Proof. By Taylor's theorem with remainder (in $\mathbb{R}^n$),

$$f(a+h) = f(a) + \nabla f(a) \cdot h + \tfrac{1}{2} h^{\mathsf{T}} H_f(a) h + o(\|h\|^2) \qquad (h \to 0).$$

Since $\nabla f(a) = 0$, the sign of $f(a+h) - f(a)$ for sufficiently small h is governed by the quadratic form $h^{\mathsf{T}} H_f(a) h$, yielding the three cases. $\qquad\square$

8.5.3 Constrained extrema and the Lagrange multiplier condition

We now optimize f subject to a constraint $g(x) = 0$. Geometrically, the constraint set $M = \{x : g(x) = 0\}$ is (under mild hypotheses) a smooth hypersurface, and the gradient ∇g is normal to its level sets. At an extremum of f on M, the level set of f must be tangent to M, forcing the gradients to be parallel.
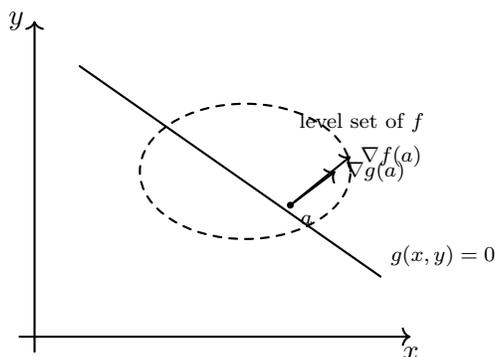
Figure 8.9. At a constrained extremum, the level set of f is tangent to the constraint set $g = 0$, so ∇f is parallel to ∇g.

Theorem 8.46 (Lagrange multiplier condition)**.** *Let* $f, g : \mathbb{R}^n \to \mathbb{R}$ *be* C^1 *in a neighborhood of* $a \in \mathbb{R}^n$. *Assume* $g(a) = 0$ *and* $\nabla g(a) \neq 0$. *If* a *is a local extremum of* f *restricted to the constraint set* $M = \{x : g(x) = 0\}$, *then there exists* $\lambda \in \mathbb{R}$ *such that*

$$\nabla f(a) = \lambda \nabla g(a).$$

Proof strategy. Use the implicit function theorem to straighten the constraint set into a graph. After this reduction, the constrained optimization problem becomes an unconstrained problem in one fewer variable, where Fermat's theorem applies directly.

Proof sketch. Since $\nabla g(a) \neq 0$, at least one partial derivative of g is nonzero at a. By the implicit function theorem, near a the constraint set M can be represented as the graph of a C^1 function over $(n-1)$ variables. Restricting f to this graph reduces the problem to an unconstrained extremum in $\mathbb{R}^{n-1}$, where Fermat's theorem applies. Translating the resulting equations back to $\mathbb{R}^n$ yields the stated proportionality of gradients. $\square$

Remark 8.47. The condition $\nabla g(a) \neq 0$ is essential: without it, the constraint set may have a corner or self-intersection at a, and the geometric tangency argument can fail.

Example 8.48 (A standard minimization problem)**.** Minimize $f(x, y) = x^2 + y^2$ subject to $g(x, y) = x + y - 1 = 0$. We have $\nabla f = (2x, 2y)$ and $\nabla g = (1, 1)$. The Lagrange condition gives $(2x, 2y) = \lambda(1, 1)$, hence $x = y$. Imposing $x + y = 1$ yields $x = y = \frac{1}{2}$. Thus the minimum of $x^2 + y^2$ on the line $x + y = 1$ occurs at $\left(\frac{1}{2}, \frac{1}{2}\right)$.

Exercises

Exercise 8.7. Prove that if $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a, then f is continuous at a.

Exercise 8.8. Give an example of a function $f : \mathbb{R}^2 \to \mathbb{R}$ whose partial derivatives exist everywhere but which is not differentiable at $(0, 0)$.

Exercise 8.9. Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable. Prove that the directional derivative in direction v equals $Df(a)\, v$.

Exercise 8.10. Prove the multivariable chain rule using linear approximation.

Exercise 8.11. Let $f : \mathbb{R}^n \to \mathbb{R}$ be C^2. Prove that if the Hessian at a is positive definite then a is a strict local minimum.

Exercise 8.12. Compute the Jacobian determinant of the polar coordinate map $(r, \theta) \mapsto (r\cos\theta, r\sin\theta)$ and deduce the change-of-variables formula for integrals over planar regions.

Exercise 8.13. State and prove the Implicit Function Theorem in $\mathbb{R}^2$ (one equation, two variables) and apply it to show that the unit circle is locally the graph of a smooth function away from the points where $y = 0$.

Exercise 8.14. Use the Inverse Function Theorem to prove that a C^1 map with everywhere invertible derivative is locally open.

Exercise 8.15. Apply Lagrange multipliers to find the extrema of $f(x,y) = xy$ subject to $x^2 + y^2 = 1$.

Exercise 8.16. Let $g : \mathbb{R}^n \to \mathbb{R}$ be C^1 with $\nabla g(a) \neq 0$. Show that $\{x : g(x) = g(a)\}$ is locally a smooth hypersurface through a.

Chapter 9

# Multiple Integration and Vector Calculus

This chapter develops the Riemann theory of double and triple integrals, repeated integration and Fubini's theorem, and change of variables. We then introduce surface and line integrals, and conclude with Green's theorem, the divergence theorem (Gauss), and Stokes' theorem.

Learning objectives.

- Pass from one-variable integration to multiple integration with full control over iterated integrals and geometry of domains.

- Compute line, surface, and volume integrals while keeping track of orientation and parametrization.

- Interpret divergence, curl, and the integral theorems of vector calculus as global manifestations of local differential data.

9.1 A brief review of the Riemann integral in one variable

Let $f : [a, b] \to \mathbb{R}$ be a bounded function and let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$, where $\{a = x_0 < x_1 < \cdots < x_n = b\}$.
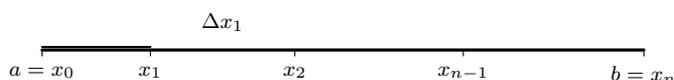
Figure 9.1. A partition P of $[a, b]$. The mesh $\|P\| = \max_i(x_i - x_{i-1})$ measures the fineness of the partition.

Let $\Delta x_i = x_i - x_{i-1}$. Define $m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}$ and $M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\}$ Write

$$L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i \text{ and } U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i.$$

Since f is bounded, there exist real numbers $m \leq M$ such that $m \leq f(x) \leq M$ for all $x \in [a, b]$. Hence

$$m(b - a) \leq L(P, f) \leq U(P, f) \leq M(b - a).$$

It is easy to see that if $P_1 \subseteq P_2$, then $U(P_1, f) \geq U(P_2, f)$ and $L(P_1, f) \leq L(P_2, f)$. It is clear that $L(P, f)$ is an increasing function over the set of all finer partitions while $U(P, f)$ is a decreasing function of P.

Definition 9.1. The function f is said to be Riemann integrable (or $f \in \mathcal{R}[a,b]$) if

$$\inf_P U(P,f) = \sup_P L(P,f).$$

Let $\omega(P,f) = U(P,f) - L(P,f)$. From the definition, it follows that

$$\inf_P \omega(P,f) = \inf_P \{U(P,f) - L(P,f)\} = 0, \tag{9.1}$$

where $\omega(P,f)$ is known as oscillatory sum of f over the partition P. Hence, if $f \in \mathcal{R}[a,b]$, then for each $\epsilon > 0$, there exists a partition P such that $\omega(P,f) < \epsilon$. On the other hand, for $\epsilon = \frac{1}{n}$, $n \in \mathbb{N}$, there exists a partition P_n such that $\omega(P_n,f) < \frac{1}{n}$. Thus, $\lim_{n\to\infty} \omega(P_n,f) = 0$.

Theorem 9.2. *Let $f : [a,b] \to \mathbb{R}$ be bounded. Then $f \in \mathcal{R}[a,b]$ if and only if there exists a sequence $\{P_n\}$ of partitions of $[a,b]$ such that $\lim_{n\to\infty} \omega(P_n,f) = 0$.*

Proof. We have already seen the forward implication. For the other one, if $\lim_{n\to\infty} \omega(P_n,f) = 0$, then for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $\omega(P_n,f) < \epsilon$, whenever $n \geq n_0$. But, then $\inf_P \omega(P,f) \leq \omega(P_{n_0},f) < \epsilon$ for all $\epsilon > 0$. Since f is bounded, both $\inf_P U(P,f)$ and $\sup_P L(P,f)$ exist, and from (9.1) it follows that $\inf_P U(P,f) = \sup_P L(P,f)$. Hence $f \in \mathcal{R}[a,b]$. $\qquad\square$

Example 9.3. Let $f : [0,1] \to \mathbb{R}$ is given by

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Then f is bounded and for $P_n = \{\frac{i}{n} : i = 0, 1, \ldots, n\}$, we have

$$\omega(P_n,f) = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i \leq 2.\frac{1}{n} \to 0,$$

since $\frac{1}{2}$ can belong to two consecutive subintervals. Hence $f \in \mathcal{R}[0,1]$.

Recall that if $P_1 \subseteq P_2$, then $U(P_1,f) \geq U(P_2,f)$ and $L(P_1,f) \leq L(P_2,f)$. Hence $\omega(P_1,f) \geq \omega(P_2,f)$. Using this fact, it is enough to workout $\lim_{n\to\infty} \omega(P_n,f) = 0$, while $\{P_n\}$ is an increasing sequence of partitions.

Theorem 9.4. *Let $f : [a,b] \to \mathbb{R}$ be bounded. Then $f \in \mathcal{R}[a,b]$ if and only if there exists an increasing sequence of partitions $\{P_n\}$ of $[a,b]$ such that $\lim_{n\to\infty} \omega(P_n,f) = 0$.*

Proof. Since $f \in \mathcal{R}[a,b]$, by Theorem 9.2, there exists a sequence of partition $\{P_n\}$ such that $\lim_{n\to\infty} \omega(P_n,f) = 0$. Let $Q_1 = P_1$ and $Q_n = P_1 \cup P_2 \cup \cdots \cup P_n$. Then $\omega(Q_n,f) \leq \omega(P_n,f) \to 0$. The converse part is obvious from Theorem 9.2. $\qquad\square$

Remark 9.5. From Theorem 9.4 it follows that $\lim_{n\to\infty} U(P_n,f) = \lim_{n\to\infty} L(P_n,f) = \int_a^b f(x)dx$.

Theorem 9.6. *If $f : [a, b] \to \mathbb{R}$ is continuous, then $f \in \mathcal{R}[a, b]$.*

Proof. Since f is continuous on the closed interval $[a, b]$, f is bounded and uniformly continuous. For each $\epsilon > 0$, there exists $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \frac{\epsilon}{2(b-a)}$. Choose a partition P of $[a, b]$ such that $\Delta x_i < \delta$. Since f attains its infimum and supremum on each subinterval, we get $M_i - m_i \leq \frac{\epsilon}{2(b-a)}$. Hence

$$\omega(P, f) = \sum_{i=1}^{n}(M_i - m_i)\Delta x_i \leq \sum_{i=1}^{n}\frac{\epsilon}{2(b-a)}\Delta x_i < \epsilon.$$

$\square$

Example 9.7. Every monotone function f on $[a, b]$ is Riemann integrable. Assume f is monotone increasing. Let $P_n = \left\{x_i = a + \frac{(b-a)i}{n} : i = 0, 1, \ldots, n\right\}$. Then the oscillatory sum

$$\omega(P_n, f) = \sum_{i=1}^{n}(M_i - m_i)\Delta x_i = \sum_{i=1}^{n}\{f(x_i) - f(x_{i-1})\}\frac{b-a}{n} = \{f(b) - f(a)\}\frac{b-a}{n} \to 0.$$

Hence by Theorem 9.4 we conclude that $f \in \mathcal{R}[a, b]$.

Continuity like condition for Riemann integrability on $[a, b]$.

We know that the oscillatory sum $\omega(P, f)$ decreases over the set of finer partitions. And f is Riemann integrable if and only if there is a sequence of partitions $\{P_n\}$ such that $\omega(P_n, f) \to 0$. Using this fact, we derive a continuity like condition for Riemann integrability of bounded function on $[a, b]$. For a given partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$, we define $|P| = \max_{1 \leq i \leq n} \Delta x_i$, where $\Delta x_i = x_i - x_{i-1}$.

Theorem 9.8. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then $f \in \mathcal{R}([a, b])$ if and only if for each $\epsilon > 0$, there exists $\delta > 0$ such that for each partition P with $|P| < \delta$ implies $\omega(P, f) < \epsilon$.*

Proof. Since f is Riemann integrable, for each $\epsilon > 0$ there exists a partition P of $[a, b]$ such that $\omega(P, f) < \epsilon$. Let $\delta > 0$ be small enough and P' be a refinement of P such that $|P'| < \delta$. As $P \subseteq P'$, it follows that $\omega(P', f) \leq \omega(P, f) < \epsilon$. The other implication is obvious by definition of $\mathcal{R}([a, b])$. $\square$

Corollary 9.9. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then $f \in \mathcal{R}([a, b])$ if and only if for each sequence of partitions $\{P_n\}$ with $|P_n| \to 0$ implies $\omega(P_n, f) \to 0$.*

Question*. Think about, how far can be a Riemann integrable function from continuous function.

9.2   Double integrals

Section overview.

- The passage from one variable to two variables introduces geometry of domains as an essential part of integration.

- Iterated integrals are not merely computational tools; they encode the structure of the domain through slicing.

- Sketching the region before computing is part of the proof-oriented method, not an optional extra.

We know that the Riemann integral of a non-negative function of one variable on a finite interval is the area of the region under the graph of the function. In a similar way, the double integral of a non-negative function $f(x, y)$ defined on a region in the plane is the volume of the region under the graph of $f(x, y)$.

First, we discuss double integral on the rectangular region, and later we consider more general region with curvilinear boundary.

Let $D = [a, b] \times [c, d]$ and $f : D \to \mathbb{R}$ be bounded. Let $P_1 = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$ and $P_2 = \{y_0, y_1, \ldots, y_m\}$ be a partition of $[c, d]$. Note that the partition $P = P_1 \times P_2$ decomposes D into mn sub-rectangles (or cells). Let $D_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$. Let $m_{ij} = \inf\{f(x, y) : (x, y) \in D_{ij}\}$. Define

$$L(P, f) = \sum_{i=1}^{n} \sum_{j=1}^{m} m_{ij} \Delta x_i \Delta y_j.$$

Similarly, we can define

$$U(P, f) = \sum_{i=1}^{n} \sum_{j=1}^{m} M_{ij} \Delta x_i \Delta y_j,$$

where $M_{ij} = \sup\{f(x, y) : (x, y) \in D_{ij}\}$. The lower integral of f is defined by $\sup_{P} L(P, f)$. The upper integral of f is defined by $\inf_{P} U(P, f)$. Note that both the integrals exist because f is bounded. We say that f is integrable on D (or $f \in \mathcal{R}(D)$) if both lower and upper integrals of f are equal. If the function f is integrable on D, then the **double integral** is denoted by

$$\iint_{D} f(x, y) dx dy \text{ or } \iint_{D} f(x, y) dA.$$

Example 9.10. Let $f : D = [0, 1] \times [0, 1] \to \mathbb{R}$ is given by

$$f(x, y) = \begin{cases} 1 & \text{if } x, y \in \mathbb{Q} \cap [0, 1], \\ 0 & \text{otherwise.} \end{cases}$$

Then f is not integrable on D, because for any partition P of D defined as above, we get $U(P, f) = 1 \neq 0 = L(P, f)$.

Theorem 9.11. *Let $f : D = [a, b] \times [c, d] \to \mathbb{R}$ be bounded. Then $f \in \mathcal{R}(D)$ if and only if for each $\epsilon > 0$ there exists a partition P of D such that $\omega(P, f) = U(P, f) - L(P, f) < \epsilon$.*

Theorem 9.12. *Let $f : D = [a, b] \times [c, d] \to \mathbb{R}$ be bounded. Then $f \in \mathcal{R}(D)$ if and only if there exists an increasing sequence of partitions $\{P_n\}$ of D such that $\lim_{n \to \infty} \omega(P_n, f) = 0$.*

Since the proof of Theorem 9.12 is similar to Theorem 9.4, we omit here.

Example 9.13. Let $f : D = [0, 1] \times [0, 1] \to \mathbb{R}$ is given by

$$f(x, y) = \begin{cases} 0 & \text{if } x \neq y \\ 1 & \text{if } x = y. \end{cases}$$

Then $\iint\limits_{D} f(x, y) dx dy = 0$. Let $P_n = \{\frac{i}{n} : i = 0, 1, \ldots, n\} \times \{\frac{i}{n} : i = 0, 1, \ldots, n\}$. In this case, $\Delta x_i = \Delta y_j = \frac{1}{n}$. The oscillatory sum of the function f on D satisfies

$$\omega(P_n, f) = \sum_{i=1}^{n} \sum_{j=1}^{n} (M_{ij} - m_{ij}) \Delta x_i \Delta y_j = \sum_{i=1}^{n} \sum_{j=1}^{n} (M_{ij} - 0) \frac{1}{n^2} = \sum_{i=j, \ j=1}^{n} 1 \cdot \frac{1}{n^2} = \frac{1}{n} \to 0.$$

Theorem 9.14. *Let $D = [a, b] \times [c, d]$. If $f : D \to \mathbb{R}$ is continuous, then f is integrable on D.*

Proof. Since f is continuous on the closed rectangle D, it follows that f is bounded and uniformly continuous on D. Hence for given $\epsilon > 0$ there exists $\delta >$ such that for (x, y), $(x', y') \in D$ with $\sqrt{(x - x')^2 + (y - y')^2} < \delta$ implies $|f(x, y) - f(x', y')| < \frac{\epsilon}{2A}$, where A is the area of the rectangle D. Let $P = \{D_{ij} : i = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, m\}$, where $D_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$. Write $d(D_{ij}) = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{j-1})^2}$. Now, suppose P satisfies $d(D_{ij}) < \delta$ for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$. Since f attains its infimum and supremum on each closed cell D_{ij}, we get $M_{ij} - m_{ij} \leq \frac{\epsilon}{2A}$. Hence

$$\omega(P, f) = \sum_{i=1}^{n} \sum_{i=1}^{m} (M_{ij} - m_{ij}) \Delta x_i \Delta y_i \leq \sum_{i=1}^{n} \sum_{i=1}^{m} \frac{\epsilon}{2A} \Delta x_i \Delta y_j < \epsilon.$$

Hence by Theorem 9.12, we conclude that $f \in \mathcal{R}(D)$. $\qquad \square$

Continuity like condition for Riemann integrability
Let $P = \{D_{ij} : i = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, m\}$ be a partition of $D = [a, b] \times [c, d]$, where $D_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$. Write $d(D_{ij}) = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{j-1})^2}$. Define $|P| = \max\{d(D_{ij}) : i = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, m\}$.

Theorem 9.15. *Let $f : D = [a, b] \times [c, d] \to \mathbb{R}$ be a bounded function. Then $f \in \mathcal{R}(D)$ if and only if for each $\epsilon > 0$ there exists $\delta > 0$ such that for each partition P of D with $|P| < \delta$ implies $\omega(P, f) < \epsilon$.*

Since the proof of Theorem 9.15 is similar to Theorem 9.8, we omit here.

Corollary 9.16. *Let $f : D = [a, b] \times [c, d] \to \mathbb{R}$ be a bounded function. Then $f \in \mathcal{R}(D)$ if and only if for each sequence of partitions $\{P_n\}$ of D with $|P_n| \to 0$ implies $\omega(P_n, f) \to 0$.*

Note that in order to show $f \notin \mathcal{R}(D)$, it is enough to show that there exists a sequence of partitions $\{P_n\}$ with $|P_n| \to 0$ but $\omega(P_n, f) \not\to 0$.

Geometric interpretation.   If $f : D = [a,b] \times [c,d] \to [0,\infty)$ is integrable, then

$$\iint\limits_{D} f(x,y)\, dx\, dy$$

represents the volume of the solid bounded by the planes $x = a$, $x = b$, $y = c$, $y = d$, and the surface $z = f(x,y)$.

9.2.1   Repeated Integrals

The next result shows that a suitable double integral can be evaluated by means of repeated integrals. This is the content of Fubini's theorem. Before stating the theorem, let us examine two instructive examples.

Example 9.17. Consider the function $f : D = [0,1] \times [0,1] \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \cap [0,1], \\ 2y, & \text{if } x \in \mathbb{Q}^c \cap [0,1]. \end{cases}$$

Then

$$\int_0^1 \left(\int_0^1 f(x,y)\, dy \right) dx = 1.$$

However, f is not integrable on D. (**Hint:** Use Corollary 9.16 to deduce that $f \notin \mathcal{R}(D)$.)

Example 9.18. Consider the function $f : D = [0,1] \times [0,1] \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} 0, & \text{if } x \neq \frac{1}{2}, \\ 1, & \text{if } x = \frac{1}{2} \text{ and } y \in \mathbb{Q} \cap [0,1], \\ -1, & \text{if } x = \frac{1}{2} \text{ and } y \in \mathbb{Q}^c \cap [0,1]. \end{cases}$$

For $x = \frac{1}{2}$, the inner integral $\int_0^1 f(x,y)\, dy$ does not exist. Nevertheless, the double integral $\iint\limits_{D} f(x,y)\, dx\, dy$ does exist.

Theorem 9.19 (Fubini's Theorem). *Let $f : D = [a,b] \times [c,d] \to [0,\infty)$ be integrable. If for each $y \in [c,d]$, the function $f(\cdot, y) \in \mathcal{R}[a,b]$, then the function F defined by $F(y) = \int\limits_a^b f(x,y)dx$ is integrable on $[c,d]$ and*

$$\iint\limits_{D} f(x,y)dxdy = \int_c^d \left(\int_a^b f(x,y)dx \right) dy.$$

Proof. Since $f \in \mathcal{R}(D)$, for each $\epsilon > 0$ there exists a partition

$$P = P_1 \times P_2 = \{D_{ij} : \ i = 1, 2, \ldots, n \text{ and } j = 1, 2, \ldots, m\}$$

of D such that $U(P, f) - L(P, f) < \epsilon$. Recall that $m_{ij} = \inf\{f(x, y) : (x, y) \in D_{ij}\}$ and $M_{ij} = \sup\{f(x, y) : (x, y) \in D_{ij}\}$. Let us define $k_j = \inf\{F(y) : y_{j-1} \leq y \leq y_j\}$ and $K_j = \sup\{F(y) : y_{j-1} \leq y \leq y_j\}$. Since $m_{ij} \leq f(x, y) \leq M_{ij}$ for each $(x, y) \in D_{ij}$, it follows that

$$\sum_{i=1}^{n} m_{ij}\Delta x_i \leq L(P_1, f(\cdot, y)) \leq \int_a^b f(x, y)dx = F(y) \leq \sum_{i=1}^{n} M_{ij}\Delta x_i \qquad (9.2)$$

for each $y \in [y_{j-1}, y_j]$. Note the **first inequality** in (9.2) follows due to the fact that infimum m_{ij} of f on D_{ij} is smaller than the infimum of f over $[x_{i-1}, x_i] \times \{y\}$.

From the above it follows that

$$L(P, f) = \sum_{i=1}^{n}\sum_{j=1}^{m} m_{ij}\Delta x_i \Delta y_j \leq \sum_{j=1}^{n} k_j \Delta y_j = L(P_2, F) \leq U(P_2, F)$$

and

$$U(P_2, F) = \sum_{j=1}^{n} K_j \Delta y_j \leq \sum_{i=1}^{n}\sum_{j=1}^{m} M_{ij}\Delta x_i \Delta y_j = U(P, f).$$

Hence

$$L(P, f) \leq L(P_2, F) \leq U(P_2, F) \leq U(P, f). \qquad (9.3)$$

Since $U(P, f) - L(P, f) < \epsilon$, from (9.3) we get $U(P_2, F) - L(P_2, F) < \epsilon$. That is, $F \in \mathcal{R}[c, d]$, and hence once again from (9.3) we infer that

$$L(P, f) \leq \int_c^d F(y)dy \leq U(P, f) \text{ and } L(P, f) \leq \iint_D f(x, y)dxdy \leq U(P, f).$$

Thus,

$$-\epsilon < \iint_D f(x, y)dxdy - \int_c^d F(y)dy \leq \epsilon$$

for each $\epsilon > 0$. Hence

$$\iint_D f(x, y)dxdy = \int_c^d F(y)dy.$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

If we instead define $G(x) = \int_c^d f(x, y)\,dy$, then an analogous statement also holds.

Corollary 9.20 (Fubini's Theorem). *Let $f : D = [a, b] \times [c, d] \to \mathbb{R}$ be a continuous function. Then*

$$\iint_D f(x, y)dxdy = \int_c^d \left(\int_a^b f(x, y)dx \right) dy = \int_a^b \left(\int_c^d f(x, y)dy \right) dx.$$

Example 9.21. Let $f(x,y) = xe^{xy}$ for $(x,y) \in D = [0,2] \times [0,1]$. Then f is continuous and hence by Fubini's theorem

$$\iint\limits_{D} f(x,y)dxdy = \int_0^2 \left(\int_0^1 xe^{xy}dy \right) dx = \int_0^2 [e^{xy}]_0^1\, dx = \int_0^2 (e^x - 1)dx = e^2 - 3.$$

9.2.2 Bounded functions with discontinuities

We know from Theorem 9.14 that if f is continuous on D then f is integrable. In this section, we discuss that the integral of a function f also exists if the set of discontinuities of f is not too large. In order to measure discontinuities, we introduce the following concept.

Definition 9.22. Let A be a bounded subset of $\mathbb{R}^2$. We say that A has *content zero* if, for every $\epsilon > 0$, there exist finitely many rectangles $\{R_i\}_{i=1}^n$ such that

$$A \subseteq \bigcup_{i=1}^n R_i \qquad \text{and} \qquad \text{Area}\left(\bigcup_{i=1}^n R_i \right) < \epsilon.$$

Example 9.23. (i) Any finite set of points in $\mathbb{R}^2$ has content zero.

(ii) Every subset of a set of content zero has content zero.

(iii) The union of finitely many bounded sets of content zero is also of content zero.

(iv) Every line segment has content zero.

Exercise 9.1. Any bounded subset of $\mathbb{R}^2$ having nonempty interior cannot have content zero.

Theorem 9.24. *Let $f : D = [a,b] \times [c,d] \to \mathbb{R}$ be a bounded function. If the set of discontinuities of f in D is a set of content zero, then f is integrable.*

Proof. Let $M > 0$ be such that $|f(x,y)| \leq M$ for all $(x,y) \in D$. Suppose E is the set of discontinuities of f in D. Let

$$P = \{D_i : D_i \text{ is a subrectangle of } D\}$$

be a partition of D, and write

$$m_i = \inf_{D_i} f, \qquad M_i = \sup_{D_i} f, \qquad A(D_i) = \text{Area}(D_i).$$

Since E has content zero, we can choose finitely many subrectangles $D_1, \ldots, D_m$ such that

$$E \subset \bigcup_{i=1}^m D_i \qquad \text{and} \qquad \sum_{i=1}^m A(D_i) < \frac{\epsilon}{4M}.$$

On each of the remaining closed subrectangles D_i $(i = m+1, \ldots, n)$, the function f is continuous and hence uniformly continuous. Therefore, by refining the partition if necessary, we may arrange that

$$M_i - m_i \leq \frac{\epsilon}{2A(D)} \qquad (i = m+1, \ldots, n).$$

Hence

$$
\begin{aligned}
\omega(P, f) &= \sum_{i=1}^{n} (M_i - m_i) A(D_i) \\
&= \sum_{i=1}^{m} (M_i - m_i) A(D_i) + \sum_{i=m+1}^{n} (M_i - m_i) A(D_i) \\
&\leq \sum_{i=1}^{m} 2M A(D_i) + \sum_{i=m+1}^{n} \frac{\epsilon}{2A(D)} A(D_i) \\
&< 2M \frac{\epsilon}{4M} + \frac{\epsilon}{2} \frac{A(D)}{A(D)} = \epsilon.
\end{aligned}
$$

Thus, for each $\epsilon > 0$ we have constructed a partition P of D such that $\omega(P, f) < \epsilon$. This implies $f \in \mathcal{R}(D)$. $\qquad\square$

9.2.3 Double integral over general bounded regions

Let D be a bounded region in $\mathbb{R}^2$, and let $f : D \to \mathbb{R}$ be bounded. Choose a rectangle Q with $D \subseteq Q$, and extend f to a function $\tilde{f} : Q \to \mathbb{R}$ by

$$
\tilde{f}(x, y) = \begin{cases} f(x, y), & \text{if } (x, y) \in D, \\ 0, & \text{if } (x, y) \in Q \setminus D. \end{cases}
$$

If $\tilde{f}$ is integrable over Q, then we say that f is integrable over D and define

$$
\iint_D f(x, y)\, dx\, dy = \iint_Q \tilde{f}(x, y)\, dx\, dy.
$$

Theorem 9.25. (Fubini's Theorem) Let f be a bounded continuous function over a bounded region D in $\mathbb{R}^2$.

  (i) If $D = \{(x, y) : a \leq x \leq b \text{ and } f_1(x) \leq y \leq f_2(x)\}$ for some continuous functions $f_1, f_2 : [a, b] \to \mathbb{R}$, then

$$
\iint_D f(x, y)\, dx\, dy = \int_a^b \left(\int_{f_1(x)}^{f_2(x)} f(x, y)\, dy \right) dx.
$$

(ii) If $D = \{(x, y) : c \leq y \leq d \text{ and } g_1(y) \leq x \leq g_2(y)\}$ for some continuous functions $g_1, g_2 : [c, d] \to \mathbb{R}$, then

$$\iint_D f(x, y)dxdy = \int_c^d \left(\int_{g_1(y)}^{g_2(y)} f(x, y)dx \right) dy.$$

For a proof of Theorem 9.25, we refer to Chapter 11, Calculus Vol. II, by Apostol.

Example 9.26. (i) Let D be the region bounded by the lines joining the points $(0, 0), (0, 1)$ and $(2, 2)$. Evaluate the integral $\iint_D (x + y)^2 dxdy$.

(ii) Evaluate the integral $\int_0^2 \left(\int_{\frac{y}{2}}^1 e^{x^2} dx \right) dy$.

Riemann integrable functions on D satisfy the following basic algebraic relations.

Theorem 9.27. *Let f and g be Riemann integrable functions on the region D in the plane and $c \in \mathbb{R}$. Then*

(i) $cf + g \in R(D), \; \iint_D \{cf(x, y) + g(x, y)\}dxdy = c \iint_D f(x, y)dxdy + \iint_D g(x, y)dxdy.$

(ii) *If* $f(x, y) \leq g(x, y)$ *for all* $(x, y) \in D$, *then* $\iint_D f(x, y)dxdy \leq \iint_D g(x, y)dxdy.$

(iii) $|f| \in \mathcal{R}(D)$ *and* $\left| \iint_D f(x, y)dxdy \right| \leq \iint_D |f(x, y)|dxdy.$

9.2.4   Change of variable

Change of variables formula is one of the most important results in multivariable calculus. The reason that many problems have a natural coordinate system, and if we look from the right perspective, the calculation gets considerably simplified. As an effect, making the function and the domain of integration simpler.

9.2.5   Change of variable in single integral

Let $f : [a, b] \to \mathbb{R}$ be an integrable function. Suppose $g : [c, d] \to [a, b]$ is continuously differentiable function (C^1 - function) such that $g'(t) \neq 0$ for all $t \in (c, d)$. Then g is one to one (by Mean Value Theorem) and hence monotone. We also assume that g is surjective. Put $x = g(t)$. Then $dx = g'(t)dt$. If g is monotone increasing, then

$$\int_a^b f(x)dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(g(t))g'(t)dt = \int_c^d f(g(t))g'(t)dt.$$

In case, if g is monotone decreasing, then $[c, d] = [g^{-1}(b), g^{-1}(a)]$. Thus, we have the formula

$$\int_a^b f(x)dx = \int_c^d f(g(t))|g'(t)|dt.$$

9.2.6 Change of variable in double integral

We now extend the one-variable change-of-variables principle to the setting of double integrals. A complete proof in higher dimensions requires additional material from advanced analysis and differential calculus, so we state the result here and focus on its interpretation and use.

Suppose S is a bounded region in the uv-plane which is transformed onto the bounded region D in the xy-plane by $x = \varphi(u, v)$ and $y = \psi(u, v)$. Now, consider mapping T from D to S which is bijective, continuously differentiable and it's inverse $T^{-1} : S \to D$ is defined by $T^{-1}(u, v) = (\varphi(u, v), \psi(u, v))$. See fig. 9.2.
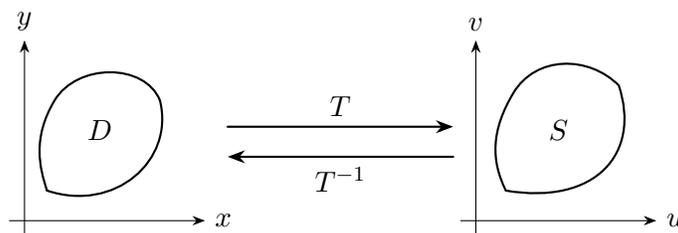
Figure 9.2. A bijective C^1 change of variables $T : D \to S$ and its inverse.

Assume further that T^{-1} is also continuously differentiable and that its derivative $(T^{-1})'$ is invertible (that is, nonsingular) on the interior of S. Then

$$\det(T^{-1})' = \begin{vmatrix} \frac{\partial \varphi}{\partial u} & \frac{\partial \varphi}{\partial v} \\ \frac{\partial \psi}{\partial u} & \frac{\partial \psi}{\partial v} \end{vmatrix} = J(u, v) \neq 0 \ ,$$

where function J is known as Jacobi of the transformation. In this way, the function $f(x, y)$ defined on D can be thought of as a function $f(\varphi(u, v), \psi(u, v))$ defined on S.

Theorem 9.28. *Let $f(x, y) : D \to \mathbb{R}$ be continuous function and J be as defined above. Then the integral of $f(x, y)$ over D and the integral of $f(\varphi(u, v), \psi(u, v))$ over S are related by*

$$\iint\limits_{D} f(x, y)dxdy = \iint\limits_{S} f[\varphi(u, v), \psi(u, v)] \, |J(u, v)|dudv.$$

Example 9.29. Find the area of the region D bounded by the hyperbolas $xy = 1$ and $xy = 2$ and the curves $xy^2 = 3$ and $xy^2 = 4$.

Note that the area of the region is given by $\iint\limits_{D} dxdy$. Let $u = xy$ and $v = xy^2$. Then $x = \frac{u^2}{v}$ and $y = \frac{v}{u}$. Also $J(u, v) = \frac{1}{v}$. Thus, $\iint\limits_{D} dxdy = \int\limits_{u=1}^{2} \int\limits_{v=3}^{4} \frac{1}{v} dvdu = \log(\frac{4}{3})$.

Example 9.30. Evaluate the double integral $\iint\limits_{D} \frac{(x-y)}{(x+y+2)^2} dxdy$ over the region D bounded by the lines $x + y = \pm 1$ and $x - y = \pm 1$.

Let $u = x + y$ and $v = x - y$. Then $x = \frac{u+v}{2}$ and $y = \frac{u-v}{2}$. Here $J(u,v) = -\frac{1}{2}$. Also, $u = \pm 1$ and $v = \pm 1$. Hence we have

$$\iint\limits_D \frac{(x-y)}{(x+y+2)^2} dx dy = \frac{1}{2} \int_{u=-1}^1 \int_{v=-1}^1 \frac{v}{(u+2)^2} du dv.$$

Polar coordinates: In this case the variables x and y are changed to r and θ by the following two equations

$$x = r \cos\theta \text{ and } y = r \sin\theta.$$

We assume that $r > 0$ and θ lies in $[0, 2\pi)$ so that the mapping $T^{-1}(r,\theta) = (r\cos\theta, r\sin\theta)$ is bijective. Then

$$J(r,\theta) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

Hence the change of variable formula in this case is

$$\iint\limits_D f(x,y) dx dy = \iint\limits_S f(r\cos\theta, r\sin\theta)\, r\, dr d\theta.$$

Example 9.31. Let $D = \{(x,y) : x^2 + y^2 \le a^2\}$ and $f : D \to \mathbb{R}$ is given by $f(x,y) = 2\sqrt{a^2 - x^2 - y^2}$. Then

$$\iint\limits_D f(x,y) dx dy = 2 \int_0^a \int_0^{2\pi} \sqrt{a^2 - r^2} r d\theta dr$$

$$= 4\pi \int_0^a r\sqrt{a^2 - r^2} dr$$

$$= 4\pi \left. \frac{(a^2 - r^2)^{\frac{3}{2}}}{-3} \right|_0^a = \frac{4\pi a^3}{3}.$$

9.3 Triple integrals

The concept of double integrals can be extended to functions defined on $D = [a,b] \times [c,d] \times [e,f]$. Consider the partition P of D is of the form $P = P_1 \times P_2 \times P_3$, where P_1, P_2 and P_3 are partitions of $[a,b], [b,c]$ and $[e,f]$, respectively. For a given partition P and a bounded function f defined on D, we can define $\inf_P L(P,f)$ and $\sup_P U(P,f)$, lower integral and upper integral of f. If lower and upper integrals are equal, then we say f is integrable and the integral is known as triple integral. It is denoted by

$$\iiint\limits_D f(x,y,z) dx dy dz \quad \text{or} \quad \iiint\limits_D f(x,y,z) dV.$$

Remark 9.32. Note that most of the results related to the integrability test of the function of two variables will analogously hold true in the case of the function of three variables. Hence we avoid mentioning those results over here.

Theorem 9.33. (Fubini's Theorem) Let R be a bounded region in $\mathbb{R}^2$ and let D be a bounded domain in $\mathbb{R}^3$ given by $D = \{(x,y,z) : (x,y) \in R \quad \text{and} \quad f_1(x,y) \le z \le f_2(x,y)\}$, where f_1, f_2 are continuous functions on R. If f is continuous on D, then

$$\iiint_D f(x,y,z)dV = \iint_R \left(\int_{f_1(x,y)}^{f_2(x,y)} f(x,y,z)dz \right) dA.$$

9.3.1 Change of variable in a triple integral

The change of variable formula for a double integral can be extended to triple integrals.

$$\iiint_S f(x,y,z)dxdydz = \iiint_T f[\varphi(u,v,w), \psi(u,v,w), \eta(u,v,w)]|J(u,v,w)|dudvdw,$$

where

$$J(u,v,w) = \begin{vmatrix} \frac{\partial \varphi}{\partial u} & \frac{\partial \varphi}{\partial v} & \frac{\partial \varphi}{\partial w} \\ \frac{\partial \psi}{\partial u} & \frac{\partial \psi}{\partial v} & \frac{\partial \psi}{\partial w} \\ \frac{\partial \eta}{\partial u} & \frac{\partial \eta}{\partial v} & \frac{\partial \eta}{\partial w} \end{vmatrix}.$$

Example 9.34. Consider the integral $\iiint_D x\,dxdydz$, where D is the region in $\mathbb{R}^3$ bounded by $x = 0, y = 0, z = 2$ and the surface $z = x^2 + y^2$. Here $D = \{(x,y,z) : (x,y) \in R, x^2 + y^2 \le z \le 2\}$ and $R = \{(x,y) : 0 \le x \le \sqrt{2}, 0 \le y \le \sqrt{2 - x^2}\}$. Therefore

$$\iiint_D xdxdydz = \iint_R \left(\int_{x^2+y^2}^2 x\,dz \right) dA$$

$$= \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} \int_{x^2+y^2}^2 x\,dzdydx$$

$$= \frac{8\sqrt{2}}{15}.$$

Cylindrical co-ordinates: In this case the variables x, y and z are changed to r, θ and z by the following three equations

$$x = r\cos\theta, y = r\sin\theta \quad \text{and} \quad z = z,$$

where $r > 0$ and $\theta \in [0, 2\pi)$. The Jacobian is

$$J(r,\theta,z) = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r.$$

Therefore the change of variable formula is

$$\iiint_S f(x, y, z)dxdydz = \iiint_T f(r\cos\theta, r\sin\theta, z)\, r\, drd\theta dz.$$

Example 9.35. Consider $\iiint_D \left(z^2 x^2 + z^2 y^2\right) dxdydz$, where D is the region determined by $x^2 + y^2 \leq 1$ and $-1 \leq z \leq 1$. We can describe D in cylindrical coordinates by $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$ and $-1 \leq z \leq 1$. Therefore,

$$\iiint_D \left(z^2 x^2 + z^2 y^2\right) dxdydz = \int_{-1}^{1} \int_{0}^{2\pi} \int_{0}^{1} \left(z^2 r^2\right) r\, drd\theta dz$$

$$= \int_{-1}^{1} \int_{0}^{2\pi} z^2 \frac{r^4}{4}\Big|_{r=0}^{1} d\theta dz$$

$$= \int_{-1}^{1} \frac{2\pi}{4} z^2 dz = \frac{\pi}{3}.$$

Spherical co-ordinates: In this case the variables x, y and z are changed to r, θ and ϕ by the following three equations

$$x = r\sin\phi\cos\theta, \quad y = r\sin\phi\sin\theta, \quad z = r\cos\phi.$$

We assume $r > 0, 0 \leq \theta < 2\pi$ and $0 \leq \phi < \pi$ to get mapping of transformation one-to-one. The Jacobian is

$$J(r, \theta, \phi) = -r^2 \sin\phi.$$

Hence the change of variable formula is

$$\iiint_S f(x, y, z)dxdydz = \iiint_T f(r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi)\, r^2 \sin\phi\, drd\theta d\phi.$$

9.4 Surface area and surface integrals

A surface is the locus of two points moving in the space $(\mathbb{R}^3)$ with two degrees of freedom. A surface can be represented in several ways, however, we discuss some of them.
Parametric surface: A parametric surface is the graph of a continuous function of two variables that taking values in $\mathbb{R}^3$. That is, given a continuous function $r : T \subseteq \mathbb{R}^2 \to \mathbb{R}^3$, defined by $r(u, v) = f(u, v)i + g(u, v)j + h(u, v)k$, the set $r(T) = \{r(u, v) : (u, v) \in T\}$ is called parametric surface determined by r. We assume that the function r is **one-to-one in the interior** of T so that the surface does not cross itself. The equations

$$x = f(u, v), \ y = g(u, v), \ z = h(u, v), \text{where } (u, v) \in T$$

are called parametric equations of the surface $r(T)$.

Example 9.36. (*i*). Let $T = \{(x, y) : x^2 + y^2 \leq 1\}$. Consider the function $r : T \to \mathbb{R}^3$ given by $r(x, y) = xi + yj + \sqrt{x^2 + y^2}k$. This represents a cone of height 1.

(*ii*). For fixed $a > 0$ with $0 \leq \theta < 2\pi$ and $0 \leq \phi < \pi$, the equations

$$x = a \sin \phi \cos \theta, \quad y = a \sin \phi \sin \theta, \quad z = a \cos \phi$$

represent a sphere. Here the parameters are θ and ϕ.

Note that a parametric surface can degenerate to a point or curve. For instance, if $x = f(u, v)$, $y = g(u, v)$, $z = h(u, v)$ are constant, then $r(T)$ is a point. On the other hand if $x = u + v, y = (u + v)^2$ and $z = (u + v)^3$, by letting $t = u + v$, $r(T)$ becomes a curve in $\mathbb{R}^3$.

If at $(u, v) \in T$, $\frac{\partial r}{\partial u}$ and $\frac{\partial r}{\partial v}$ are continuous and the fundamental product $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \neq 0$, then the point $r(u, v)$ is called a **regular point** of the surface $r(T)$. If one of $\frac{\partial r}{\partial u}$ and $\frac{\partial r}{\partial u}$ is not continuous or $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = 0$ at $(u, v,)$ we say (u, v) is a **singular point**. A surface $r(T)$ is called **smooth** if each point of the surface is regular.

9.4.1 Area of a parametric surface

Let $S = r(T)$ be be a **smooth** parametric surface defined on the domain T. That is, $\frac{\partial r}{\partial u}$ and $\frac{\partial r}{\partial v}$ are continuous and $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$ is never zero on T.

If we fix v and allow u to run, then the image of r reduces to a curve in $\mathbb{R}^3$. Hence the distance travel along the curve $r(\cdot, v)$ in a small time interval Δu is $\|\frac{\partial r}{\partial u}\|\Delta u$. Similarly, if we fix u, then the graph of r is a curve in $\mathbb{R}^3$, and the distance traveled along this curve in a small time interval Δv is $\|\frac{\partial r}{\partial v}\|\Delta v$. See fig. 9.3.
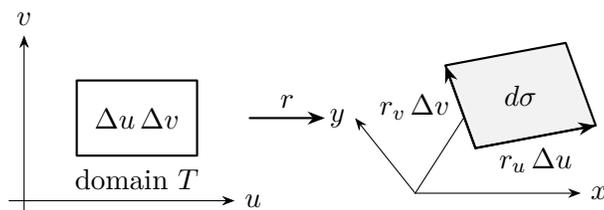
Figure 9.3. A small rectangle in the parameter domain maps to a parallelogram on the surface, with area element $d\sigma = \|r_u \times r_v\| \, du \, dv$.

Thus, we see that a small rectangle in T of area $\Delta u \Delta v$ in the uv-plane is transferred to a parallelogram on the surface $r(T)$ with area $\left\|\frac{\partial r}{\partial u}\Delta u \times \frac{\partial r}{\partial v}\Delta v\right\| = \left\|\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right\| \Delta u \Delta v$.

Note that the point, where $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = 0$, the parallelogram on $r(T)$ will collapse to a curve or a point. Now at each regular point of $r(T)$, the vectors $\frac{\partial r}{\partial u}$ and $\frac{\partial r}{\partial v}$ determine a plane having $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$ as the normal vector to the surface at the point (u, v). We know that $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = \left\|\frac{\partial r}{\partial u}\right\| \left\|\frac{\partial r}{\partial v}\right\| \sin \theta \, \hat{n}$, where $\hat{n}$ is the unit normal to the surface $r(T)$ at $(u, v) \in T$. Hence, the plane determined by $\frac{\partial r}{\partial u}$ and $\frac{\partial r}{\partial v}$ is called tangent plane of the surface. Note that the continuity of $\frac{\partial r}{\partial u}$ and $\frac{\partial r}{\partial v}$ implies the continuity of $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}$. Hence the tangent plane varies continuously on the smooth surface. Thus, the continuity of $\frac{\partial r}{\partial u}$ and $\frac{\partial r}{\partial v}$ prevent the occurrence of sharp edges or corners on the surface.

Let us denote the area of small parallelogram obtained by transferring the small rectangle of areas $\Delta u \Delta v$ in the domain T by $d\sigma$. Let $r_u = \frac{\partial r}{\partial u}$ and $r_v = \frac{\partial r}{\partial v}$. Then $d\sigma = \|r_u \times r_v\| \Delta u \Delta v$. Hence the surface area of $r(T)$ denoted by $a(S)$ is given by

$$a(S) = \iint_T \|r_u \times r_v\| \, du dv.$$

Area of a surface defined by a graph: Suppose a surface S is given by

$$z = f(x, y), \quad \text{for } (x, y) \in T.$$

That is, S is the graph of the function $f(x, y)$. Then S can be considered as a parametric surface defined by:

$$r(x, y) = xi + yj + f(x, y)k, \quad (x, y) \in T.$$

In this case, $r_x = i + f_x k$, $r_y = j + f_y k$. Further, $r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = -f_x i - f_y j + k.$

Hence the surface area becomes

$$a(S) = \iint_T \sqrt{1 + f_x^2 + f_y^2} \, dx dy.$$

Example 9.37. Let us find the area of the surface of the portion of the sphere $x^2 + y^2 + z^2 = 4a^2$ that lies inside the cylinder $x^2 + y^2 = 2ax$. Consider $f(x, y) = \sqrt{4a^2 - x^2 - y^2}$, then

$$f_x = \frac{-x}{\sqrt{4a^2 - x^2 - y^2}}, \quad f_y = \frac{-y}{\sqrt{4a^2 - x^2 - y^2}}$$

$$\text{and} \quad \sqrt{1 + f_x^2 + f_y^2} = \sqrt{\frac{4a^2}{4a^2 - x^2 - y^2}}.$$

Then

$$a(S) = 2 \iint_T \sqrt{\frac{4a^2}{4a^2 - x^2 - y^2}} \, dx dy$$

$$= 2 \times 2 \int_0^{\frac{\pi}{2}} \int_0^{2a \cos\theta} \frac{2ar}{\sqrt{4a^2 - r^2}} \, dr d\theta.$$

Remark 9.38. Note that

$$\|r_u \times r_v\|^2 = \|r_u\|^2 \|r_v\|^2 \sin^2\theta$$
$$= \|r_u\|^2 \|r_v\|^2 \left(1 - \cos^2\theta\right)$$
$$= \|r_u\|^2 \|r_v\|^2 - (r_u \cdot r_v)^2.$$

Let $E = r_u \cdot r_u$, $G = r_v \cdot r_v$ and $F = r_u \cdot r_v$, then

$$a(S) = \iint\limits_T \sqrt{EG - F^2} \, dudv.$$

9.4.2 Surface integrals

Let S be a parametric surface defined by $r(u, v)$ over T. Suppose r_u and r_v are continuous. Let $g : S \to \mathbb{R}$ be bounded. The surface integral of g over S, denoted by $\iint\limits_S g d\sigma$, is defined by

$$\iint\limits_S g d\sigma = \iint\limits_T g(r(u, v)) \, \|r_u \times r_v\| \, dudv = \iint\limits_T g(r(u, v)) \sqrt{EG - F^2} \, dudv$$

provided double integral in the RHS exists.

Remark 9.39. (i).

$$\iint\limits_S g d\sigma = \iint\limits_T g(r(u, v)) \sqrt{EG - F^2} \, dudv.$$

(ii). If S is defined by $z = f(x, y)$, then

$$\iint\limits_S g d\sigma = \iint\limits_T g[x, y, f(x, y)] \sqrt{1 + f_x^2 + f_y^2} \, dxdy,$$

where T is the projection of the surface S over the xy-plane.

Example 9.40. Let S be the hemispherical surface $z = \left(a^2 - x^2 - y^2\right)^{1/2}$. Evaluate

$$\iint\limits_S \frac{d\sigma}{[x^2 + y^2 + (z + a)^2]^{1/2}}.$$

Consider

$$S := r(\theta, \phi) = (a \sin \phi \cos \theta, \, a \sin \phi \sin \theta, \, a \cos \phi),$$

where $0 \le \theta \le 2\pi$, $0 \le \phi \le \frac{\pi}{2}$. Note that

$$\sqrt{EG - F^2} = a^2 \sin \phi \quad \text{and} \quad \left[x^2 + y^2 + (z + a)^2\right]^{1/2} = 2a \cos \frac{\phi}{2}.$$
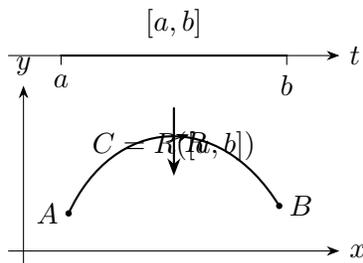
Hence

$$\iint\limits_S \frac{d\sigma}{[x^2 + y^2 + (z + a)^2]^{1/2}} = \int_0^{2\pi} \int_0^{\pi/2} \frac{a^2 \sin \phi}{2a \cos \frac{\phi}{2}} d\phi d\theta.$$

9.5 Line Integrals

Let $R : [a, b] \to \mathbb{R}^3$ be a differentiable function and the curve C is parameterized by $R(t)$. Suppose $f : C \to \mathbb{R}^3$ is a bounded function. The line integral of f along C is denoted by the symbol $\int_C f \cdot dR$ and is defined by

$$\int_C f \cdot dR = \int_a^b f(R(t)) \cdot R'(t) dt$$

provided the integral in the right-hand side exists. See fig. 9.4.

Figure 9.4. A curve C parameterized by $R : [a, b] \to \mathbb{R}^3$; the line integral $\int_C F \cdot dR$ is defined as $\int_a^b F(R(t)) \cdot R'(t) \, dt$.

Remark 9.41. Suppose $f = (f_1, f_2, f_3)$ and $R(t) = (x(t), y(t), z(t))$. Then the line integral $\int_C f \cdot dR$ is also written as

$$\int_C f_1 dx + f_2 dy + f_3 dz \quad \text{or} \quad \int_C f_1(x, y, z) dx + f_2(x, y, z) dy + f_3(x, y, z) dz.$$

Example 9.42. Let $f = x^2 i + yj + (xz - y)k$. Compute the line integral $\int_C f \cdot dR$, along the curve C joining $(0, 0, 0)$ with $(1, 2, 4)$.
(i) C is the straight line joining theses points,
(ii) C is the curve given by $R(t) = (t^2, 2t, 4t^2)$.

The second FTC for line integral: We know that if $f : [a, b] \to \mathbb{R}$ is continuously differentiable on $[a, b]$, then $\int_a^b f(t)' dt = f(b) - f(a)$. Since f is continuous, $F(x) = \int_\alpha^x f(t) dt$ is differentiable and by FTC it follows that $F(x)' = f(x)$, where $\alpha \in [a, b]$. Hence $\int_a^b F'(x) dx = \int_a^b f(x) dx = \int_\alpha^b f(x) dx - \int_\alpha^a f(x) dx = F(b) - F(a)$. This says that the value of integral of continuously differentiable function depends only on end points and not on the points inside the interval.

We generalize the above second FTC to the line integral.

Theorem 9.43. *Let D be a solid domain in $\mathbb{R}^3$, and $f : D \to \mathbb{R}$ be continuously differentiable. Suppose A, B are two points in D. Let $C = \{R(t) : t \in [a, b]\}$ be a curve lying in D and joining the points A and B. If $R(t)$ is continuously differentiable on $[a, b]$, then*

$$\int_C \nabla f \cdot dR = f(B) - f(A).$$

Proof. Let $h(t) = f(R(t))$. Then by chain rule, we get $h'(t) = (f \circ R)' = \nabla f(R(t)).R'(t)$. Hence

$$\int_C \nabla f \cdot dR = \int_a^b \nabla f(R(t)).R'(t)dt = \int_a^b h'(t)dt = h(b) - h(a) = f(B) - f(A).$$

$\square$

Remark 9.44. Line integral of gradient of a function is independent of the choice of path joining the points A and B in the domain D.

Definition 9.45. Let $R : [a,b] \to \mathbb{R}^3$ be a continuous function that represents a curve C. The curve C is said to be

(i) **simple** if R is one-to-one on $(a,b]$.

(ii) **Closed** if $R(a) = R(b)$.

(iii) **Smooth** if R' exists and continuous.

(iv) **Piecewise smooth** if the interval $[a,b]$ can be partitioned into a finite number of subintervals and in each of which the curve is smooth.

Theorem 9.46. *(Green's Theorem) Let C be a piecewise smooth simple closed curve in the xy-plane and let D denote the closed region enclosed by C. Suppose $M, N, \frac{\partial N}{\partial x}$ and $\frac{\partial M}{\partial y}$ are real valued continuous functions in an open set containing D. Then*

$$\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = \oint_C (Mi + Nj)dR = \oint_C Mdx + Ndy, \tag{9.4}$$

where the line integral is taken around C in the counterclockwise direction.

Since the identity (9.4) holds true for every choice of M and N (satisfying the assumption of Green's Theorem), by letting $M = 0$ and N arbitrary and vice-versa, the identity (9.4) is equivalent to two identities $\iint_D \frac{\partial N}{\partial x}dxdy = \oint_C Ndy$ and $-\iint_D \frac{\partial M}{\partial y}dxdy = \oint_C Mdy$. We shall present the proof of Green's Theorem for two special cases I and II as shown in fig. 9.5.
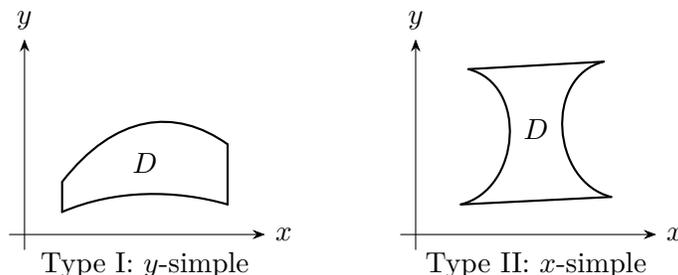
Figure 9.5. Two standard classes of planar domains used in the proof of Green's theorem.

Proof. (*i*) Let $D = \{(x, y) : a \leq x \leq b \text{ and } f(x) \leq y \leq g(x)\}$, where f and g are continuous functions on $[a, b]$. Since $\frac{\partial M}{\partial y}$ is continuous, by Fubini's Theorem, the double integral

$$-\iint\limits_{D} \frac{\partial M}{\partial y} dx dy = \int_a^b \left[\int_{f(x)}^{g(x)} \frac{\partial M}{\partial y} dy \right] dx = \int_a^b M[x, f(x)] dx - \int_a^b M[x, g(x)] dx. \tag{9.5}$$

On the other hand, we can write

$$\int_C M dx = \int_{C_1} M dx + \int_{C_2} M dx, \tag{9.6}$$

since the line integral along each of vertical segment is zero. Note that C_1 and C_2 can be represented by $r_1(t) = ti + f(t)j$ and $r_2(t) = ti + g(t)j$ respectively. Hence

$$\int_{C_1} M dx = \int_a^b M[t, f(t)] dt \text{ and } \int_{C_2} M dx = -\int_a^b M[t, g(t)] dt. \tag{9.7}$$

Negative sign appeared in the second equation since the curve C_2 traverses in the reverse direction. Thus, from (9.5-9.7) we conclude that the identity (9.4) holds for the type I region. Similarly, we can obtain the result for the type II region. Further, we can obtain the result for any region which can be decomposed into finitely many regions of the above two types. $\qquad\square$

Area expressed as a line integral: Let C be a simple (piecewise smooth) closed curve and D be the region enclosed by C. Let $M(x, y) = -\frac{y}{2}$ and $N(x, y) = \frac{x}{2}$. Then by Green's Theorem the area of D is

$$a(D) = \iint\limits_{D} dx dy = \iint\limits_{D} (N_x - M_y) \, dx dy = \int_a^b M dx + N dy = \frac{1}{2} \int_C -y dx + x dy.$$

Example 9.47. Note that the integral

$$\int_C xy^2 dx + \left(x^2 y + x\right) dy = \iint\limits_{D} dx dy = \text{Area(D)}$$

(By Green's Theorem), where D is the region enclosed by C. Hence the integral is depending only on the region enclosed by C but not its location.

Example 9.48. Find the area bounded by the ellipse $C = \{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\}$.
Consider the parametric form of $C = \{(a \cos t, b \sin t) : 0 \leq t < 2\pi.\}$ Then the area is

$$\frac{1}{2} \int_C -y dx + x dy = \frac{1}{2} \int_0^{2\pi} -(b \sin t)(-a \sin t) dt + (a \cos t)(b \cos t) dt = \frac{1}{2} \int_0^{2\pi} ab \, dt = ab\pi.$$

Example 9.49. Let C_1 and C_2 be two simple (piecewise smooth) closed curves as shown in fig. 9.6.
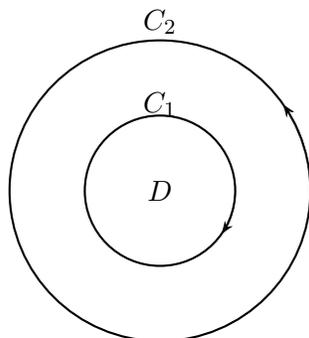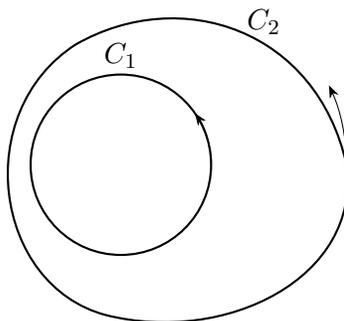
Figure 9.6. A multiply connected domain D bounded by an outer curve C_2 and an inner curve C_1 (annular-type region).

Consider the region D bounded by the curves C_1 and C_2. Note that $D = D_1 \cup D_2$ and D_1 is enclosed by the curves γ_i; $i = 1, 2, 3, 4$ and D_2 is enclosed by curves γ_j; $i = 1, 3, 5, 6$.

$$\iint\limits_{D} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = \iint\limits_{D_1} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy + \iint\limits_{D_2} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

$$= \left(\int_{\gamma_1} \alpha + \int_{\gamma_2} \alpha + \int_{\gamma_3} \alpha + \int_{\gamma_4} \alpha \right) + \left(\int_{\gamma_6} \alpha - \int_{\gamma_3} \alpha + \int_{\gamma_5} \alpha - \int_{\gamma_1} \alpha \right) = \oint_{C_2} \alpha - \oint_{C_1} \alpha,$$

where $\alpha = Mdx + Ndy$.

Example 9.50. Let C_1 be unit circle and C_2 be any simple closed curve as shown in fig. 9.7.

Figure 9.7. An inner unit circle C_1 and an outer simple closed curve C_2 bounding a multiply connected domain.

Find $\int_{C_2} \frac{xdy - ydx}{x^2 + y^2}$. Let D be the domain lies between C_1 and C_2. A simple calculation shows that $N_x - M_y = 0$ on D. By applying Green's Theorem for multiply-connected domain D, we get

$$\oint_{C_2} (Mdx + Ndy) - \oint_{C_1} (Mdx + Ndy) \iint\limits_{D} (N_x - M_y) \, dxdy = 0.$$

Since $C_1 = \{(\cos t, \sin t) : 0 \leq t \leq 2\pi\}$, we get

$$\int_{C_1} \frac{-ydx + xdy}{x^2 + y^2} = \int_0^{2\pi} \frac{\sin^2 t + \cos^2 t}{\sin^2 t + \cos^2 t} dt = 2\pi.$$

Hence,

$$\oint_{C_2} (Mdx + Ndy) = 2\pi.$$

9.5.1 Exactness of the line integral

Let Q be a cube in $\mathbb{R}^3$. Suppose C is a curve in Q which is parameterized by $R(t) = (x(t), y(t), z(t))$, where $R : [a, b] \to \mathbb{R}^3$ is continuously differentiable. Now, is there a function $F : Q \subset \mathbb{R}^3 \to \mathbb{R}$ such that $\int_C f \cdot dR = \int_C dF$ for every curve C in Q? Suppose there exists F such that $\int_C f \cdot dR = \int_C dF$ for every curve C in Q. Then by Theorem 9.43 (second FTC for line integral), it follows that

$$\int_C f \cdot dR = F(R(b)) - F(R(a)) = F(B) - F(A) = \int_C \nabla F \cdot dR.$$

That is,

$$\int_C (f - \nabla F) \cdot dR = 0$$

for all curves C in Q. It is easy to see that $f = \nabla F$ on Q.

Remark 9.51. Note that it is difficult to prove that $f = \nabla F$ on a general domain D. However, the following exercise can be done with small effort.

Exercise 9.2. Let $D = \{(x, y) : x^2 + y^2 < 1\}$. If $f : D \to R^2$ is a continuously differentiable function such that $\int_\Gamma f \cdot dR = 0$ for every curve Γ in D, then f constant.

Example 9.52. Show that the line integral

$$\int_C 2x \sin y \ dx + (x^2 \cos y - 3y^2)dy$$

is path independent joining the points $(-1, 0)$ and $(5, 1)$.

9.6 Curl and Divergence

Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field given by $F(x, y, z) = P(x, y, z)i + Q(x, y, z)j + R(x, y, z)k$.

Definition 9.53. (Curl of F) The curl of F is another vector field denoted by curl F and defined by the vector

$$\text{curl}\, F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \nabla \times f,$$

where $\nabla = \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k$.

Definition 9.54. (Divergence of F) The divergence of F is a scalar valued function denoted by div F and is defined by div $F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$. We can rewrite the div F as div $F = \nabla \cdot F$.

Now, we recall Green's Theorem to get a motivation for Stokes' Theorem. Let C be the piece-wise smooth curve which encloses the domain D in $\mathbb{R}^2$. Let $F : D \to \mathbb{R}^2$ be a vector field in the plane given by $F(x, y) = M(x, y)i + N(x, y)j + 0k$. By Green's Theorem

$$\iint\limits_{D} (N_x - M_y) \, dxdy = \oint_C M \, dx + N \, dy,$$

where $C = \{R(t) : t \in [a, b]\}$. The above identity can be represented as

$$\iint\limits_{D} \operatorname{curl} F \cdot k \; dxdy = \oint_C F \cdot dR, \tag{9.8}$$

where $\operatorname{curl} F = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) k$. Stokes' Theorem is a generalization of the identic (9.8) in $\mathbb{R}^3$. Before we make a formal statement for Stokes' Theorem, we discuss unit normal vector on some special surfaces.
(i) Suppose the surface S is given by $f(x, y, z) = c$, where f is differentiable function on some domain D in $\mathbb{R}^3$. See fig. 9.8.
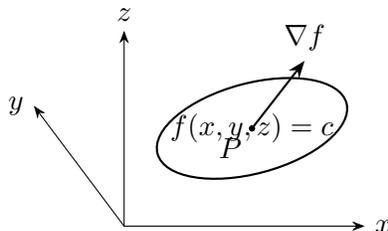

Figure 9.8. A level surface $f(x, y, z) = c$ with normal direction given by $\nabla f(P)$.

Consider a smooth curve C given by $R : [a, b] \to \mathbb{R}^3$ which lies on the surface S and passes through a point P on S. Then $f(R(t)) = c$. By the chain rule, we get $f'(R(t)).R'(t) = 0$. That is, $\nabla f(R(t)).R'(t) = 0$. Since $R'(t)$ is the tangent vector at point P, the vector $\nabla f(R(t))$ is the normal vector at P. Hence the unit normal vector $\hat{n}$ is given by $\hat{n} = \frac{\nabla f}{\|\nabla f\|}$. Note that $\hat{n} : D \subset \mathbb{R}^3 \to \mathbb{R}^3$. If $\hat{n}$ is continuous and never vanishes on D, then the surface S is called **orientable.**
(ii) Let D be a domain in $\mathbb{R}^2$. Let $F : D \to \mathbb{R}^3$ given by $F(s, t) = x(s, t)i + y(s, y)j + z(s, t)k$ is a parametrization of surface S, where F is smooth (continuously differentiable). Let $P = F(s_o, t_o)$ be a point on the surface S. Then $F(s, t_o)$ and $F(s_o, t)$ are curves on S passing through P as shown in fig. 9.9.

Recall that the fundamental product $F_s \times F_t$ is the normal to the surface S at P. Hence unit normal vector to the surface S, in this case, is given by $\hat{n} = \frac{F_s \times F_t}{\|F_s \times F_t\|}$. Note that $\hat{n} : D \subset \mathbb{R}^2 \to \mathbb{R}^3$.
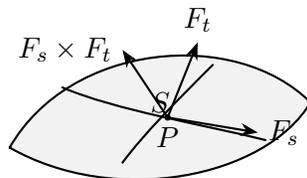
Figure 9.9. Coordinate curves on a parametrized surface $F(s,t)$; the tangent vectors F_s and F_t span the tangent plane, and $F_s \times F_t$ is normal.

(iii) If the surface S is given by the graph of smooth function $f : D \subset \mathbb{R}^2 \to \mathbb{R}^3$. That is, $F(x,y) = xi + yj + f(x,y)k$. Then unit normal vector is given by

$$\hat{n} = \frac{F_x \times F_y}{\|F_x \times F_y\|} = \frac{-f_x i - f_y j + k}{\| - f_x i - f_y j + k\|} = \frac{-f_x i - f_y j + k}{\sqrt{1 + f_x^2 + f_y^2}}.$$

Definition 9.55. A surface S is called orientable if unit normal vector to the surface S is continuous and never vanishes.

Hence **orientable surface** is a two-sided surface. Möbius strip is not an orientable surface.

Theorem 9.56 (Stokes' Theorem)**.** *Let S be a piecewise smooth orientable surface and C be the piecewise smooth boundary of S. Let $F(x,y,z) = P(x,y,z)i + Q(x,y,z)j + R(x,y,z)k$ be a vector field such that P, Q and R are continuously differentiable on an open set containing S. If $\hat{n}$ is a unit normal vector to S, then*

$$\iint_S \mathrm{curl}\, F \cdot \hat{n}\, d\sigma = \oint_C F \cdot dR, \tag{9.9}$$

where the line integral is evaluated around C in the direction of the orientation of C with respect to $\hat{n}$.

See fig. 9.10.
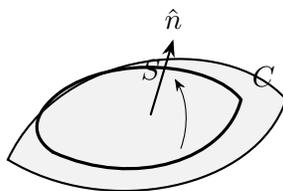
Figure 9.10. Stokes' theorem relates the circulation along the oriented boundary $C = \partial S$ to the flux of $\mathrm{curl}\, F$ through S.

   (i) The value of the surface integral in (9.9) depends only on the boundary C and not on the particular shape of the surface S.

(ii) If S is a plane surface, then identity (9.9) reduces to identity (9.8). Thus, Stokes' theorem may be viewed as a direct extension of Green's theorem.

(iii) If S is a closed smooth surface, such as a sphere or a torus, then $\partial S = \varnothing$, and therefore

$$\iint\limits_S \operatorname{curl} F \cdot \hat{n} \, d\sigma = 0.$$

(iv) Stokes' theorem extends to smooth surfaces whose boundary consists of more than one simple smooth closed curve.

Remark 9.57. If a surface S is given by the graph of a smooth function f defined on a domain $D \subset \mathbb{R}^2$, then

$$\oint_C F \cdot dR = \iint\limits_D (-f_x i - f_y j + k) \cdot \operatorname{curl} F \, dx \, dy.$$

Example 9.58. Let S be the portion of the surface $z = 1 - x^2$ with $0 \leq x \leq 1$ and $-2 \leq y \leq 2$. Let $C = \partial S$, and let $F(x, y, z) = yi + yj + k$. Use Stokes' theorem to compute the line integral $\int_C F \cdot dR$.

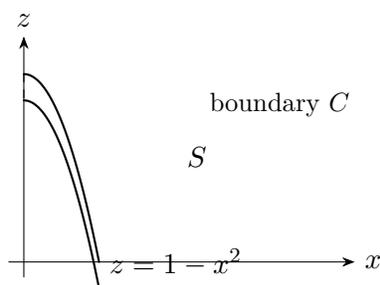Here $\operatorname{curl} F = -\vec{k}$. Write $z = f(x, y) = 1 - x^2$. Then the upward unit normal to S is

$$\hat{n} = \frac{-f_x \vec{i} - f_y \vec{j} + \vec{k}}{\sqrt{1 + f_x^2 + f_y^2}} = \frac{2x\vec{i} + \vec{k}}{\sqrt{1 + 4x^2}},$$

and the surface element is

$$d\sigma(x, y) = \sqrt{1 + f_x^2 + f_y^2} \, dx \, dy.$$

See fig. 9.11.

Figure 9.11. A schematic portion of the cylinder-like surface $z = 1 - x^2$ over a rectangle in the (x, y)-plane; its boundary curve C is oriented consistently with $\hat{n}$.

By Stokes' theorem and Remark 9.57,

$$\oint_C F \cdot dR = \iint\limits_D (-f_x i - f_y j + k) \cdot \operatorname{curl} F \, dx \, dy = \int_{-2}^{2} \int_{0}^{1} (-1) \, dx \, dy = -4.$$

Let us return to Green's theorem in the plane. Let $F(x, y) = M(x, y)i + N(x, y)j$ be smooth on the domain $D \subset \mathbb{R}^2$, where D is enclosed by the simple and smooth curve $C = \{R(t) : t \in [a, b]\}$. Then $R'(t) = x'(t)i + y'(t)j$ is the tangent vector to the curve. Hence $n = y'(t)i - x'(t)j$ is a normal vector to the curve C. By Green's Theorem

$$\oint_C (F \cdot n)dt = \oint_C Mdy - Ndx = \iint_D \left(\frac{\partial M}{\partial x} - \left(-\frac{\partial N}{\partial y} \right) \right) dxdy = \iint_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy.$$

Hence

$$\iint_D \operatorname{div} F \, dxdy = \oint_C (F \cdot n)ds. \tag{9.10}$$

The generalization of the identity (9.10) is known as the divergence theorem.

Theorem 9.59 (Divergence Theorem). *Let D be a solid region in $\mathbb{R}^3$ bounded by a piecewise smooth orientable surface S. Let $F(x, y, z) = P(x, y, z)i + Q(x, y, z)j + R(x, y, z)k$ be a vector field that is continuously differentiable on an open set containing D. If $\hat{n}$ denotes the outward unit normal to S, then*

$$\iiint_D \operatorname{div} F \, dV = \iint_S F \cdot \hat{n} \, d\sigma.$$

See fig. 9.12.
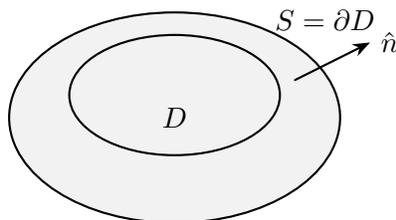
Figure 9.12. A bounded domain D with boundary surface $S = \partial D$ and outward unit normal $\hat{n}$ (Gauss–divergence theorem).

Example 9.60. Let $F(x, y, z) = (x + y)i + z^2 j + x^2 k$. Let $\hat{n}$ be the outward unit normal to the hemisphere

$$S = \{(x, y, z) : x^2 + y^2 + z^2 = 1, \; z > 0\}.$$

Compute the surface integral $\iint_S F \cdot \hat{n} \, d\sigma$ using the divergence theorem.

Write $F(x, y, z) = (x + y, z^2, x^2)$. Then $\operatorname{div} F = 1$. Since S is not closed, let

$$S_1 = \{(x, y, 0) : x^2 + y^2 \leq 1\}.$$

Then $S \cup S_1$ is a closed surface, so the divergence theorem applies; see fig. 9.13.

By the divergence theorem,

$$\iint_S F \cdot \hat{n} \, d\sigma + \iint_{S_1} F \cdot \hat{n}_1 \, d\sigma_1 = \iiint_D \operatorname{div} F \, dV = \frac{2\pi}{3}.$$

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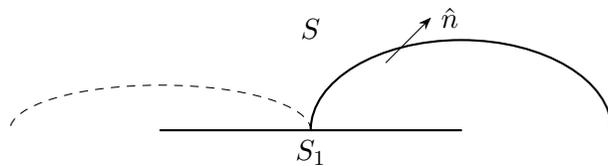
Figure 9.13. A hemisphere S closed by the base disk S_1 to form a closed surface, enabling the divergence theorem.

Here

$$\iint\limits_{S_1} F \cdot \hat{n}_1 \, d\sigma_1 = \iint\limits_{x^2+y^2\leq 1} F(x, y, 0) \cdot (-k) \, dx \, dy.$$

Problem Sets

These problem sets are designed in three layers: a first pass of diagnostic questions that test command of definitions, a second pass of proof-oriented exercises that consolidate the main theorems, and a final pass of synthesis problems that ask you to combine ideas across sections. A productive workflow is to attempt each set in that order, returning to the later questions only after the structural lemmas and standard examples have become familiar.

Problem Set 1

Problem-set architecture.

- Begin with Item 1 and Item 3 to audit your command of the definitions and to build a bank of counterexamples.

- Use the middle part of the set to compare genuine metric phenomena with artifacts of particular norms or coordinates.

- Reserve the later questions for synthesis: these require you to combine completeness, continuity, and structure of function spaces rather than treating them separately.

1. State whether each of the following statements is true or false, and justify your answer.

   (a) There does not exist a monotone function $f : \mathbb{R} \to \mathbb{Q}$ which is onto.

   (b) There exists a monotone function $f : (0, \infty) \to \mathbb{R}$ such that each $c \in (0, \infty)$ satisfies $|f(c+) - f(c-)| = \frac{1}{c}$.

   (c) There exists a sequence of differentiable functions f_n on $(0, \infty)$ such that f_n' is uniformly convergent on $(0, \infty)$ but f_n is nowhere pointwise convergent.

   (d) There exists a metric space having exactly 36 open sets.

   (e) It is impossible to define a metric d on $\mathbb{R}$ such that only finitely many subsets of $\mathbb{R}$ are open in $(\mathbb{R}, d)$.

   (f) If A and B are open (closed) subsets of a normed vector space X, then $A + B = \{a + b : a \in A, b \in B\}$ is open (closed) in X.

   (g) If A and B are closed subsets of $[0, \infty)$ (with the usual metric), then $A + B$ is closed in $[0, \infty)$.

   (h) It is possible to define a metric d on $\mathbb{R}$ such that the sequence $(1, 0, 1, 0, \dots)$ converges in $(\mathbb{R}, d)$.

(i) It is possible to define a metric d on $\mathbb{R}^2$ such that $((\frac{1}{n}, \frac{n}{n+1}))$ is not a Cauchy sequence in $(\mathbb{R}^2, d)$.

(j) It is possible to define a metric d on $\mathbb{R}^2$ such that in $(\mathbb{R}^2, d)$, the sequence $((\frac{1}{n}, 0))$ converges but the sequence $((\frac{1}{n}, \frac{1}{n}))$ does not converge.

(k) Let $A \subset (1, \infty)$ be a closed set. Then $A^2 := \{a^2 : a \in A\}$ is a closed set.

(l) Let $A_n = \{(x, y) \in \mathbb{R}^2 : 0 < \frac{1}{x} < y < \frac{1}{n}\}$. Determine whether the set $\bigcap\limits_{n=1}^{\infty} A_n$ is open or closed.

(m) There exists a set $A \subset (\mathbb{R}, u)$ such that $\delta(A^\circ \cup \{0\}) = 0$ but $\delta((\overline{A})^\circ) = 1$, where δ denotes the diameter.

(n) If (x_n) is a sequence in a complete normed vector space X such that $\|x_{n+1} - x_n\| \to 0$ as $n \to \infty$, then (x_n) must converge in X.

(o) If (f_n) is a sequence in $C[0, 1]$ such that $|f_{n+1}(x) - f_n(x)| \le \frac{1}{n^2}$ for all $n \in \mathbb{N}$ and for all $x \in [0, 1]$, then there must exist $f \in C[0, 1]$ such that $\int_0^1 |f_n(x) - f(x)|\, dx \to 0$ as $n \to \infty$.

(p) If (x_n) is a Cauchy sequence in a normed vector space, then $\lim\limits_{n \to \infty} \|x_n\|$ must exist.

(q) $\{f \in C[0, 1] : \|f\|_1 \le 1\}$ is a bounded subset of the normed vector space $(C[0, 1], \|\cdot\|_\infty)$.

(r) For $x, y \in \ell^\infty$, let $d(x, y) = \min\{1, \limsup_{n \to \infty} |x_n - y_n|\}$. Determine whether this defines a metric on ℓ^∞.

(s) The sequence $f_n(t) = e^{-n^2 \sin \pi t}$ converges uniformly to 0 on $(0, 1)$.

(t) If the sequence (x_n) in $\mathbb{R}$ satisfies

$$0 \le \inf x_n \le \sup x_n < \infty,$$

determine whether (x_n) must have a convergent subsequence.

(u) If $f : \mathbb{R} \to \mathbb{R}$ is continuous, bounded, and monotone, must the limits $\lim\limits_{x \to \pm\infty} f(x)$ be finite?

2. What is the cardinality of the set $\{f : \mathbb{R} \to \mathbb{R}, \ f \text{ is nowhere continuous}\}$?

3. For a monotone increasing function $f : [a, b] \to \mathbb{R}$, define $g(x) = \sup\{f(y) : y < x\}$. If f has limit at c, then show that $f(c) = g(c)$.

4. Examine whether (X, d) is a metric space, where

(a) $X = \mathbb{R}$ and $d(x, y) = \frac{|x - y|}{1 + |xy|}$ for all $x, y \in \mathbb{R}$.

(b) $X = \mathbb{R}$ and $d(x, y) = |x - y|^p$ for all $x, y \in \mathbb{R}$ $(0 < p < 1)$.

(c) $X = \mathbb{R}$ and $d(x, y) = \min\{\sqrt{|x - y|}, |x - y|^2\}$ for all $x, y \in \mathbb{R}$.

(d) $X = \mathbb{R}$ and for all $x, y \in \mathbb{R}$, $d(x, y) = \begin{cases} 1 + |x - y| & \text{if exactly one of } x \text{ and } y \text{ is positive,} \\ |x - y| & \text{otherwise.} \end{cases}$

(e) $X = \mathbb{R}^2$ and $d(x, y) = (|x_1 - y_1| + |x_2 - y_2|^{\frac{1}{2}})^{\frac{1}{2}}$ for all $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$.

(f) $X = \mathbb{R}^n$ and $d(x, y) = [(x_1 - y_1)^2 + \frac{1}{2}(x_2 - y_2)^2 + \cdots + \frac{1}{n}(x_n - y_n)^2]^{\frac{1}{2}}$ for all $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n) \in \mathbb{R}^n$.

(g) $X = \mathbb{C}$ and for all $z, w \in \mathbb{C}$, $d(z, w) = \begin{cases} \min\{|z| + |w|, |z - 1| + |w - 1|\} & \text{if } z \neq w, \\ 0 & \text{if } z = w. \end{cases}$

(h) $X = \mathbb{C}$ and for all $z, w \in \mathbb{C}$, $d(z, w) = \begin{cases} |z - w| & \text{if } \frac{z}{|z|} = \frac{w}{|w|}, \\ |z| + |w| & \text{otherwise.} \end{cases}$

(i) $X = \mathbb{C}$ and $d(z, w) = \frac{2|z - w|}{\sqrt{1 + |z|^2}\sqrt{1 + |w|^2}}$ for all $z, w \in \mathbb{C}$.

(j) $X = $ The class of all finite subsets of a nonempty set and $d(A, B) = $ The number of elements of the set $A \triangle B$ (the symmetric difference of A and B).

5. Let $1 \leq p \leq \infty$ and let d_i, $i = 1, 2$, be two metrics on a nonempty set X. Show that $d_p = (d_1^p + d_2^p)^{1/p}$ is a metric on X for $1 \leq p < \infty$. Determine whether $d_\infty = \max\{d_1, d_2\}$ is a metric on X.

6. Examine whether $\|\cdot\|$ is a norm on $\mathbb{R}^2$, where for each $(x, y) \in \mathbb{R}^2$,

   (a) $\|(x, y)\| = (|x|^p + |y|^p)^{\frac{1}{p}}$, where $0 < p < 1$.

   (b) $\|(x, y)\| = \sqrt{\frac{x^2}{9} + \frac{y^2}{4}}$.

   (c) $\|(x, y)\| = \begin{cases} \sqrt{x^2 + y^2} & \text{if } xy \geq 0, \\ \max\{|x|, |y|\} & \text{if } xy < 0. \end{cases}$

7. Let $\|f\| = \min\{\|f\|_\infty, 2\|f\|_1\}$ for all $f \in C[0, 1]$. Prove that $\|\cdot\|$ is not a norm on $C[0, 1]$.

8. Let X be a normed linear space. Prove that norm of any $x \in X$, can be expressed as $\|x\| = \inf\{|\alpha| : \alpha \in \mathbb{C} \setminus \{0\} \text{ with } \|x\| \leq |\alpha|\}$.

9. Let $(X, \|\cdot\|)$ be a normed linear space. Show that $\|x\| = \sup\{|\alpha| : |\alpha| < \|x\|\}$.

10. Let $(X, \|\cdot\|)$ be a normed linear space and let p be a seminorm on X. Show that $p : (X, \|\cdot\|) \to \mathbb{R}$ is continuous if and only if there exists $\alpha > 0$ such that $p(x) \leq \alpha \|x\|$ for all $x \in X$.

11. If $1 \leq p < q \leq \infty$, then show that $\|x\|_q \leq \|x\|_p$ for all $x \in \ell^p$.

12. If $\mathbf{x} \in \mathbb{R}^n$, then show that $\lim_{p \to \infty} \|\mathbf{x}\|_p = \|\mathbf{x}\|_\infty$. And if $x \in \ell^p$, then show that $\liminf_{p \to \infty} \|\mathbf{x}\|_p \geq \|\mathbf{x}\|_\infty$.

13. Let d be a metric on a real vector space X satisfying the following two conditions:

    (i) $d(x + z, y + z) = d(x, y)$ for all $x, y, z \in X$,

    (ii) $d(\alpha x, \alpha y) = |\alpha| d(x, y)$ for all $x, y \in X$ and for all $\alpha \in \mathbb{R}$.

    Show that there exists a norm $\|\cdot\|$ on X such that $d(x, y) = \|x - y\|$ for all $x, y \in X$.

14. Let f be a non-negative function on a linear space X such that $f(\alpha x) = |\alpha| f(x)$ for all $\alpha \in \mathbb{C}$. Show that f is norm on X if and only if f is a convex map which can vanish at most at one point.

15. Let $f : (X, d) \to [0, 1]$ be continuous map. Show that $f^{-1}(0)$ is a closed G_δ set.

16. Let (x_n) be a sequence in a normed linear space X which converges to a non-zero vector $x \in X$. Show that $\frac{x_1 + \cdots + x_n}{n^\alpha} \to x$ if and only if $\alpha = 1$. What are admissible values of α if $x_n \to 0$?

17. For $x = (x_n) \in l^2$, write $\|x\| = (\sum\limits_{n=1}^{\infty} a_n |x_n|^2)^{1/2}$. Find all possible sequences (a_n) such that $\| \cdot \|$ is a norm on l^2.

18. Let $\mathbb{R}^\infty$ be the real vector space of all sequences in $\mathbb{R}$, where addition and scalar multiplication are defined componentwise. Let $d((x_n), (y_n)) = \sum\limits_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{|x_n - y_n|}{1 + |x_n - y_n|}$ for all $(x_n), (y_n) \in \mathbb{R}^\infty$. Show that d is a metric on $\mathbb{R}^\infty$ but that no norm on $\mathbb{R}^\infty$ induces d.

19. Let $(X, \| \cdot \|)$ be a nonzero normed vector space. Consider the metrics d_1, d_2 and d_3 on X:

$$d_1(x, y) := \min\{1, \|x - y\|\},$$
$$d_2(x, y) := \frac{\|x - y\|}{1 + \|x - y\|},$$
$$d_3(x, y) := \begin{cases} 1 + \|x - y\| & \text{if } x \neq y, \\ 0 & \text{if } x = y, \end{cases}$$

for all $x, y \in X$. Prove that none of d_1, d_2 and d_3 is induced by any norm on X.

20. Let X be a normed vector space containing more than one point, let $x, y \in X$ and let $\varepsilon, \delta > 0$. If $B_\varepsilon[x] = B_\delta[y]$, show that $x = y$ and $\varepsilon = \delta$. Does the result remain true if X is assumed to be a metric space? Justify.

21. Let $A = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 < 1\}$ and $B = \{(x, y, z) \in \mathbb{R}^3 : z = 0\}$. Examine whether $A \cap B$ is a closed/an open subset of $\mathbb{R}^3$ with respect to the usual metric on $\mathbb{R}^3$.

22. Let F_n be a sequence of closed sets in $\mathbb{R}$ such that $F_n \subset (n, n+1]$ and $F_n \cap F_m = \emptyset$, whenever $m \neq n$. Show that $F = \bigcup\limits_{n=1}^{\infty} F_n$ is a closed set in $\mathbb{R}$.

23. For all $x, y \in \mathbb{R}$, let $d_1(x, y) = |x - y|$, $d_2(x, y) = \min\{1, |x - y|\}$ and $d_3(x, y) = \frac{|x-y|}{1+|x-y|}$. If G is an open set in any one of the three metric spaces $(\mathbb{R}, d_i)$ $(i = 1, 2, 3)$, then show that G is also open in the other two metric spaces.

24. Let X be a normed vector space and let $Y(\neq X)$ be a subspace of X. Show that Y is not open in X.

25. Let (x_n) and (y_n) be Cauchy sequences in a metric space (X, d). Show that the sequence $(d(x_n, y_n))$ is convergent.

26. Let d_o be the discrete metric on non-empty set X. Show that (X, d_o) is complete.

27. Let (x_n) be a sequence in a complete metric space (X, d) such that $\sum\limits_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$. Show that (x_n) converges in (X, d).

28. Let (x_n) be a sequence in a metric space X such that each of the subsequences (x_{2n}), (x_{2n-1}) and (x_{3n}) converges in X. Show that (x_n) converges in X.

29. Show that the following are incomplete metric spaces.

    (a) $(\mathbb{N}, d)$, where $d(m, n) = |\frac{1}{m} - \frac{1}{n}|$ for all $m, n \in \mathbb{N}$

    (b) $((0, \infty), d)$, where $d(x, y) = |\frac{1}{x} - \frac{1}{y}|$ for all $x, y \in (0, \infty)$

    (c) $(\mathbb{R}, d)$, where $d(x, y) = |\frac{x}{1+|x|} - \frac{y}{1+|y|}|$ for all $x, y \in \mathbb{R}$

    (d) $(\mathbb{R}, d)$, where $d(x, y) = |e^x - e^y|$ for all $x, y \in \mathbb{R}$

30. Examine whether the following metric spaces are complete.

    (a) $([0, 1), d)$, where $d(x, y) = |\frac{x}{1-x} - \frac{y}{1-y}|$ for all $x, y \in [0, 1)$

    (b) $((-1, 1), d)$, where $d(x, y) = |\tan \frac{\pi x}{2} - \tan \frac{\pi y}{2}|$ for all $x, y \in (-1, 1)$

    (c) $((0, 2], d)$, where $d(x, y) = |\frac{1}{x} - \frac{1}{y}|$

31. For $X(\neq \emptyset) \subset \mathbb{R}$, let $d(x, y) = \frac{|x-y|}{1+|x-y|}$ for all $x, y \in X$. Examine the completeness of the metric space (X, d), where X is

    (a) $[0, 1] \cap \mathbb{Q}$.

    (b) $[-1, 0] \cup [1, \infty)$.

    (c) $\{n^2 : n \in \mathbb{N}\}$.

32. Let $X = C[0, 1]$ be the space all the continuous functions on interval $[0, 1]$. Prove that norms $\| \cdot \|_\infty$ and $\| \cdot \|_1$ on X are not equivalent.

33. Let $C^1[0, 1]$ denote the space of all continuously differentiable functions on $[0, 1]$. For $f \in C^1[0, 1]$, define $\|f\| = \|f\|_\infty + \|f'\|_\infty$. Show that space $(C^1[0, 1], \| \cdot \|)$ is a Banach space.

34. The space $(C^1[0, 1], \| \cdot \|)$, where $\|f\| = (\|f\|_2^2 + \|f'\|_2^2)^{\frac{1}{2}}$ is complete.

35. Let $f \in C^1[0, 1]$ and $\|f\| = \|f'\|_2 + \|f\|_\infty$. Then verify if $(C^1[0, 1], \| \cdot \|)$ is complete.

36. Let $f \in C^1[0, 1]$. Then verify if $\|f\| = \min(\|f'\|_2, \|f\|_\infty)$ defines a norm on $C^1[0, 1]$.

37. Let $X = \{f \in C^1[0, 1] : f(0) = 0\}$. Then $\|f\| = \|f'\|_2$ is a norm on $C^1[0, 1]$ but not complete.

38. Let $D = \{z \in \mathbb{C} : |z| < 1\}$. Let X be the class of all functions f which are analytic on D and continuous on $\bar{D}$. Define $\|f\| = \sup\{|f(e^{it})| : 0 \leq t \leq 2\pi\}$. Show that $(X, \| \cdot \|)$ is complete.

39. Examine whether the sequence (f_n) is convergent in $(C[0, 1], d_\infty)$, where for all $n \in \mathbb{N}$ and for all $t \in [0, 1]$,

    (a) $f_n(t) = \frac{nt^2}{1+nt}$.

    (b) $f_n(t) = 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!}$.

(c) $f_n(t) = \begin{cases} nt & \text{if } 0 \le t \le \frac{1}{n}, \\ \frac{1}{nt} & \text{if } \frac{1}{n} < t \le 1. \end{cases}$

(d) $f_n(t) = \begin{cases} nt & \text{if } 0 \le t \le \frac{1}{n}, \\ \frac{n}{n-1}(1-t) & \text{if } \frac{1}{n} < t \le 1. \end{cases}$

40. Find the pointwise limit of the sequence $f_n(t) = e^{-nt^2} \sin nt$. Examine for uniform convergence of f_n on $\mathbb{R}$.

41. Let $f_n, f : \mathbb{R} \to (0, \infty)$ be such that $f_n \to f$ uniformly on $\mathbb{R}$. Examine for $e^{f_n} \to e^f$ uniformly on $\mathbb{R}$.

42. Let $f_n(t) = \sqrt{t^2 + n}$. Examine for the uniform convergence of f_n' on $\mathbb{R}$.

43. Let X be the class of all continuous functions $f : \mathbb{R} \to \mathbb{C}$ such that for each $\epsilon > 0$, there exists a compact set $K \subset \mathbb{R}$ such that $|f(x)| < \epsilon$, for all $x \in \mathbb{R} \setminus K$. Show that $(X, \ \| \ . \ \|_\infty)$ is complete.

44. Let $1 \le p < \infty$. Let X_p be a class of all the Riemann integrable functions on $[0, 1]$. Prove that $\|f\|_p = \left(\int_0^1 |f|^p \right)^{\frac{1}{p}} < \infty$. Prove that $(X_p, \ \| \ . \ \|_p)$ is a normed linear space but not complete.

45. Show that $\{(x_n) \in l^2 : |x_n| < \frac{1}{n} \text{ for all } n \in \mathbb{N}\}$ is a convex set with empty interior.

46. Suppose that $x \in \ell^p$ for some $p \ge 1$. Show that $\liminf\limits_{p \to \infty} \|x\|_p \ge \|x\|_\infty$. Prove/disprove that $\lim\limits_{p \to \infty} \|x\|_p = \|x\|_\infty$.

47. Let M be a subspace of a normed linear space X. Then show that M is closed if and only if $\{y \in M : \ \|y\| \le 1\}$ is closed in X.

48. Let $T : (C[0, \frac{\pi}{2}], \|.\|_\infty) \to (C[0, \frac{\pi}{2}], \|.\|_\infty)$ be defined by $(Tf)(x) = \int_{s=0}^x f(s) \sin s \, ds$. Show that T is not a contraction but T^2 is a contraction.

49. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous and let there exist $\alpha > 0$ such that $\|f(\mathbf{x}) - f(\mathbf{y})\| \ge \alpha \|\mathbf{x} - \mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Show that $f(\mathbb{R}^n)$ is complete.

50. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a contraction and let $g(\mathbf{x}) = \mathbf{x} - f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$. Show that $g : \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one and onto. Also, show that both g and $g^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ are continuous.

51. Find a neighborhood of $x = 0$ in which initial value problem $y' = \frac{x}{1+y^2}$ with $y(0) = 0$ has a unique solution.

Problem Set 2

Problem-set architecture.

- Start with the true/false diagnostics to separate completeness, compactness, and total boundedness cleanly in your mind.

- The next tier asks for proofs that should explicitly invoke the correct compactness criterion rather than an informal heuristic.

- The final questions are best approached after revisiting Arzelà–Ascoli, uniform convergence, and the basic examples of noncompact bounded families.

1. State whether each of the following statements is true or false, and justify your answer.

   (a) If X is a finite metric space, then $C(X)$, the space of continuous functions on X, is a finite-dimensional normed linear space.

   (b) If every countable closed subset of a metric space (X, d) is complete, then X is complete.

   (c) Total boundedness is preserved under homeomorphisms.

   (d) Let $f_n : [0, 1] \to \mathbb{R}$ be defined by $f_n = \chi_{[0,1/n]}$, and suppose that f_n converges pointwise to f. Then the set $\{f, f_n : n \in \mathbb{N}\}$ is compact in $B[0, 1]$.

   (e) Let $f_n \in C^1[0, 1]$. Then the set $\{f_n : n \in \mathbb{N}\}$ is compact in $C[0, 1]$.

   (f) Determine whether the set $\{x = (x_1, x_2, \ldots) \in \ell^2 : |x_n| \leq 1/n\}$ is totally bounded in ℓ^2.

2. Prove that if every countable closed subset of a metric space X is complete, then X is complete.

3. Show that a subset A of a metric space X is closed if and only if $A \cap K$ is compact for every compact subset K of X.

4. Find a subset of ℓ^∞ that is closed and bounded but not totally bounded.

5. Show that a subset A of a metric space (X, d) is totally bounded if and only if every sequence (x_n) in A has a subsequence (x_{n_k}) satisfying

$$d(x_{n_k}, x_{n_{k+1}}) \leq 2^{-k} \qquad (k \in \mathbb{N}).$$

6. Let K and F be nonempty subsets of a metric space (X, d). If K is compact and F is closed, show that $\text{dist}(K, F) > 0$ whenever $K \cap F = \emptyset$. Does the same conclusion remain valid if K is closed but not compact?

7. A function $f : (X, d) \to \mathbb{R}$ is called lower semicontinuous if, for each $\alpha \in \mathbb{R}$, the set $\{x \in X : f(x) > \alpha\}$ is open in X.

   (a) Show that f is lower semicontinuous if and only if

$$f(x) \leq \liminf_{n \to \infty} f(x_n)$$

   whenever $x_n \to x$.

   (b) If X is a compact metric space, prove that every lower semicontinuous function on X is bounded below and attains its minimum.

8. Let $f : (X, d) \to \mathbb{R}$ be lower semicontinuous. Show directly that for every $x \in X$ and every sequence $x_n \to x$,

$$f(x) \leq \liminf_{n \to \infty} f(x_n).$$

9. Let X be a compact metric space, and let $f : X \to X$ satisfy $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. Show that f is onto. Is compactness of X necessary?

10. Let X be a compact metric space, and let $f : X \to X$ satisfy $d(f(x), f(y)) \geq d(x, y)$ for all $x, y \in X$. Show that f is an onto isometry.

11. Let X be a compact metric space, and let $f : X \to X$ be bijective with

$$d(f(x), f(y)) \leq d(x, y) \qquad (x, y \in X).$$

Show that f is an isometry.

12. Let X be a compact metric space, and let $\mathcal{F} \subset C(X)$.

(a) Prove that an equicontinuous family $\mathcal{F}$ is pointwise bounded if and only if it is uniformly bounded.

(b) Prove that $\mathcal{F}$ is pointwise equicontinuous if and only if it is uniformly equicontinuous.

13. Let X be a compact metric space, and let (f_n) be a sequence in $C(X)$.

(a) Suppose that (f_n) is equicontinuous and converges pointwise. Show that (f_n) converges uniformly.

(b) If (f_n) decreases pointwise to 0, show that (f_n) is equicontinuous.

(c) If (f_n) is equicontinuous, show that the set

$$\{x \in X : (f_n(x)) \text{ converges}\}$$

is closed in X.

14. For fixed $k > 0$ and $0 < \alpha \leq 1$, define

$$\text{Lip}_{k,\alpha} = \{f \in C[0, 1] : |f(x) - f(y)| \leq k|x - y|^\alpha\}.$$

Show that $\{f \in \text{Lip}_{k,\alpha} : f(0) = 0\}$ is a compact subset of $C[0, 1]$. Determine whether the set $\{f \in \text{Lip}_{k,\alpha} : \int_0^1 f(t) \, dt = 1\}$ is compact.

15. Let $K(x, t)$ be continuous on the square $[0, 1] \times [0, 1]$. For $f \in C[0, 1]$, define

$$Tf(x) = \int_0^1 f(t) K(x, t) \, dt.$$

Show that T maps bounded sets into equicontinuous sets.

16. Let $f_n \in C[0,1]$ satisfy $\|f_n\|_\infty \leq 1$. Define

$$F_n(x) = \int_0^x f_n(t)\, dt.$$

    Show that (F_n) has a convergent subsequence.

17. If $f \in B[0,1]$, show that $B_n(f)(x) \to f(x)$ at each point of continuity of f.

18. Give an example of a sequence of functions $f_n \in C[0,1]$ that decreases pointwise to a limit function f, but not uniformly.

19. For a given polynomial p and $\varepsilon > 0$, show that there exists a polynomial q with rational coefficients such that $\|p - q\|_\infty < \varepsilon$ on $[0,1]$.

20. Let (x_i) be a sequence in $(0,1)$ such that

$$\frac{1}{n}\sum_{i=1}^n x_i^k$$

    converges for each $k = 0, 1, 2, \ldots$. Show that

$$\frac{1}{n}\sum_{i=1}^n f(x_i)$$

    converges for every $f \in C[0,1]$.

21. For $f \in C^1[0,1]$ and $\varepsilon > 0$, show that there exists a polynomial p such that

$$\|f - p\|_\infty < \varepsilon \qquad \text{and} \qquad \|f' - p'\|_\infty < \varepsilon.$$

22. Let $f : [1, \infty) \to \mathbb{R}$ be continuous, and suppose that $\lim_{x\to\infty} f(x)$ exists. Given $\varepsilon > 0$, show that there exists a polynomial p such that

$$|f(x) - p(1/x)| < \varepsilon \qquad (x \geq 1).$$

Problem Set 3

Problem-set architecture.

- Read this set in two tracks: first the separation/connectedness questions, then the continuity and image-of-an-interval consequences.

- Many problems can be solved elegantly by choosing the right continuous map and pushing connectedness through the image.

- The later questions are deliberately synthesis-heavy and reward a clear stock of standard examples, especially subsets of $\mathbb{R}^2$.

1. State whether each of the following statements is true or false, and justify your answer.

   (a) It is possible to write $\mathbb{R}^2$ as a countable union of path-connected sets.

   (b) The set of all real polynomials whose zero-set complement is connected is countable.

   (c) There exists a nonempty open connected set $A \subset \mathbb{R}^n$ such that every real-valued function on A is continuous.

   (d) If a metric space X is path-connected, then there exists a continuous map $f : [0, 1] \to X$ that is onto.

   (e) Let $f : (X, d) \to \mathbb{R}$ be such that the graph $G_f = \{(x, f(x)) : x \in X\}$ is connected. Does this imply that X is connected?

   (f) There exists a discontinuous function $f : \mathbb{R} \to \mathbb{R}$ such that the graph G_f is connected in $\mathbb{R}^2$, while $\text{int}(\overline{G_f}) \neq \emptyset$.

2. Let A be a connected subset of a metric space X, and let B be a clopen subset of X such that $A \cap B \neq \emptyset$. Show that $A \subseteq B$.

3. If E is a connected subset of a metric space X and $E \subseteq A \cup B$, where A and B are disjoint open subsets of X, show that either $E \subseteq A$ or $E \subseteq B$.

4. Prove that $E \subseteq X$ is disconnected if and only if there exist nonempty open sets A and B such that $E = A \cup B$, $A \cap \overline{B} = \emptyset$, and $\overline{A} \cap B = \emptyset$.

5. If every pair of points in X is contained in some connected subset of X, show that X itself is connected.

6. If E and F are connected subsets of X with $E \cap F \neq \emptyset$, show that $E \cup F$ is connected.

7. If E and F are nonempty subsets of X such that $E \cup F$ is connected, show that $\overline{E} \cap \overline{F} \neq \emptyset$.

8. Show that the complement of any countable subset of $\mathbb{R}^2$ is path-connected.

9. Prove that X is disconnected if and only if there exists a continuous function $f : X \to \mathbb{R}$ such that $f^{-1}(\{0\}) = \emptyset$, while both $f^{-1}((-\infty, 0))$ and $f^{-1}((0, \infty))$ are nonempty.

10. If X is connected and has at least two points, show that X is uncountable.

11. Let $f : [a, b] \to \mathbb{R}$ have the intermediate value property, and assume that $f^{-1}(\{y\})$ is closed for every $y \in \mathbb{R}$. Show that f is continuous.

12. If $f : [a, b] \to [a, b]$ is continuous, show that f has a fixed point.

13. Let $f : [0, 2] \to \mathbb{R}$ be continuous with $f(0) = f(2)$. Show that there exists $x \in [0, 1]$ such that $f(x) = f(x + 1)$.

14. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and open, show that f is strictly monotone.

15. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and one-to-one, show that f is strictly monotone.

16. Prove that there does not exist a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(\mathbb{Q}) \subseteq \mathbb{R} \smallsetminus \mathbb{Q}$ and $f(\mathbb{R} \smallsetminus \mathbb{Q}) \subseteq \mathbb{Q}$.

17. If A and B are closed subsets of X such that both $A \cup B$ and $A \cap B$ are connected, show that A and B are connected.

18. Let $I = (\mathbb{R} \smallsetminus \mathbb{Q}) \cap [0, 1]$ and $Q = \mathbb{Q} \cap [0, 1]$. Show that there exists a continuous map from I onto Q, but that there does not exist a continuous map from $[0, 1]$ onto Q.

19. Suppose that $f : \mathbb{R} \to \mathbb{R}$ has the intermediate value property. If its graph G_f is closed, show that f is continuous.

20. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable, show that f' has the intermediate value property.

21. Show that the set
$$\{(x, y) \in \mathbb{R}^2 : x^2 + y^3 \in \mathbb{R} \smallsetminus \mathbb{Q}\}$$
is disconnected in the usual topology of $\mathbb{R}^2$.

22. Show that $GL_n(\mathbb{C})$ is path-connected, using the fact that every polynomial over $\mathbb{C}$ has only finitely many zeros. Is $GL_n(\mathbb{C})$ open in $M_n(\mathbb{C})$?

Problem Set 4

Problem-set architecture.

- Begin with the differentiability diagnostics and small Jacobian computations before attempting the inverse/implicit function problems.

- Several questions are most transparent if you first identify the correct linearization and only then compute coordinates.

- The final tier emphasizes local invertibility, homogeneous maps, and geometric applications; these are intended as capstone problems for the chapter.

1. State whether each of the following statements is true or false, and justify your answer.

   (a) There exists a one-to-one continuous function from $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ onto $\mathbb{R}^2$.

   (b) There exists a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ which is differentiable only at $(1, 0)$.

   (c) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that $f_x(0, 0) = 0$. Then there exists some $\delta > 0$ such that $f(x, 0)$ is continuous on $(-\delta, \delta)$.

   (d) If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable with $f(0, 0) = (1, 1)$ and $[f'(0, 0)] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then there cannot exist a differentiable function $g : \mathbb{R}^2 \to \mathbb{R}^2$ with $g(1, 1) = (0, 0)$ and $(f \circ g)(x, y) = (y, x)$ for all $(x, y) \in \mathbb{R}^2$.

(e) A continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}^2$ cannot be one-to-one and onto if $\det[f'(x, y)] = 0$ for some $(x, y) \in \mathbb{R}^2$.

(f) The equation $\sin(xyz) = z$ defines x implicitly as a differentiable function of y and z locally around the point $(x, y, z) = (\frac{\pi}{2}, 1, 1)$.

2. Let Ω be an open subset of $\mathbb{R}^n$ and let $f : \Omega \to \mathbb{R}^m$ and $g : \Omega \to \mathbb{R}^m$ be continuous at $\mathbf{x}_0 \in \Omega$. If for each $\varepsilon > 0$, there exist $\mathbf{x}, \mathbf{y} \in B_\varepsilon(\mathbf{x}_0)$ such that $f(\mathbf{x}) = g(\mathbf{y})$, then show that $f(\mathbf{x}_0) = g(\mathbf{x}_0)$.

3. Let $A(\neq \emptyset) \subset \mathbb{R}^n$ be such that every continuous function $f : A \to \mathbb{R}$ is bounded. Show that A is a closed and bounded subset of $\mathbb{R}^n$.

4. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be linear and let $f(\mathbf{x}) = T(\mathbf{x}) \cdot \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Find $f'(\mathbf{x})$, where $\mathbf{x} \in \mathbb{R}^n$.

5. Examine the differentiability of f at $\mathbf{0}$, where

   (a) $f : \mathbb{R}^n \to \mathbb{R}$ satisfies $|f(\mathbf{x})| \leq \|\mathbf{x}\|_2^2$ for all $\mathbf{x} \in \mathbb{R}^n$.

   (b) $f : \mathbb{R}^n \to \mathbb{R}$ is defined by $f(\mathbf{x}) = \|\mathbf{x}\|_2$ for all $\mathbf{x} \in \mathbb{R}^n$.

   (c) $f : \mathbb{R}^n \to \mathbb{R}^n$ is defined by $f(\mathbf{x}) = \|\mathbf{x}\|_2 \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

6. Let Ω be a nonempty open subset of $\mathbb{R}^n$. Let $f : \Omega \to \mathbb{R}$ be differentiable at $\mathbf{x}_0 \in \Omega$, let $f(\mathbf{x}_0) = 0$ and let $g : \Omega \to \mathbb{R}$ be continuous at $\mathbf{x}_0$. Prove that $fg : \Omega \to \mathbb{R}$, defined by $(fg)(\mathbf{x}) = f(\mathbf{x})g(\mathbf{x})$ for all $\mathbf{x} \in \Omega$, is differentiable at $\mathbf{x}_0$.

7. Let Ω be a nonempty open subset of $\mathbb{R}^n$ and let $g : \Omega \to \mathbb{R}^n$ be continuous at $\mathbf{x}_0 \in \Omega$. If $f : \Omega \to \mathbb{R}$ is such that $f(\mathbf{x}) - f(\mathbf{x}_0) = g(\mathbf{x}) \cdot (\mathbf{x} - \mathbf{x}_0)$ for all $\mathbf{x} \in \Omega$, then show that f is differentiable at $\mathbf{x}_0$.

8. The directional derivatives of a differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ at $(0, 0)$ in the directions of $(1, 2)$ and $(2, 1)$ are 1 and 2 respectively. Find $f_x(0, 0)$ and $f_y(0, 0)$.

9. Let $A \in GL(\mathbb{R}^n)$ and $\alpha \geq 2$. If $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $\|f(x)\| \leq k\|x\|^\alpha$, for some $k > 0$. Determine whether the map $g = f + A$ is continuously differentiable at $\mathbf{0}$ and g is invertible in the neighborhood of $\mathbf{0}$.

10. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable such that $f(1, 1) = 1$, $f_x(1, 1) = 2$ and $f_y(1, 1) = 5$. If $g(x) = f(x, f(x, x))$ for all $x \in \mathbb{R}$, determine $g'(1)$.

11. Let $A \in GL_n(\mathbb{C})$. Show that the set $E = \{B \in L_n(\mathbb{C}) : \|B - A\| < \frac{1}{2\|A^{-1}\|}\}$ is open in $GL_n(\mathbb{C})$. Hence deduce that E is path-connected in $L_n(\mathbb{C})$.

12. Prove that a differentiable function $f : \mathbb{R}^n \setminus \{\mathbf{0}\} \to \mathbb{R}^m$ is homogeneous of degree $\alpha \in \mathbb{R}$ (*that is $f(t\mathbf{x}) = t^\alpha f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ and for all $t > 0$*) if and only if $f'(\mathbf{x})(\mathbf{x}) = \alpha f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$.

13. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is satisfying $f(rx) = r^{\frac{3}{2}} f(x)$ for all $(x, r) \in \mathbb{R}^n \times (0, \infty)$. Determine whether f is differentiable at $\mathbf{0}$.

14. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable such that $f_x(a, b) = f_y(a, b)$ for all $(a, b) \in \mathbb{R}^2$ and $f(a, 0) > 0$ for all $a \in \mathbb{R}$. Show that $f(a, b) > 0$ for all $(a, b) \in \mathbb{R}^2$.

15. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be such that $f(tx) = t^2 f(x)$ for every $t > 0$ and $x \in \mathbb{R}^n$. Does it imply that f is differentiable at 0?

16. Let Ω be an open subset of $\mathbb{R}^n$ such that $\mathbf{a}, \mathbf{b} \in \Omega$ and $S = \{(1 - t)\mathbf{a} + t\mathbf{b} : t \in [0, 1]\} \subset \Omega$. If $f : \Omega \to \mathbb{R}^m$ is differentiable at each point of S, then show that there exists a linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ such that $f(\mathbf{b}) - f(\mathbf{a}) = L(\mathbf{b} - \mathbf{a})$.

17. Let $f(x, y) = (2ye^{2x}, xe^y)$ for all $(x, y) \in \mathbb{R}^2$. Show that there exist open sets U and V in $\mathbb{R}^2$ containing $(0, 1)$ and $(2, 0)$ respectively such that $f : U \to V$ is one-to-one and onto.

18. Let $f(x, y) = (3x - y^2, 2x + y, xy + y^3)$ and $g(x, y) = (2ye^{2x}, xe^y)$ for all $(x, y) \in \mathbb{R}^2$. Examine whether $(f \circ g^{-1})'(2, 0)$ exists (with a meaningful interpretation of g^{-1}) and find $(f \circ g^{-1})'(2, 0)$ if it exists.

19. For $n \geq 2$, let $B = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_2 < 1\}$ and let $f(\mathbf{x}) = \|\mathbf{x}\|_2^2 \mathbf{x}$ for all $\mathbf{x} \in B$. Show that $f : B \to B$ is differentiable and invertible but that $f^{-1} : B \to B$ is not differentiable at $\mathbf{0}$.

20. Let $f : \mathbb{R} \to \mathbb{R}^n$ be a differentiable function with $\|f'(x)\| \leq 1$. Show that f satisfies $\|f(x) - f(y)\| \leq |x - y|$ for every $x, y \in \mathbb{R}$. (*Hint.* Use the one-dimensional mean value theorem.)

21. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable. Find a suitable condition under which the equation $f(x, y) = 0$ can be solved for x in a neighborhood of $(0, 0)$.

22. Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and $f'(0) \neq 0$. Show that $F(x, y) = (x - yf(y), f(y))$ is locally invertible in some neighborhood of $(0, 0)$. Does there exist some f for which F is globally invertible?

23. Using implicit function theorem, show that the system of equations

$$x^3(y^3 + z^3) = 0,$$
$$(x - y)^3 - z^2 = 7,$$

can be solved locally near the point $(1, -1, 1)$ for y and z as a differentiable function of x.

24. Using implicit function theorem, show that in a neighbourhood of any point $(x_0, y_0, u_0, v_0) \in \mathbb{R}^4$ which satisfies the equations

$$x - e^u \cos v = 0,$$
$$v - e^y \sin x = 0,$$

there exists a unique solution $(u, v) = \varphi(x, y)$ satisfying $\det[\varphi'(x, y)] = v/x$.

25. Show that around the point $(0, 1, 1)$, the equation $xy - z \log y + e^{xz} = 1$ can be solved locally as $y = f(x, z)$ but cannot be solved locally as $z = g(x, y)$.

26. Find the 3rd order Taylor polynomial of $f(x, y, z) = x^2 y + z$ about the point $(1, 2, 1)$.

27. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable. Show that f is not one-to-one.

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