Special Theory of Relativity

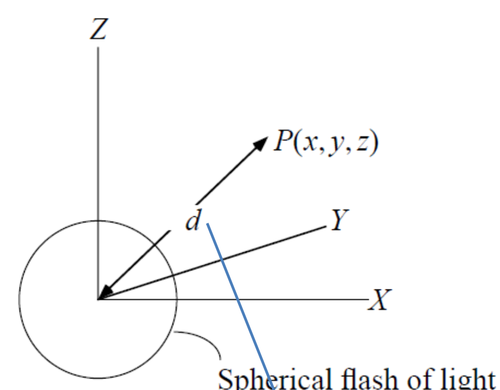
PH101

Lec-3

Clock Synchronization

- In order to measure the time at which an event occurred at a point in space, we assumed that all of space are filled with clocks, one for each point in space.
  - ✓There are separate set of clocks for each set of rulers !
  - ✓All the clocks in each frame of reference should be synchronized in some way !

We have to be very clear about what we are doing when we are comparing the times of occurrence of events, particularly when the events occur at two spatially separate points.



How this synchronization is to be achieved ?

- Imagine that at the spatial origin of the frame of reference we have a master clock, and that at some instant $t_0 = 0$ indicated by this clock a spherical flash of light is emitted from the source.

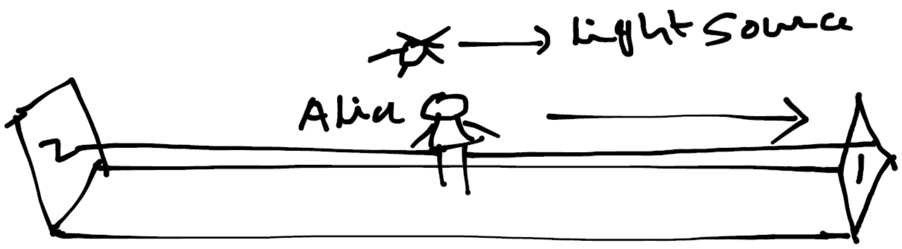
✓ When this flash reaches P, the clock at that position is adjusted to read $t = d/c$!

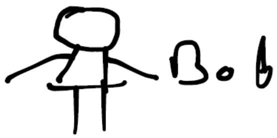
$d^2 = x^2 + y^2 + z^2$

This procedure is followed for all the clocks throughout the frame of reference, and applies for other F.O.R.

Therefore, $x^2 + y^2 + z^2 = c^2 t^2$

Simultaneity





✓Consider a train moving with constant velocity, v with 2 light sensored doors at either end of the carriage, and a light bulb and observer positioned directly in the centre of the carriage.

For the observer inside the train

- When the light bulb turns on, Alice will see the light travel with velocity **c** to reach the doors 1 and 2 at the same time.

✓For Alice events will appear simultaneous

From the observer on the platform

- Bob will see the light bulb turn on as the light will travel towards the doors.
- However, the train is also moving with v m/s thus door 2 will move towards the light source while 1 moves away
  - **light will reach door 2 before 1**

✓For Bob Events are not simultaneous

❖This suggests that each coordinate system has its own observers with "clocks" that are synchronized…

Lorentz transformation

- The special set of linear transformations that:
  - ✓Preserve the constancy of the speed of light (*c*) between inertial observers;
  - ✓Account for the problem of simultaneity between these observers !

- We will derive the general form that the transformation law must take purely from kinematic/symmetry considerations.

Based on two assumptions:

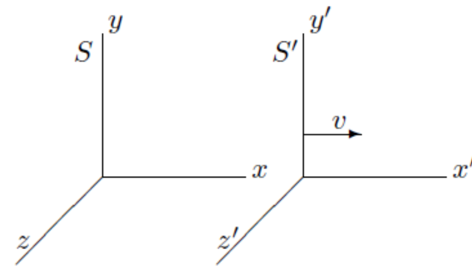
Homogeneity: The intrinsic properties of empty space are the same everywhere and for all time.

Spatial Isotropy: The intrinsic properties of space is the same in all directions.

No Memory: The extrinsic properties of the rulers and clocks may be functions of their current states of motion, but not of their states of motion at any other time.

Lorentz transformation

- The starting point is to consider two inertial frames S and S′ where S′ is moving with a velocity *v* relative to *S* .



✓Suppose that the origins coincide at t = t’ = 0, and that at this time there is a burst of light at the origins.

✓ Light travels with speed c in both frames, at time t in frame S, the wavefront of the light forms a sphere of radius r = ct centered on the origin of S, and in S’ the wavefront forms a similar sphere of radius r’ =ct’

Spherical wavefronts in S’: $x'^2 + y'^2 + z'^2 = c^2 t'^2$ $\neq$ $x^2 + y^2 + z^2 = c^2 t^2$ Spherical wavefronts in S

Galilean transformation

$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t = t'$$

We now try t’ = t + f x

$$x^2 - \cancel{2xvt} + v^2 t^2 + y^2 + z^2 = c^2(t^2 + \cancel{2tfx} + f^2 x^2)$$

Terms will cancel if f= - v/c^2

Hence for t’ = t – v x/c^2

$$x^2\left(1 - \frac{v^2}{c^2}\right) + y^2 + z^2 = c^2 t^2\left(1 - \frac{v^2}{c^2}\right)$$

Lorentz transformation

- we can remove the $(1 - v^2/c^2)$ terms by altering our transformations :

$$x' = \frac{x - vt}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}}$$ and $$t' = \frac{t - \frac{vx}{c^2}}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}},$$

Letting $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{(1-\beta^2)^{\frac{1}{2}}}$

$$x' = \gamma(x - vt)$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$y' = y,\ z' = z$$

Also,

$$x = \gamma(x' + vt')$$

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right)$$

$$y = y',\ z = z'$$

✓Transformation equations represent an intrinsic property of space and time.

✓Other material objects, which may have nothing to do with light, will also be influenced by this fundamental property of space and time.

✓It is is saying something about the properties of space and time, and the consequent behaviour that matter and forces must have in order to be consistent with these properties.

Consequences

- The Lorentz transformation leads to a number of important consequences for our understanding of the motion of objects in space and time without concern for how the matter was set into motion, i.e. the kinematics of matter.
  - ✓We will look at the consequences for our understanding of the laws of motion themselves, that is relativistic dynamics.
  - ✓Perhaps the most startling aspect of the Lorentz Transformation is the appearance of a transformation for time.

Length Contraction : A length gets shortened when measured by observer in motion with respect to it !

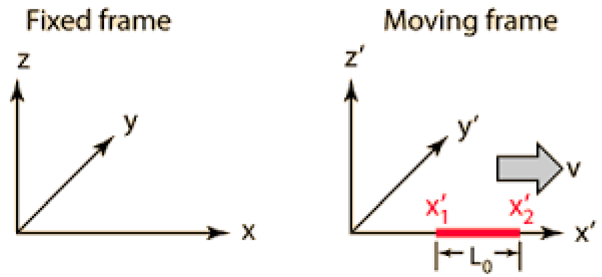
Time Dilation : Clocks slow down when seen by observer moving w.r.t them !

Simultaneity : Two events occurring at the same time !

Length contraction

✓ Length no longer has an absolute meaning: the length of an object depends on its motion relative to the frame of reference in which its length is being measured.

➢ Let us consider a rod moving with a velocity ***v*** relative to a frame of reference S , and lying along the x axis.



✓ As the rod is stationary in S′ , the ends of the rod will have coordinates x'_1 and x'_2 which remain fixed as functions of the time in *S′* . The length of the rod, as measured in S′ is $\mathbf{L_0 = x'_2 - x'_1}$

Proper length of the rod

✓ The length of the rod as measured with respect to *S* ?

$$L_0 = x'_2 - x'_1 = \frac{x_2 - vt_2 - x_1 + vt_1}{\sqrt{1-\frac{v^2}{c^2}}}$$

✓ Two measurements made in the fixed frame and are simultaneously in that frame, $\mathbf{t_1 = t_2}$

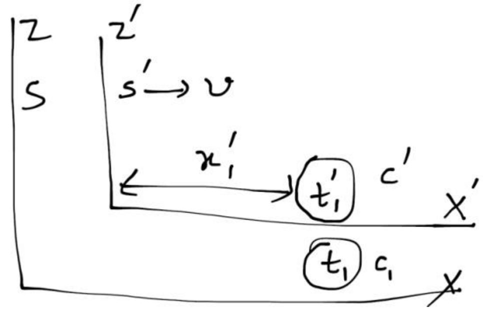
$$L = L_0\sqrt{1-\frac{v^2}{c^2}} = \frac{L_0}{\gamma}$$

Length contraction

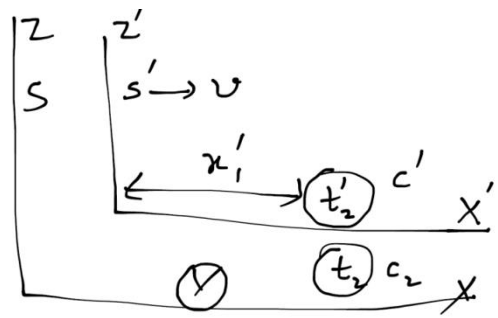
$\mathbf{L < L_0}$

Time dilation

- Perhaps the most unexpected consequence of the Lorentz transformation is the way in which our 'commonsense' concept of time has to be drastically modified.



- Consider a clock C' placed at rest in a frame of reference S' at some point x_1' on the X axis.
- Suppose once again that this frame is moving with a velocity ***v*** relative to some other frame of reference S .
- Some time later, clock C' will read the time t'_2 at which instant a different clock C_2 in S will pass the position x'_1 in S'.



$$T_0 = t'_2 - t'_1$$ ⇨ Time interval measured in the moving frame !

<u>Using Lorenz Transformation</u>

$$T = t_2 - t_1 = \frac{t'_2 + \frac{vx'_2}{c^2} - t'_1 - \frac{vx'_1}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}}$$

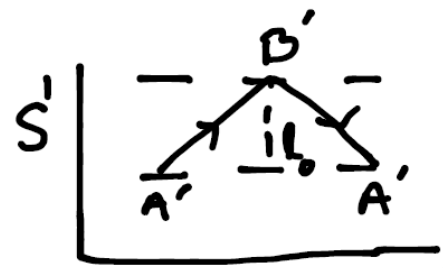
⇨ In the moving frame, time measurements are done in the same place , $x'_2 = x'_1$

⇨ $$T = \frac{T_0}{\sqrt{1-\frac{v^2}{c^2}}} = T_0\gamma$$

Time dilation *($T > T_0$)*

Light pulse Clock Method

Time dilation : Consider a light source moving with a velocity v with respect to a frame S' along the x-axis.



$\tau \Rightarrow$ time interval in s'

Light Pulse travels from A' to B' and back

$\tau_0 = \frac{2l_0}{c}$

$$\tau = 2\left[l_0^2 + \left(\frac{v\tau}{2}\right)^2\right]^{\frac{1}{2}}$$

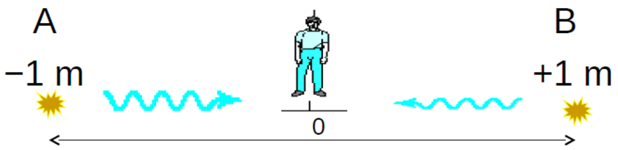
$$\therefore \tau^2 = 4\left[l_0^2 + \frac{v^2\tau^2}{4}\right]$$

$$\therefore \tau = \frac{2l_0}{(c^2 - v^2)^{\frac{1}{2}}} = \frac{\frac{2l_0}{c}}{\sqrt{1-\frac{v^2}{c^2}}}$$

Hence $\boxed{\tau = \gamma \tau_0}$

Simultaneity

- Simultaneity is not an absolute concept anymore



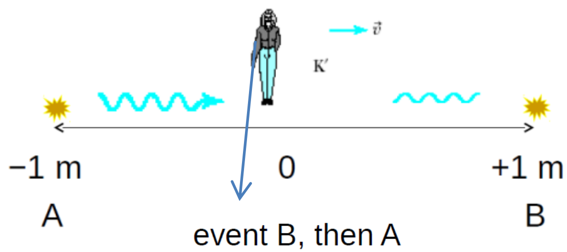
Precise new definition needed:

Two events in an inertial frame of reference are simultaneous if light signals from the locations of these events reach an observer positioned halfway between them at the same time !

✓Consider two events **1 and 2** which are simultaneous in S i.e. $\mathbf{t_1 = t_2}$, but which occur at **two different places $\mathbf{x_1}$ and $\mathbf{x_2}$**

➢Then, in **S'**, the time interval between these two events is

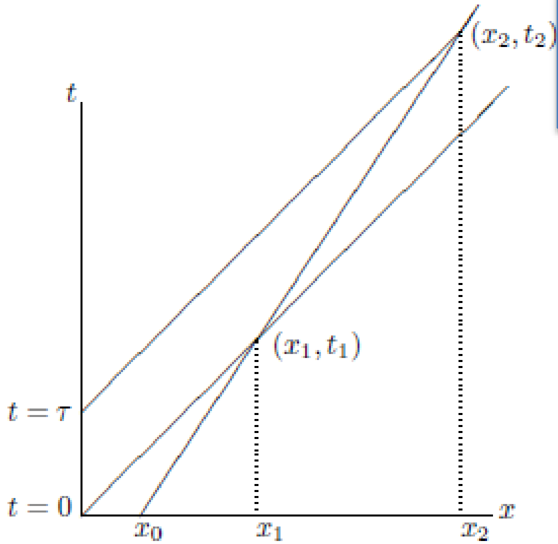
$$t'_2 - t'_1 = \gamma\,(t_2 - v\,x_2/c^2) - \gamma\,(t_1 - v\,x_1/c^2)$$
$$= \gamma\,(x_1 - x_2)\,v/c^2 \neq 0$$



✓Here t'_1 is the time registered on the clock in S' which coincides with the position x_1 in S at the instant t_1 that the event 1 occurs and similarly for t'_2 .

✓ Note that if the two events occur at the same point (i.e. $\mathbf{x_1 = x_2}$) then the events will occur simultaneously in all frames of reference.

Relativistic Doppler Effect



Consider now an observer in a frame S' moving along the x-axis with a velocity v away from the source of light ($x = x_0$ at $t = t'=0$)

$$x_1 = ct_1 = x_0 + vt_1$$
$$x_2 = c(t_2 - \tau) = x_0 + vt_2$$

$$x_0 = ct_1 - vt_1 = c(t_2 - \tau) - vt_2$$
$$\Rightarrow t_1(c-v) = t_2(c-v) - c\tau$$
$$\Rightarrow t_2 - t_1 = \frac{c\tau}{c-v}$$

$$x_2 - x_1 = v(t_2 - t_1)$$
$$\Rightarrow x_2 - x_1 = \frac{vc\tau}{c-v}$$

Consider a source of light at rest in a frame S at x=0. It emits a crest of light at t=0 and a second crest at time $t=\tau$ = period.

Using Lorentz transformations in S'

$$\tau' = t_2' - t_1' = \gamma\left[t_2 - t_1 - \frac{v}{c^2}(x_2 - x_1)\right] \Rightarrow \tau' = \gamma\left[\frac{c\tau}{c-v} - \frac{v}{c^2}\frac{vc\tau}{c-v}\right]$$

Further simplifications

$$\Rightarrow \tau' = \frac{\tau(1+\beta)}{(1-\beta^2)^{\frac{1}{2}}} = \frac{\tau(1+\beta)}{(1-\beta)^{\frac{1}{2}}(1+\beta)^{\frac{1}{2}}}$$
$$\Rightarrow \tau' = \left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}\tau$$

Change in frequency f-> f'

$$f' = \left(\frac{1-\beta}{1+\beta}\right)^{\frac{1}{2}} f$$