Special Theory of Relativity

PH101

Lec-4

Time interval Revisited

- ✓ Proper length of an object is the length measured in the rest frame of the object.
- ✓ Proper time interval $\Delta\tau_0$ between two events is time interval measured between two events which occur at the same place.

Often stated as the time interval that can be measured by the same clock

- ➢ The measurements of the time interval between two events depends upon the reference frame !

$$(c\Delta t)^2 - \Delta x^2 = (c\Delta t')^2 - \Delta x'^2$$

Coordinate time interval

$$\Delta\tau_o^2 = \Delta t^2 - \left(\frac{v\Delta t}{c}\right)^2 = \Delta t^2 - \left(\frac{\Delta x}{c}\right)^2 = \Delta t^2\left[1 - \left(\frac{v}{c}\right)^2\right]$$

Inertial proper time interval

Invariant

spacetime interval

The coordinate time interval depends upon the relative speeds of the two different frames of reference, and the distance between the two events as measured in the corresponding frame of reference. It must be different in different frames of reference.

There are a large number of coordinate time intervals which could be measured for the same two events, but only one spacetime interval !

Implication of cause and effects

- ✓ Consider two events A and B that occur at coordinate locations x_A and x_B and at times t_A and t_B, respectively in the laboratory or Home reference frame !

Push of a button which initiates a light pulse

Reception of that light pulse which triggers the detonation of a bomb

✓ Clearly in the Home Frame, one event causes the other

➢ We demand that all inertial frames "see" the same events occuring in the same order, and not reversed

T = $(t_B - t_A) > 0$ -> time interval in HOME frame !

$$(t'_B - t'_A) = \gamma\left[(t_B - t_A) - \frac{v}{c^2}(x_B - x_A)\right]$$

$$= \gamma\left[(t_B - t_A) - \frac{v}{c^2}c(t_B - t_A)\right]$$

$$= \gamma(t_B - t_A)\left[1 - \left(\frac{v}{c}\right)\right]$$

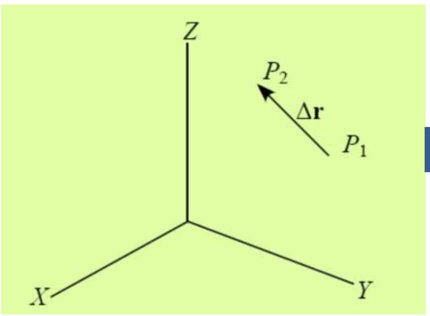
The time interval in moving frame will follow the same order of events, i.e $(t_B' - t_A') > 0$, only if

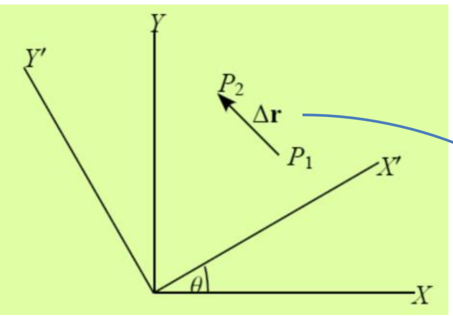
$$\left(\frac{v}{c}\right) \leq 1$$

✓ The speed of any inertial reference frame which will preserve cause and effect must be less than or equal to the speed of light !

Geometrical properties of 3D space

- ✓ Imagine a suitable set of rulers so that the position of a point *P* can be specified by the three coordinates (*x*, *y*, *z*) with respect to this coordinate system, which we will call *R*.





$$\begin{pmatrix}\Delta x'\\ \Delta y'\\ \Delta z'\end{pmatrix}_{R'} = \begin{pmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\Delta x\\ \Delta y\\ \Delta z\end{pmatrix}_{R}$$

- ✓ We consider two such points P_1 with coordinates (x_1, y_1, z_1) and P_2 with coordinates (x_2, y_2, z_2) then the line joining these two points defines a vector **r** .

$$\Delta\mathbf{r} \doteq \begin{pmatrix}\Delta x\\ \Delta y\\ \Delta z\end{pmatrix}_{R} \doteq \begin{pmatrix}\Delta x'\\ \Delta y'\\ \Delta z'\end{pmatrix}_{R'}$$

$$(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 = (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2$$

(Length)^2 in both frame

$$\Delta x_1\Delta x_2 + \Delta y_1\Delta y_2 + \Delta z_1\Delta z_2 = \Delta x_1'\Delta x_2' + \Delta y_1'\Delta y_2' + \Delta z_1'\Delta z_2'$$

Angles in both frame

- ✓ Thus the transformation is consistent with the fact that the length and relative orientation of these vectors is independent of the choice of coordinate systems.

Space time four vector

- ✓ we will consider two events E_1 and E_2 occurring in spacetime. For event E_1 with coordinates (x_1, y_1, z_1, t_1) in frame of reference S and (x_1', y_1', z_1', t_1') in S'

<u>Lorentz transformations</u>

$$ct_1' = \gamma c t_1 - \frac{\gamma v_x}{c} x_1$$
$$x_1' = -\frac{\gamma v_x}{c} c t_1 + \gamma x_1$$
$$y_1' = y_1$$
$$z_1' = z_1$$

<u>Separation of two events in spacetime</u>

$$c\Delta t' = c(t_2' - t_1') = \gamma c \Delta t - \frac{\gamma v_x}{c} \Delta x$$
$$\Delta x' = x_2' - x_1' = -\frac{\gamma v_x}{c} c\Delta t + \gamma \Delta x$$
$$\Delta y' = \Delta y$$
$$\Delta z' = \Delta z$$

$$\begin{pmatrix} c\Delta t' \\ \Delta x' \\ \Delta y' \\ \Delta z' \end{pmatrix}_{S'} = \begin{pmatrix} \gamma & -\gamma v_x/c & 0 & 0 \\ -\gamma v_x/c & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}_{S}$$

$$(\Delta s)^2 = (c\Delta t)^2 - \left[(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2\right] = (c\Delta t')^2 - \left[(\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2\right]$$

interval between the two events

Analogous to, but fundamentally different from, the length of a three-vector in that it can be positive, zero, or negative.

$$(c\Delta t_1)(c\Delta t_2) - [\Delta x_1 \Delta x_2 + \Delta y_1 \Delta y_2 + \Delta z_1)\Delta z_2]$$

Analogous to the scalar product for three-vectors.

$$\Delta \vec{s} = \begin{pmatrix} c\Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

Four-vector

Transformation of Velocities

- Suppose, relative to a frame S, a particle has a velocity $\mathbf{u} = u_x\,\mathbf{i} + u_y\,\mathbf{j} + u_z\,\mathbf{k}$

  Where $u_x = dx/dt$

  ✓ What we require is the velocity of this particle as measured in the frame of reference S' moving with a velocity υ relative to S ?

  ✓ If the particle has coordinate x at time t in S, then the particle will have coordinate x' at time t' in S', such that $x' = \gamma\,(x - \upsilon t)$ and $t' = \gamma\,(t - \upsilon x/c^2)$

- Let the particle is displaced to a new position **x + dx** at time **t + dt** in S , then in S' it will be at the position **x' + dx'** at time **t' + dt'**

$$X' + dx' = \gamma\,(x + dx - \upsilon\,(t + dt)) \quad \text{and} \quad t' + dt' = \gamma\,(t + dt - \upsilon/c^2\,(x + dx))$$

$$\frac{dx'}{dt'} = \frac{\gamma(dx - vdt)}{\gamma(dt - \frac{vdx}{c^2})} = \frac{\frac{dx}{dt} - v}{1 - \frac{v\frac{dx}{dt}}{c^2}} \longrightarrow u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}$$

$$u'_y = \frac{u_y}{\gamma\left(1 - \frac{u_x v}{c^2}\right)} \qquad u'_z = \frac{u_z}{\gamma\left(1 - \frac{u_x v}{c^2}\right)}$$

The inverse transformation follows by replacing $\upsilon \to -\upsilon$ *interchanging the primed and unprimed* variables

$$u_x = \frac{u'_x + v}{1 + \frac{u'_x v}{c^2}} \qquad u_y = \frac{u'_y}{\gamma\left(1 + \frac{u'_x v}{c^2}\right)} \qquad u_z = \frac{u'_z}{\gamma\left(1 + \frac{u'_x v}{c^2}\right)}$$