Special Theory of Relativity

PH101

Lec-5

Transformation of Velocities

- In particular, if $u_x = c$ and $u_y = u_z = 0$, we find that $$u'_x = \frac{c - v}{1 - \frac{v}{c}} = c$$
- If the particle has the speed c in S , it has the same speed c in S' !
- Now consider the case in which the particle is moving with a speed that is less that c, *i.e*. suppose $u_y = u_z = 0$ and $|u_x| < c$, therefore

$$u'_x - c = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} - c = \frac{(v + c)(u_x - c)}{c(1 - \frac{u_x v}{c^2})}$$

✓ Now, if $v < c$ then along with $|u_x| < c$, it is not difficult to show that the $u'_x < c$!

$$u'_x + c = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} + c = \frac{(c - v)(u_x + c)}{c(1 - \frac{u_x v}{c^2})}$$

Therefore, $u'_x > -c$

$|u'_x| < c$

Therefore, if a particle has a speed less than c in one frame of reference, then its speed is always less than c in any other frame of reference, provided this other frame of reference is moving at a speed less than c !

Relativistic Dynamics

- Till now we have only been concerned with kinematics !!
  - ➢ we need to look at the laws that determine the motion

✓ The relativistic form of Newton's Laws of Motion ?

Newton's Second Law may need revision

✓ Following Newton's Second Law a particle can be accelerated to a velocity up to and then beyond the speed of light.

➢ Now if υ > c then we find that the factor γ in the Lorentz Transformation becomes imaginary i.e. the real space and time will transform into imaginary quantities.

✓ In an isolated system, the momentum **p** = m **u** of all the particles involved is constant !

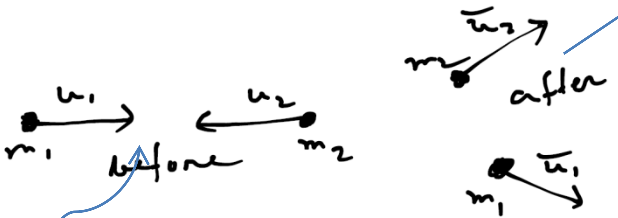
➢ With momentum defined in this way, is momentum conserved in all inertial frames of reference?

✓ We could study the collision of two bodies !

Relativistic momentum

Collision Problem

- Collision between two particles of masses m_1 and m_2 !



$$m_1 u_1 + m_2 u_2 = m_1 \bar{u}_1 + m_2 \bar{u}_2$$

We will check whether or not this relation holds in all inertial frame of reference ?

✓The velocities must be transformed according to the relativistic laws !

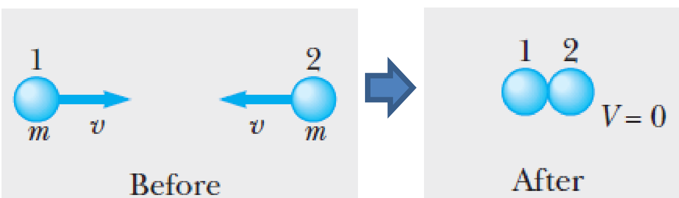
✓However, if we retain the Newtonian principle that the mass of a particle is independent of the frame of reference in which it is measured we find that the above equation does *not hold true in all frames of reference !*

Relativistic Momentum

- Any relativistic generalization of Newtonian momentum must satisfy two criteria:

    1. Relativistic momentum must be conserved in all frames of reference.
    2. Relativistic momentum must reduce to Newtonian momentum at low speeds.

Center of mass frame



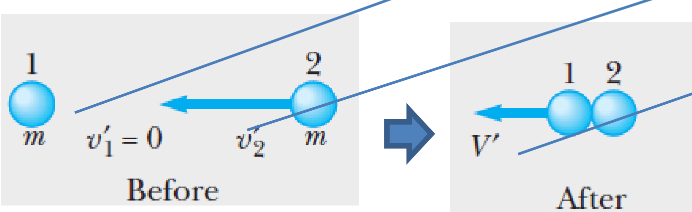
$$p_{\text{before}} = mv + m(-v) = 0$$
$$p_{\text{after}} = 0$$

An inelastic collision between two equal point Masses, momentum is conserved according to S

$$v_1' = \frac{v_1 - v}{1 - (v_1 v/c^2)} = \frac{v - v}{1 - (v^2/c^2)} = 0$$

$$v_2' = \frac{v_2 - v}{1 - (v_2 v/c^2)} = \frac{-v - v}{1 - [(-v)(v)/c^2]} = \frac{-2v}{1 + (v^2/c^2)}$$

$$V' = \frac{V - v}{1 - (Vv/c^2)} = \frac{0 - v}{1 - [(0)v/c^2]} = -v$$



Lab frame

$$p'_{\text{before}} = \frac{-2mv}{1 + v^2/c^2}$$
$$p'_{\text{after}} = -2mv$$

The same collision viewed from S', momentum is *not* conserved according to S'

Relativistic Momentum

✓ Thus the Newtonian definition of momentum and the Newtonian law of conservation of momentum are inconsistent with the Lorentz transformation!!

✓However , at very low speeds (i.e. $v << c$) these Newtonian principles are known to yield results in agreement with observation to an exceedingly high degree of accuracy.

✓ So, instead of abandoning the momentum concept entirely in the relativistic theory, a more reasonable approach is to search for a generalization of the Newtonian concept of momentum in which the law of conservation of momentum is obeyed in all frames of reference.

$$\mathbf{p} = \frac{m\mathbf{u}}{\sqrt{1 - (u^2/c^2)}}$$

✓Using this definition of momentum it can be shown that momentum is conserved in both S and S'

Relativistic definition of momentum

Relativistic momentum

A more general definition of momentum must be something slightly different from the mass of an object times the object's velocity as measured in a given reference frame, but must be similar to the Newtonian momentum since we must preserve Newtonian momentum at low speeds.

- ✓ The Lorentz transformation equations for the transverse components of position and velocity are not the same !
  - ➢ Time intervals measured in one reference frame are not equal to time intervals measured in another frame of reference.

If the momentum is to transform like the position, and not like velocity, we must divide the perpendicular components of the vector position by a quantity that is invariant. ➡ space-time interval

$$c\Delta\tau = \sqrt{(c\Delta t)^2 - \Delta R^2} = c\Delta t\sqrt{1-\left(\frac{\Delta R}{c\Delta t}\right)^2} = c\Delta t\sqrt{1-\beta^2}$$

$$c\Delta\tau = \sqrt{(c\Delta t')^2 - \Delta R'^2} = c\Delta t\sqrt{1-\left(\frac{\Delta R'}{c\Delta t'}\right)^2} = c\Delta t'\sqrt{1-\beta'^2}$$

Now, if the displacement of an object measured in a given intertial frame is divided by the space-time interval, we obtain

$$\frac{\Delta x}{c\Delta\tau} = \frac{\Delta x}{c\Delta t\sqrt{1-\beta^2}} = \frac{u_x/c}{\sqrt{1-\beta^2}} \quad \Rightarrow \quad m\frac{\Delta x}{\Delta\tau} = \frac{mu_x}{\sqrt{1-\beta^2}}$$

Velocity four vector

A further four-vector is the velocity four-vector

✓ How the velocity four-vector relates to our usual understanding of velocity ?

➢ Consider a particle in motion relative to the inertial reference frame S => We can identify two events , E_1 at (x,y,z) at time t and E_2 at (x+dx, y+dy, z+dz) at time t+dt !

✓ The displacement in time dt can be represented by four vector **ds**

The velocity ➡

$$\mathbf{u} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}$$

$$dt = \frac{d\tau}{\sqrt{1-(u/c)^2}}$$

$d\tau$ → *proper time interval*

This is the time interval measured by a clock in its own rest frame as it makes its way between the two events an interval *ds* apart.

$$\vec{u} \doteq \begin{pmatrix} c\,dt/d\tau \\ dx/d\tau \\ dy/d\tau \\ dz/d\tau \end{pmatrix} = \frac{1}{\sqrt{1-(u/c)^2}} \begin{pmatrix} c \\ dx/dt \\ dy/dt \\ dz/dt \end{pmatrix} = \frac{1}{\sqrt{1-(u/c)^2}} \begin{pmatrix} c \\ u_x \\ u_y \\ u_z \end{pmatrix}$$

➡ Four velocity associated with the two events E_1 and E_2

If $u << c$, the three spatial components of the four velocity reduces to the usual components of ordinary three-velocity.

Relativistic kinetic energy

Relativistic Force : $F = dp/dt$, Relativistic Work : $dW = \mathbf{F}.dr$

Hence, the rate of doing work : $P = \mathbf{F.\ u} = \mathbf{dT/dt}$ → Relativistic kinetic energy (K.E.)

$$\frac{dT}{dt} = \vec{F}\cdot\vec{u} = \vec{u}\cdot\frac{d\vec{p}}{dt} = \vec{u}\cdot\frac{d}{dt}\left(\frac{m_0\vec{u}}{\sqrt{1-\frac{u^2}{c^2}}}\right)$$

$$= \frac{m_0\vec{u}\cdot\frac{d\vec{u}}{dt}}{\sqrt{1-\frac{u^2}{c^2}}} + m_0\,\vec{u}\cdot\vec{u}\,u\frac{du}{dt}$$

$$\vec{u}\cdot\frac{d\vec{u}}{dt} = u\frac{du}{dt}$$

$$= \frac{m_0}{\sqrt{\left(1-\frac{u^2}{c^2}\right)^3}}\,u\frac{du}{dt}$$

$$= \frac{d}{dt}\left[\frac{m_0c^2}{\sqrt{1-\frac{u^2}{c^2}}}\right].$$

Integrating with respect to *t gives*

$$T = \frac{m_0c^2}{\sqrt{1-u^2/c^2}} + \text{Constant}$$

$T = 0$ for $u = 0$

Therefore, $T = \dfrac{m_0c^2}{\sqrt{1-u^2/c^2}} - m_0c^2$

If we suppose $u<<c$

$$\frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \left(1-\frac{u^2}{c^2}\right)^{-\frac{1}{2}} \approx 1 + \frac{u^2}{2c^2}$$

So that, $T \approx m_0c^2\left(1+\frac{u^2}{2c^2}\right) - m_0c^2$

$$= \frac{m_0u^2}{2}$$

- Classical Newtonian expression for the kinetic energy of a particle of mass moving with a velocity **u**

Total Relativistic Energy

- We can now define a quantity E by

$$E = T + m_0c^2 = \frac{m_0c^2}{\sqrt{1-u^2/c^2}}, \quad \text{Now } u = \frac{u'+v}{1+\frac{u'v}{c^2}}$$

$$= \frac{m_0c^2}{\sqrt{1-u'^2/c^2}} \frac{1+u'_x v/c^2}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$= \gamma\left[\frac{m_0c^2}{\sqrt{1-\frac{u'^2}{c^2}}} + \frac{m_0 u'_x v}{\sqrt{1-\frac{u'^2}{c^2}}}\right]$$

$$= \gamma\left[E' + p'_x v\right]$$

Also, $E^2 = \frac{m_0^2c^4}{1-u^2/c^2} = \frac{m_0^2c^4\left[1-\frac{u^2}{c^2}+\frac{u^2}{c^2}\right]}{1-\frac{u^2}{c^2}}$

$\therefore E^2 = m_0^2c^4 + \frac{m_0u^2}{1-\frac{u^2}{c^2}} \cdot c^2$

Now $p^2 = \frac{m_0u^2}{1-\frac{u^2}{c^2}}$ $\quad \boxed{\therefore E^2 = p^2c^2 + m_0^2c^4}$

This results allows us to formally take the limit $m_0 \to 0$ while keeping E and P fixed.

In this limit, $E = pc = |\vec{P}|c$

$\therefore E\sqrt{1-\frac{u^2}{c^2}} = m_0c^2$

Hence $m_0 \to 0$ with $E \neq 0$ we must have

$$\sqrt{1-\frac{u^2}{c^2}} \to 0$$

or, $\boxed{u = c}$

Thus, if there exists particles of zero rest mass, we see that their energy and momentum are related and that they always travel at the speed of light. Examples are Photon, Neutrinos ?