A STOCHASTIC APPROACH TO SOLVE CONSOLIDATION PROBLEM THROUGH FEYNMAN-KAC FORMULATION

Naina Deb[1], Ayush Thakur[1], Budhaditya Hazra, Ph.D.[1], and Arindam Dey, Ph.D.[1]

[1]Research Scholar, Department of Civil Engineering, Indian Institute of Technology, Guwahati, Assam, India, 781039; email: nainadeb21@iitg.ac.in

[1]Research Scholar, Department of Civil Engineering, Indian Institute of Technology, Guwahati, Assam, India, 781039; email: ayushthakur1196@gmail.com

[1]Associate Professor, Department of Civil Engineering, Indian Institute of Technology, Guwahati, Assam, India, 781039; email: budhaditya.hazra@iitg.ac.in

[1]Associate Professor, Department of Civil Engineering, Indian Institute of Technology, Guwahati, Assam, India, 781039; email: arindam.dey@iitg.ac.in (corresponding author)

ABSTRACT

Terzaghi's analytical solution for one-dimensional (1D) consolidation is primarily developed for thin samples and thus provides reliable results within restricted domain sizes. However, extending its applicability to samples of varying thickness, considering both single and double drainage boundary conditions, has posed challenges. In this paper, a novel probabilistic meshless framework based on the Feynman-Kac formulation is proposed to solve the 1D consolidation equation numerically. The proposed framework uses a stochastic differential equation (SDE) as a generator of the parabolic partial differential equation (PDE). A zero-drift SDE corresponds to the 1D consolidation equation, for which the Feynman-Kac formulation is established using the backward Kolmogorov equation and Ito-Taylor series. The proposed framework is capable of incorporating random coefficients of consolidation (c_v) to obtain robust numerical solutions depicting realistic variations of excess pore-water pressure (EPWP) with time and depth. Monte

Carlo Simulations (MCS) are employed to simulate 1D Brownian motion, from which the solution trajectory of EPWP at a particular time instant is obtained by taking the ensemble average of the trajectories formed over paths of Brownian motion at that time instant. Remarkably, even with a relatively small number of simulations (1000), the proposed method demonstrates high accuracy, as evidenced by low magnitudes of root mean square error (RMSE) when compared to analytical and finite-difference-based techniques. Strong alignment with experimental results further validates the method's efficacy, which remains robust across a wide range of c_v values. Such achievements make the current proposition computationally efficient and quite realistic while being the first of its kind for geotechnical engineering applications.

INTRODUCTION

A key problem in geotechnical engineering is the consolidation of saturated porous media. The settlement of structures on compressible soils is recognized as an important design problem due to the time-dependent nature of consolidation. The settlement behavior of structures on compressible soils necessitates research on consolidation theory to determine the rate of consolidation and excess pore water pressure (EPWP). Terzaghi (Terzaghi 1925) proposed the one-dimensional (1D) consolidation theory based on several key assumptions, which led to the discovery of the 1D consolidation equation. Although widely employed, the theory is limited by constraints such as thin soil layers, small deformations, linear stress-strain behavior, constant soil properties, and its restriction to one-dimensional vertical flow. Terzaghi's consolidation theory assumed the coefficient of consolidation, c_v to be constant (Das 2019) based on constant permeability and volume compressibility of the soil. However, in reality, c_v is not constant (Freeze 1977; Chang 1985) and can exhibit significant variation even within a uniform clay layer. Variations in the void ratio and effective stress influence the permeability and volume compressibility of a soft consolidating medium (Huang and Griffiths 2010). Due to the non-homogeneous distribution of voids within the soil layer, a non-uniform distribution of permeability arises, resulting in uncertainty in the adopted magnitudes of c_v. These fluctuations in c_v can greatly impact the development and dissipation of EPWP and the subsequent rate at which settlement occurs (Duncan 1993).

Researchers have developed various analytical solutions for the 1D consolidation equation under different boundary and loading conditions (Xie and Leo 2004; Mei et al. 2014; Ho and Fatahi 2016; Jiang et al. 2022). Lekha et al. derived analytical solutions for 1D consolidation, accounting for variations in compressibility and permeability under suddenly applied loading (Lekha et al. 2003). A nonlinearity parameter was identified that significantly influenced the consolidation rate. Li et al. (Li et al. 2018b) developed an analytical solution for the 1D consolidation of a clay layer with variable compressibility and permeability under a ramp load assuming a constant initial effective stress throughout the depth. Under the same assumption, Kim et al. (Kim et al. 2020) introduced an analytical solution for the non-linear 1D consolidation of a saturated clay layer, considering variable compressibility and permeability and subjected to different cyclic loading conditions. Wang et al. (Wang et al. 2021) presented a simplified solution to the 1D consolidation problem by introducing a threshold gradient under single drainage boundary conditions and investigated its influence on EPWP dissipation. In a recent advancement, Li et al. (Li et al. 2023a) derived analytical solutions for the 1D nonlinear consolidation of soft soils under a time-dependent drainage boundary, employing variable substitution and Laplace transform techniques. Nonlinearity was incorporated through biologarithmic models for compressibility and permeability.

Significant studies in the numerical solution of the consolidation equation have been done in the past few decades, as shown in (Sandhu and Wilson 1969; Huang and Griffiths 2010; Fox and Pu 2015; Baqersad et al. 2016). Abbasi et al. (Abbasi et al. 2007) developed a 1D non-linear partial differential equation (PDE) for the evaluation of consolidation characteristics of soft clays considering variable c_v, and solved the developed non-linear equation using a finite difference method (FDM). Li et al. (Li et al. 2018a) formulated a 1D nonlinear consolidation equation by integrating property relationships associated with pore evolution. Using the Galerkin-iterative method, an asymptotic solution for the developed equation was derived. Ma et al. (Ma et al. 2020) obtained solutions for the 1D non-linear consolidation equation of soft ground subjected to uniform load using the FDM. Boumezerane (Boumezerane 2021) addressed parameter uncertainty in 1D clay consolidation by using possibility distribution for the variable c_v. The alpha-level cut discretization technique was applied, and an interval-based FDM was utilized to obtain solutions of the 1D consolidation equation. Recently, Li et al. (Li et al. 2023b) introduced a 1D finite strain self-weight consolidation model that incorporates the creep behavior of soft clay by expanding upon Yin and Graham's 1D Elastic Visco-Plastic model (Yin and Graham 1989; Yin and Graham 1994). This model takes into account the nonlinear compressibility and permeability of soft clays, and the resulting nonlinear partial differential equations were solved using the Crank-Nicholson FDM.

The Feynman-Kac formula establishes a link between the solutions of specific partial differential equations (PDEs) and the expected values of functionals of stochastic processes (Del Moral 2004). Feynman's path integral approach in quantum mechanics established a foundational connection between quantum physics and probability theory (Feynman 1948). Concurrently, Kac formulated a probabilistic framework linking stochastic processes and PDEs, solving specific PDEs by taking expectations over Brownian motion paths (Kac 1949). Bertini and Cancrini (Bertini and Cancrini 1995) applied a generalized Feynman-Kac formula to solve the stochastic heat equation and analyzed the statistical properties by expressing moments through local times of independent Brownian motions. Hu et al. (Hu et al. 2011) developed a Feynman-Kac formula for the multidimensional stochastic heat equation with fractional white noise and obtained the Wiener chaos expansion for the solution. Kharroubi and Pham (Kharroubi and Pham 2015) introduced a probabilistic Feynman-Kac representation for the nonlinear Hamilton-Jacobi-Bellman equation, using backward SDEs and a forward simulation technique. Zhou and Cai (Zhou and Cai 2016) showed that Feynman-Kac solutions give an advantage over the grid-based finite element method. In a recent development, Wang et al. (Wang et al. 2022) established the Feynman-Kac formula for a set of path-dependent PDEs by establishing connections with forward-backward stochastic Volterra integral equations.

Analytical solutions provide precise insights by framing problems in familiar mathematical forms and delivering exact solutions. The 1D consolidation equation, within its specific domain of depth, time, and boundary conditions, has a well-defined analytical solution. However, extending the domain or altering boundary conditions often complicates the problem to the point where an analytical solution is no longer feasible or fails to converge. Numerical solution techniques, on the other hand, can handle complex scenarios where analytical methods fall short, such as irregular geometries, variable material properties, and non-standard boundary conditions. Since the 1D consolidation equation evolves in time and is parabolic, various numerical approaches such as finite element analysis, and FDM are used to solve the equation and produce approximate solutions. However, in FDM, finer discretization improves resolution but can lead to computational inefficiency. In addition, very fine discretization can cause numerical instability, leading the solution to diverge. Therefore, ensuring stability is crucial, achieved by choosing appropriate time steps and spatial intervals and using stability criteria to maintain the accuracy and stability of numerical solutions.

In the present work, a probabilistic meshless approach to solving the 1D consolidation equation is developed using the Feynman-Kac formulation under both single drainage and double drainage conditions. To model randomness in c_v, a normally distributed sample of c_v ranging from 0.90-1.5 m^2/yr is taken and Monte Carlo simulations are performed using the Feynman-Kac framework. Validation against existing FDM and analytical solutions is conducted by calculating absolute errors and second-order sample moments for EPWP. The Feynman-Kac formula relates SDE with the corresponding PDE, utilizing the backward Kolmogorov equation, conditional probability, and the Markov process. By applying Ito's lemma and the Taylor series expansion of multivariable, the relationship between PDE and the expected value of the SDE at the terminal point is established. The stochastic process starts at any time $t < T_e$ and is simulated until the process reaches the exit time T_e. The expected value of the realizations taken at time T_e is calculated to give the solution of the 1D consolidation equation at a given depth and time from where the process gets initiated. The proposed framework has the following advantages over other techniques:

- The proposed Feynman-Kac framework, utilizing a meshless path-dependent mechanism, is the first to utilize 1D Brownian motion and provides a novel method to represent the solution of the 1D consolidation equation as the expected value of trajectories formed over paths of Brownian motion.

- It offers an innovative numerical solution for the 1D consolidation equation by clearly integrating the variability of c_v, which arises from the inherent randomness in soil permeability and compressibility.
- In contrast to FDM, which relies on specific space-time discretization satisfying stability criteria to yield stable solutions, the proposed framework is meshless and ensures numerical stability irrespective of discretization in spatial and temporal domains.
- Unlike closed-form analytical solutions for single and double drainage conditions, which are valid only for restrictive standard domains (thin layer systems), the Feynman-Kac framework is more robust and efficient for handling large domain problems (thick layer systems) as the range of time and spatial variables increases.
- The framework offers a probabilistic view of consolidation, employing Monte Carlo simulations to assess the impact of soil property variations and enhance understanding of its time-dependent behavior.

BACKGROUND

Theory of 1D Consolidation

The governing equation of Terzaghi's 1D consolidation problem of a homogeneous layer of fine-grained soil subjected to a time-dependent surface load $f(t)$ is given as:

$$\frac{\partial p_w}{\partial t} = c_\mathrm{v}\frac{\partial^2 p_w}{\partial z^2} + \frac{\partial f}{\partial t} \tag{1}$$

where, p_w is EPWP, t is the elapsed time, z is the depth of the soil layer and c_v is the coefficient of consolidation. Here, c_v depends on the saturated permeability coefficient in z direction (k_z), coefficient of volume compressibility (m_v) and unit weight of water (γ_w) and is defined as,

$$c_\mathrm{v} = \frac{k_z}{m_v\gamma_w} \tag{2}$$

At the instant of application of the excess load p'_z, the load is carried completely by water present in voids of the soil. As time passes, the EPWP dissipates, and the effective vertical stress in the layer correspondingly increases. At any point within the consolidating layer, the EPWP at any given time (Murthy 2003) can be given by,

$$p_w = p - p'_z \tag{3}$$

where, p_w is EPWP at any time t and depth z, p is total stress on top of soil layer, and p'_z is effective pressure transferred to the soil grains at depth z and time t. In the present work, a time-independent surcharge loading condition is assumed. Hence, the 1D consolidation equation becomes a simple Fourier equation as,

$$\frac{\partial p_w}{\partial t} = c_{\mathrm{v}} \frac{\partial^2 p_w}{\partial z^2} \tag{4}$$

In the present study, Feynman-Kac solutions for the 1D consolidation equation are obtained for both single drainage and double drainage conditions. The pictorial representation of both single and double drainage conditions are shown in Fig. 1 and Fig. 2 respectively. For single drainage conditions, as shown in Fig. 1, the top surface of the layer allows drainage, and the bottom surface represents a closed drainage system. In the case of double drainage condition as in Fig. 2, both the top and bottom surfaces allow drainage and dissipation of the EPWP.

Background on Feynman-Kac formula

The Feynman-Kac formula offers a stochastic representation for solutions to certain second-order PDEs, particularly in the context of diffusion processes (Allouba 2002). The Feynman-Kac formula has demonstrated its versatility in various disciplines, including finance (Black and Scholes 1973), statistics (Kot 2001), and physics (Kleinert 2009). It serves as a comprehensive illustration of the interconnectedness between the solutions of parabolic and elliptical equations and their corresponding stochastic diffusion processes (Del Moral 2004). The Feynman-Kac formula gives a relationship between numerical solutions of PDE and the expected value of the corresponding SDE simulated from any time t' and space z' to the exit time of the stochastic process T_e (Särkkä and Solin 2019). Consider a general second-order inhomogeneous parabolic PDE of the form

$$\frac{\partial p}{\partial t}+f(z)\frac{\partial p}{\partial z}+\frac{1}{2}L^2(z)\frac{\partial^2 p}{\partial z^2}-rp=0 \tag{5}$$

with $p(z,T_e)=\Phi(z)$. Here, $f(z)$, $L(z)$, and $\Phi(z)$ are some functions and r is a positive constant. T_e is a fixed-time instance where the stochastic process exits. An exit time T_e can be defined as a random variable in a measurable space $(\Omega,\mathcal{F})$ that T_e can take a value between $[0,\infty]$. In order to investigate the PDE-SDE connection, let us consider an SDE of the form (Calin 2015),

$$dz_t=f(t,z)dt+L(t,z)dB_t \tag{6}$$

where, f and L are the corresponding drift and diffusion terms and B is the Brownian motion. The stochastic process starts with an initial condition, $z(0)=z_0$, where, z_0 is a constant and f and L are continuous functions satisfying the following conditions,

1. $|f(t,z)|+|L(t,z)|\leq P\,(1+|z|) \quad z\in\mathbb{R}, t\in[0,T_e]$
2. $|f(t,z)-f(t,w)|+|L(t,z)-L(t,w)|\leq S\,|z-w| \quad z,w\in\mathbb{R}, t\in[0,T_e]$

Here, P and S are positive constants. The first condition states that drift and diffusion terms satisfy linear growth conditions for some constant P and the second condition states that the functions are Lipschitz continuous in the second argument, z which guarantees the uniqueness of the solution. Using Ito's lemma (Appendix A) for diffusion and Ito-Taylor expansion (Appendix B) (Särkkä and Solin 2019) for $p(t,z)$, one can write:

$$dp=\frac{\partial p}{\partial t}dt+\frac{\partial p}{\partial z}dz_t+\frac{1}{2}\frac{\partial^2 p}{\partial z^2}(dz_t)^2 \tag{7}$$

For inhomogeneous equation $(rp\neq 0)$

For an inhomogeneous equation as in Eq. (5), Ito's formula is applied to $\exp(-rt)p(t,z)$ instead of $p(t,z)$. In practice, r is often called a risk-free interest rate in mathematical finance (Hölzermann 2024) and a cooling term in heat conduction problems (Hu et al. 2011). Now, Eq.

(7) can be written as

$$d\left\{e^{-rt}p\right\} = \frac{\partial\{e^{-rt}p\}}{\partial t}dt + \frac{\partial\{e^{-rt}p\}}{\partial z}dz_t + \frac{1}{2}\frac{\partial^2\{e^{-rt}p\}}{\partial z^2}(dz_t)^2 \tag{8}$$

Using the properties of quadratic variation such that $[dt]^2 = 0$ and $dtdB_t = 0$ and $[dB_t]^2$ converges to dt (Appendix. C), and substituting Eq. (6) in Eq. (8) and simplifying gives,

$$\begin{aligned}
d\left\{e^{-rt}p\right\} &= \frac{\partial\left\{e^{-rt}p\right\}}{\partial t}dt + \frac{\partial\left\{e^{-rt}p\right\}}{\partial z}dz_t + \frac{1}{2}\frac{\partial^2\left\{e^{-rt}p\right\}}{\partial z^2}(dz_t)^2 \\
&= e^{-rt}\left(\frac{\partial p}{\partial t} - rp\right)dt + e^{-rt}\frac{\partial p}{\partial z}\left\{f\left(t,z\right)dt + L\left(t,z\right)dB_t\right\} + e^{-rt}\frac{1}{2}\frac{\partial^2 p}{\partial z^2}\{f\left(t,z\right)dt + L\left(t,z\right)dB_t\}^2 \\
&= e^{-rt}\left(\frac{\partial p}{\partial t} + f\left(t,z\right)\frac{\partial p}{\partial z} + \frac{1}{2}L^2\left(t,z\right)\frac{\partial^2 p}{\partial z^2} - rp\right)dt + e^{-rt}\frac{\partial p}{\partial z}L\left(t,z\right)dB_t
\end{aligned} \tag{9}$$

Hence, for inhomogeneous Eq. (5), Eq. (9) can be written as

$$d\left\{e^{-rt}p\right\} = e^{-rt}\frac{\partial p}{\partial z}L\left(t,z\right)dB_t \tag{10}$$

Again, integrating from any arbitrary point t' to the exit time T_e gives

$$e^{-rT_e}p(T_e, z(T_e)) - e^{-rt'}p(t', z(t')) = \int_{t'}^{T_e} e^{-rt}\frac{\partial\{p(t, z(t))\}}{\partial z}L(t,z)dB_t \tag{11}$$

Taking expectation on both sides, recalling the properties of Ito's integral that the expectation of any Ito integral is zero, and substituting the value at the terminal point gives

$$e^{-rT_e}E[\Phi(z(T_e))] - e^{-rt'}p(t', z(t')) = 0 \tag{12}$$

Hence, for inhomogeneous second-order equations, the Feynman-Kac formula (Särkkä and Solin 2019) can be written as

$$p(t', z') = e^{-r(T_e - t')}E[\Phi(z(T_e))] \tag{13}$$

Eq. (13) represents the Feynman-Kac formula for an inhomogeneous parabolic partial differential equation. The Feynman-Kac formula for the homogeneous equation is given in Appendix D. To derive the Feynman-Kac formula, it is instructive to note the proof of the existence of the backward Kolmogorov operator using the properties of martingales and Markov process. Further, the Feynman-Kac formula can be derived using the properties of zero mean martingale and the Markov process as discussed in the subsequent section.

Existence of Backward Kolmogorov Equation and Feynman-Kac Solution using Martingales and Markov Process

Consider a more general inhomogeneous backward Kolmogorov equation and let $p : \mathbb{R}^+ \times \mathbb{R}^d$ be a $C^{1,2}$ function such that C^1 in the first argument is time and C^2 in the second argument is space and $h(z)$ be a bounded function in C^2. With the process ranging from $0 \leq t \leq T_e$, the backward Kolmogorov PDE can be stated as:

$$\frac{\partial p(t,z)}{\partial t} + \Im_t p(t,z) = 0$$

with the terminal condition as $p(T_e, z) = h(z)$. Here, $\Im_t$ is a second-order partial differential operator corresponding to m-dimensional Langevin equation driven by n-dimensional Brownian motion $B(t)$, where $t \geq 0$ given (Särkkä and Solin 2019) as

$$\Im_t = \sum_{i=1}^{m} b_i(t,z)\frac{\partial}{\partial z_i} + \frac{1}{2}\sum_{i,j=1}^{m}\sum_{k=1}^{n}\beta_{ik}(t,z)\beta_{jk}(t,z)\frac{\partial^2}{\partial z_i \partial z_j}$$

Here, for simplicity, consider a 1D SDE with coefficients $b(t,z)$ and $\beta(t,z)$ satisfying Lipschitz continuity and linear growth conditions. The Langevin equation adhering the properties with the initial condition z_0 is given as

$$dz(t) = b(t,z)dt + \beta(t,z)dB(t), \quad z(0) = z_0$$

The coefficients $b(t,z)$ and $\beta(t,z)$ are bounded and continuous such that $\beta^2(t,z) \geq k > 0$ and $b(t,z)$ and $\beta^2(t,z)$ obeys Holder continuity. For $r,t > 0$, $0 < \alpha \leq 1$, $C \in \mathbb{R}$ and $z, y \in \mathbb{R}$, satisfying Holder condition given as,

$$|b(t,y) - b(r,z)| + \left|\beta^2(t,y) - \beta^2(r,z)\right| \leq C\left(|y-z|^\alpha + |t-r|^\alpha\right)$$

then the transitional probability density function (PDF) $f(y; T_e | z; t)$ can be the fundamental solution of backward Kolmogorov PDE. $f(y; T_e | z; t)$ can be interpreted as Green's kernel for the PDE and for any bounded terminal condition $h(y)$ on $\mathbb{R}$, one may write the solution as;

$$p(t,z) = \int_{\mathbb{R}} h(y) f(y; T_e | z; t) dy$$

A Martingale Viewpoint of Feynman-Kac solution

A stochastic process in probability theory is modeled using a *probability space* $(\Omega, \mathcal{F}, \mathcal{P})$, where Ω is the sample space, $\mathcal{F}$ is defined as the event space and $\mathcal{P}$ is the corresponding probability measure associated with the event (Calin 2015). A σ-algebra is assigned to each sample outcome and is followed by a Borel-measurable subset of Euclidean space as $Z : (\Omega, \mathcal{F}) \to (\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$, where $\mathscr{B}(\mathbb{R}^d)$ is Borel-σ measure on $\mathbb{R}^d$. Consider an adapted process Z on filtration $\mathcal{F}$ such that $\forall\, 0 \leq r < t < \infty$. The expected value of the process up to time s is $E(Z_T | \mathcal{F}) = Z_s$. One should claim that the conditional expectation $E\left[h\left(Z_{T_e}\right) \middle| Z_t = z\right]$ satisfy Eq. 14 with $\lim_{t \to T_e} p(t,z) = h(z)$. To prove that $E[h\left(Z_{T_e}\right) | Z_t = z]$ is the solution of the backward Kolmogorov equation, one should show that the solution $p(t,z)$ is a martingale.

Consider a right-continuous martingale $\mathcal{M}_p(t)$ such that $E(\mathcal{M}_p^2(t)) < \infty$. A general discussion on the backward Kolmogorov equation is that if $p(t,z)$ is C^1 in t and C^2 in z having bounded first derivative in z, then one has to show:

$$\mathcal{M}_p(t) := p(t, Z_t) - \int_0^t \left(\frac{\partial p(s, Z_s)}{\partial t} + \mathfrak{I}_s p(s, Z_s)\right) ds \tag{14}$$

is a martingale. The integral form of Ito's formula with the initial condition as $p(0, Z_0)$ can be written as,

$$p(t,z) = p(0,Z_0) + \int_0^t \left(\frac{\partial p(s,Z_s)}{\partial s} + \Im_s p(s,Z_s) \right) ds + \int_0^t \frac{\partial p(s,Z_s)}{\partial z} \beta(s,Z_s) dB(s) \tag{15}$$

Substituting Eq. 14 in the integral form, Eq. 15 gives the semi-martingale as,

$$\mathcal{M}_p(t) = p(0,Z_0) + \int_0^t \frac{\partial p(s,Z_s)}{\partial z} \beta(s,Z_s) dB(s)$$

By assumption, $\frac{\partial p(s,z)}{\partial z}$ is bounded in first derivative for all $Z(s) = z$ and s. Accordingly for $Q_1 \in \mathbb{R}$, $\left(\frac{\partial p(s,z)}{\partial z}\right)^2 < Q_1$ leads to,

$$\int_0^t E\left[\frac{\partial p(s,Z_s)}{\partial z} \beta(s,Z_s)\right]^2 ds \leq Q_1 \int_0^t E\left[\beta^2(s,Z_s)\right] ds \tag{16}$$

As the coefficients of the SDE satisfy the linear condition, $|b(t,z)| + |\beta(t,z)| \leq K_1(1+|z|)$, given in (Calin 2015). This implies that

$$\beta^2(s,Z_s) \leq K_1^2(1 + \sup_{s \leq T_e} Z^2(s))$$

Hence, one can write for Eq. 16 as,

$$\int_0^t E\left[\frac{\partial p(s,Z_s)}{\partial z} \beta(s,Z_s)\right]^2 ds \leq 2Q_1 {K_1}^2 T_e (1 + E\left[\sup_{s \leq T_e} Z^2(s)\right]) \tag{17}$$

From the mean square convergence theory and the existence and uniqueness of the solution of the diffusion equation, $E\left[Z^2(0)\right] < \infty$. Therefore, for $K_2 \in \mathbb{R}$ and K_2 depends on K_1 and T_e, the following inequality:

$$E\left[\sup_{s \leq T_e} Z^2(s)\right] < K_2(1 + E\left[Z^2(0)\right]) < \infty$$

holds. Following the above inequality, one can conclude that the left-hand side of Eq. 17, $\int_0^t E\left[\frac{\partial p(s,Z_s)}{\partial z}\beta(s, Z_s)\right]^2 ds$ is finite. Now using the properties of Ito isometry and martingale property of Ito integrals and Ito isometry, one can write Eq. 17 as,

$$E\left[\int_0^t \frac{\partial p(t,z)}{\partial z}\beta(r, Z_r)dB(r)\right]^2 = \int_0^t E\left[\frac{\partial p(t,z)}{\partial z}\beta(r, Z_r)\right]^2 dr < \infty \tag{18}$$

Hence, $\mathcal{M}_p(t)$ is a right continuous semi-martingale with mean $p(0, Z_0)$.

Markov Process and Feynman-Kac solution

Using the properties of martingales and the Markov process, it can be shown that the solutions of backward Kolmogorov equation acting on filtration $\mathcal{F}_t$, can be extended for the functional $h(Z_{T_e})$ as,

$$p(t,z) = E\left[h(Z_{T_e})\middle| Z_{T_e} = z\right]$$

Consider an event that is $\mathcal{F}_t$−measurable such that for any random variable, τ is defined on a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in T_e}, \mathcal{P}\right)$. For the exit time, the condition $\{\tau \le t\} \in \mathcal{F}_t \,\forall\, t \in T_e$ holds. Using the properties of martingale for the filtration $\mathcal{F}_t$, the solution is

$$p\,(t,z) = E\left[\,p\,(T_e, Z_T)\right|\mathcal{F}_t]$$

For the terminal condition, $p(T_e, Z_{T_e}) = h(Z_{T_e})$,

$$E(\,h(Z_{T_e})\big|\,\mathcal{F}_t) = p(t,z)$$

Therefore, by using the Markov property of $Z(t)$, one can write,

$$E\left[h(Z_{T_e})|Z_t = z\right] = E(\,h(Z_{T_e})\big|\,\mathcal{F}_t) = p(t,z)$$

This demonstrates the connection between the Feynman-Kac formula and the Markov process.

SOLVING 1D CONSOLIDATION EQUATION USING FEYNMAN-KAC FORMULA

Analytical solution of 1D consolidation

Based on 1D flow through porous media, the differential equation for Terzaghi's 1D consolidation under constant loading is given (Murthy 2003) by

$$\frac{\partial p_w}{\partial t} = c_{\mathrm{v}} \frac{\partial^2 p_w}{\partial z^2} \tag{19}$$

where, p_w is the EPWP, t is the time of consolidation, z is the depth of the soil layer, and c_v is the coefficient of consolidation. More details about the development and derivation of Eq. 19 can be found in (Das 2019; Murthy 2003).

The initial condition of the 1D consolidation is given by

$$p_w\,(z, 0) = (p_w)_0\,; \quad t = 0 \quad 0 \leq z \leq H \tag{20}$$

where $(p_w)_0$ represents the constant initial pore water pressure under fully saturated conditions, where no water table is present in the soil profile. This value is equivalent to the applied surcharge loading on the soil.

For single drainage conditions with only the top boundary allowing drainage and the bottom boundary considered impermeable, the boundary conditions for the 1D consolidation equation are given by

$$p_w\,(0, t) = 0\,; \quad z = 0\,; \quad t > 0 \tag{21}$$

$$\frac{\partial p_w}{\partial z}\,(H, t) = 0\,; \quad z = H\,; \quad t > 0 \tag{22}$$

The analytical solution of the 1D consolidation Eq. 19 considering single drainage boundary conditions Eq. 21, Eq. 22 as in (Rahman and Ülker 2018) is given by:

$$p_w(z, t) = \sum_{n=0}^{\infty} \frac{2 - 2\cos(n\pi)}{n\pi} (p_w)_0 \sin\left(\frac{n\pi z}{H}\right) e^{\left(-c_v \left(\frac{n\pi}{H}\right)^2 t\right)} \tag{23}$$

where H represents the length of the drainage path which is equal to the total thickness of the soil layer.

In case of double drainage condition with both top and bottom boundaries of soil layer permitting drainage, the boundary conditions are given as:

$$p_w\,(0,t) = 0\,; \quad z = 0\,; \quad t > 0 \tag{24}$$

$$p_w\,(2H,t) = 0\,; \quad z = 2H\,; \quad t > 0 \tag{25}$$

where H is equal to half of the total thickness of the soil layer. The analytical solution of 1D consolidation equation (Eq. 19) with double drainage boundary conditions, Eq. 24, Eq. 25 as in (Arora 2008) is given by:

$$p_w(z,t) = \frac{4}{\pi}(p_w)_0 \sum_{N=0}^{\infty} \frac{1}{(2N+1)} \sin\left(\frac{(2N+1)\pi z}{H}\right) e^{\left(-\left(\frac{(2N+1)^2\pi^2}{H^2}\right)c_v t\right)} \tag{26}$$

Proposed Feynman-Kac framework for 1D consolidation equation

The 1D consolidation equation is a non-trivial parabolic PDE with a wide range of possible solutions. However, it is not always possible to develop a closed-form solution for consolidation due to several variables being involved, such as different coefficients of permeability, different types of loading, pore pressure distributions, and varying values of the coefficient of consolidation. Given the above, numerical solutions provide better flexibility in solving the 1D consolidation problem. Furthermore, in contrast to a variety of other engineering problems, the Feynman-Kac formula has not been investigated in the context of 1D consolidation, an important phenomenon controlling the long-term deformation in soft soils.

For the derivation of the Feynman-Kac formula for 1D consolidation, as given in Eq. 19, the backward Kolmogorov equation corresponding to the 1D consolidation equation subjected to terminal conditions $p_w(T_e, z) = \Psi(z(T_e))$ is of the form,

$$-\frac{\partial p_w(t,z)}{\partial t} = \frac{1}{2}\left(\sqrt{2c_v}\right)^2 \frac{\partial^2 p_w(t,z)}{\partial z^2} \tag{27}$$

Let B_t be a 1D Brownian motion. Assume that the process z_t evolves according to the SDE such that,

$$dz_t = \sqrt{2c_v}\, dB_t \tag{28}$$

Eq. 28 represents the stochastic differential equation corresponding to the homogeneous PDE of 1D consolidation phenomenon with zero drift and having diffusion term $\sqrt{2c_v}$. Using Taylor series expansion for $p_w(t,z)$ one can write,

$$dp_w(t,z) = \frac{\partial p_w(t,z)}{\partial t}dt + \frac{\partial p_w(t,z)}{\partial z}dz_t + \frac{1}{2}\frac{\partial^2 p_w(t,z)}{\partial z^2}(dz_t)^2 \tag{29}$$

Substituting the Eq. 28 in Eq. 29 and simplifying using the properties of quadratic variation (Appendix C) yields,

$$\begin{aligned}
dp_w(t,z) &= \frac{\partial p_w(t,z)}{\partial t}dt + \frac{\partial p_w(t,z)}{\partial z}dz_t + \frac{1}{2}\frac{\partial^2 p_w(t,z)}{\partial z^2}(dz_t)^2 \\
&= \frac{\partial p_w(t,z)}{\partial t}dt + \frac{\partial p_w(t,z)}{\partial z}(\sqrt{2c_v})dB_t + \frac{1}{2}\frac{\partial^2 p_w(t,z)}{\partial z^2}(\sqrt{2c_v}dB_t)^2 \\
&= \frac{\partial p_w(t,z)}{\partial t}dt + c_v\frac{\partial^2 p_w(t,z)}{\partial z^2}(dB_t)^2 + \frac{\partial p_w(t,z)}{\partial z}(\sqrt{2c_v})dB_t \\
&= \frac{\partial p_w(t,z)}{\partial t}dt + c_v\frac{\partial^2 p_w(t,z)}{\partial z^2}dt + (\sqrt{2c_v})\frac{\partial p_w(t,z)}{\partial z}dB_t \\
&= \left(\frac{\partial p_w(t,z)}{\partial t} + c_v\frac{\partial^2 p_w(t,z)}{\partial z^2}\right)dt + (\sqrt{2c_v})\frac{\partial p_w(t,z)}{\partial z}dB_t
\end{aligned} \tag{30}$$

Since the backward Kolmogorov equation corresponding to 1D consolidation equation is a homogeneous equation, from Eq. 27 we get,

$$\left(\frac{\partial p_w(t,z)}{\partial t} + c_v\frac{\partial^2 p_w(t,z)}{\partial z^2}\right) = 0 \tag{31}$$

Substituting Eq. 31 in Eq. 30 gives,

$$dp_w(t,z) = (\sqrt{2c_v})\frac{\partial p_w(t,z)}{\partial z}dB_t \tag{32}$$

Integrating from any arbitrary time t' to the exit time T_e gives,

$$p_w(T_e, z(T_e)) - p_w(t', z') = \int_{t'}^{T_e} (\sqrt{2c_v})\frac{\partial p_w}{\partial z}dB_t \tag{33}$$

Substituting $p_w(T_e, z(T_e))$ at the exit point in Eq. 33 gives,

$$\Psi(z(T_e)) - p_w\,(t', z') = \int_{t'}^{T_e} \left(\sqrt{2c_v}\right)\frac{\partial p_w}{\partial z}dB_t \tag{34}$$

Taking expectation on both sides and recalling the properties of Ito integral *(mean of Brownian motion is zero)*, one can get the Feynman-Kac formula as,

$$p_w(t', z') = E\,[\Psi(z(T_e))] \tag{35}$$

The Feynman-Kac formula given in Eq. 35 implies that one can get the solution at any arbitrary point (t', z') by simulating the process in Eq. 28 from time t' and z' allowing the process to run until it reaches exit time as T_e. The solution $p_w(z', t')$ is the mathematical expectation of values of $\Psi(z)$ over the multiple realizations of the SDE (Eq. 28) at the exit point of the process.

The algorithm for the proposed Feynman-Kac framework for the 1D consolidation problem is shown in Algorithm 1. In this work, Feynman-Kac solutions will be compared with the analytical solution and the solution obtained from the FDM. By using the approximate finite difference relations established at discrete nodal points of a grid or mesh, the derivatives of the governing equations are substituted in this approach. The resulting discretization of the governing differential equations yields a set of algebraic equations that can be solved using programming language like MATLAB®. At any grid point $O(i, k)$, the finite difference formulation of the 1D consolidation equation considering an explicit scheme (Desai and Zaman 2013) can be written as in Eq. 36,

$$(p_w)_{i,k+1} = (p_w)_{i,k} + \alpha\left[(p_w)_{i-1,k} - 2(p_w)_{i,k} + (p_w)_{i+1,k}\right] \tag{36}$$

where α is $c_v \Delta t / \Delta z^2$. For stable FDM solution, $\alpha \leq (1/2)$.

Algorithm 1: Feynman-Kac Framework for the 1D Consolidation Problem

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Input: Coefficient of consolidation c_v, initial condition $p_w(z, 0)$, boundary conditions, time step Δt, total time T, domain $[0, H]$, number of MC trials N_s.

Output: EPWP solution $p_w(z, t)$ over space and time.

Step 1: Initialize

Discretize the domain and set the initial condition $p_w(z, 0)$.

Step 2: Simulate EPWP Trajectories for N_s Trials

for $s = 1$ *to* N_s **do**

    for $t_k = k\Delta t$ *until* T **do**

        Simulate the zero-drift SDE as in Eq. (28).

        Boundary Conditions:

        if *Dirichlet condition as in Eq. (21), Eq. (24), and Eq. (25)* **then**

            Allow the Brownian particles to exit the domain.

        if *Neumann condition as in Eq. (22)* **then**

            Reflect the Brownian particles back into the domain.

Step 3: Compute Mean EPWP trajectory

Compute the mean of EPWP trajectories.

Approximate and average over N_s trials.

Step 4: Comparison with FDM/Analytical Solution

Calculate RMSE between the Feynman-Kac results and the FDM or analytical solution.

Step 5: Results Visualization

Plot variation of EPWP with depth and time.

COMPARISON AND DISCUSSION OF NUMERICAL RESULTS

In continuation with the Feynman-Kac formulation for the 1D consolidation, the results for both single and double drainage conditions are generated using the MATLAB® programming language. Further, the results are then compared with solutions of the FDM, the closed-form analytical solutions of the 1D consolidation equation, and the experimental results of the 1D consolidation experiment. To contemplate randomness in c_v, Monte Carlo Simulations (MCS) are performed using the proposed Feynman-Kac formula, and RMSE is calculated for both single and double drainage conditions. In the present study, for the case of random c_v, Mexico City Clay is considered for the simulations, whose c_v ranges from 0.9-1.5 m^2/yr (Bardet 1997).

Case 1: Constant Coefficient of Consolidation

The Feynman-Kac solution method for the 1D consolidation equation incorporates a constant coefficient of consolidation (c_v) and accommodates both single and double drainage boundary conditions. This solution is achieved by simulating the corresponding zero-drift SDE, represented by Eq. 28 using MCS. A depth of 2 m and a time of 1 year are discretized into a 50x100 grid. The Feynman-Kac formula is employed to solve the 1D consolidation equation probabilistically. This method involves simulating particle movements in Brownian motion to approximate the solution. The simulation entails multiple Monte Carlo runs to estimate particle behavior over time. In the context of consolidation, these particle movements signify the diffusion of water within the soil matrix over time. Random fluctuations in particle positions capture the inherent uncertainty and randomness in the consolidation process.

MCS generates paths of a Brownian motion representing the stochastic process, and the EPWP for a particular time instant is computed by averaging the paths implying pore pressure trajectories generated over the particle positions updated using the boundary conditions. At first, the particle positions are initialized with respect to the mesh grid values. Then, at each Monte Carlo run of the stochastic process, the position of particles is updated in each iteration according to the stochastic trajectories. Once the particles hit the boundaries of the domain, the particles are either removed or reflected back to the domain following the boundary conditions applied under single or double drainage conditions at the top and bottom boundaries. The stochastic process runs until the exit time of the process is reached, i.e., until the particles exit the domain boundaries. After updating positions, the EPWP for a particular time is calculated by taking the mean of the pore pressure trajectories formed over the particle positions, and the EPWP values at the boundaries are updated using the boundary conditions. This process is repeated for a specified number of Monte Carlo runs. In this instance, c_v is set at 0.90 m^2/yr, and 1000 MCS simulations are conducted to simulate the SDE.

Boundary conditions are crucial in this simulation. Under a single drainage condition, at the top boundary, ($z = 0$), any particles reaching this boundary are eliminated from the simulation and no longer considered in the next iterations. This reflects the boundary's permeability, facilitating drainage. Conversely, particles encountering the bottom boundary ($z = H$) are reflected back into the domain as the bottom boundary is impermeable and water particles cannot flow out through this boundary. The EPWP is updated at the boundaries using the boundary conditions as in Eq. 21 and Eq. 22. In the case of double drainage conditions allowing drainage at both the top and bottom boundaries, particles that have crossed the boundaries are no longer considered in the next iterations, and the EPWP is updated using the boundary conditions as in Eq. 24 and Eq. 25. This acknowledges the permeable nature of both boundaries, enabling water to flow out without obstruction.

Results obtained from the Feynman-Kac formula are then compared with those from the FDM and analytical solutions. FDM involves defining the domain, initial, and boundary conditions. FDM also requires specific grid discretization and time step size (Δt) following the stability criteria in Eq. (36) to produce stable solutions of the 1D consolidation PDE. Failure to satisfy this criterion leads to numerical instability and inaccurate solutions. In contrast, the Feynman-Kac framework is meshless and does not require a grid structure to approximate the solution. Since FDM operates on a grid-by-grid basis, adjusting the element order for higher accuracy is straightforward. For comparison with the Feynman-Kac solution, a grid of 50x1600 is employed to discretize the domain in depth z and time t. Analytical solutions of 1D consolidation subjected to initial condition (Eq.

20) and boundary conditions for single drainage (as given in Eq. 21, Eq. 22) and double drainage (as given in Eq. 24, Eq. 25) are obtained using Eq. 23 and Eq. 26. For the double drainage case, only the Dirichlet boundary condition is present, while for single drainage, both Dirichlet and Neumann conditions exist.

Plots of normalized EPWP vs. normalized depth for various time instants using FDM and the analytical approach are shown in Fig. 3, Fig. 4 respectively. The EPWP profiles of FDM and analytical solutions show convergence with the increase in time instant representing idealistic solutions. In the case of single drainage, the EPWP increases with the decrease in the depth of the soil layer and the lower boundary shows the highest pore water pressure. In the double drainage case, the EPWP increases with the increase in depth up to the middle of the drainage path length and then again decreases with the depth of the soil layer and the top and bottom boundary show the lowest pore water pressure. RMSE to calculate the second-order sample moments between the proposed Feynman-Kac framework and FDM/analytical solutions for EPWP is given by

$$RMSE = \sqrt{\frac{\sum_{i=1}^{N}\left[(p_w)_{Feynman-Kac} - (p_w)_{FDM/Analytical}\right]^2}{N}}$$

where N is the number of sample points where p_w is calculated. The RMSE between the FDM and the Feynman-Kac solution for single drainage and double drainage conditions for the constant c_v is found to be 0.0194 and 0.0178, respectively. The Mean Squared Error (MSE) for these comparisons is 3.77×10^{-4} and 3.18×10^{-4} respectively. The RMSE between the analytical solution and the Feynman-Kac solution for single drainage and double drainage conditions for c_v equal to 0.90 m^2/yr is 0.0250 and 0.0233, respectively, and the corresponding MSE values for these scenarios are 6.23×10^{-4} and, 5.44×10^{-4} respectively.

Case 2: Random Coefficient of Consolidation

In general, the coefficient of consolidation, c_v varies with the type of soil, permeability, volume compressibility, and unit weight of water. Based on the laboratory tests, it is customary practice to consider one single value of c_v to represent the time rate of consolidation. Since the laboratory test for consolidation involves a progressively increasing vertical stress applied on the sample, for the same specimen, the value of c_v changes for every stress increment. Hence, it is inferred that c_v is not a deterministic and intrinsic constant for a consolidating medium and should vary with the applied stress. Thus, in any large-scale consolidating system, changes in the overburden with time would lead to different time rates of consolidation. As the applied stress can randomly change over time, so does c_v.

In order to incorporate randomness in the coefficient of consolidation, normally distributed values of c_v between 0.90-1.5 m^2/yr are considered. 100 values are simulated for c_v for which 100 trials of MCS are carried out along with the 1000 MCS for simulating the SDE to obtain the Feynman-Kac solution of 1D consolidation. For each random value of c_v, the SDE runs until the exit time of the process is reached and the EPWP is estimated subsequently. Furthermore, the average of the EPWP is estimated considering each of these individual values. The different possible solution trajectories representing the variation of EPWP profiles with normalized depth at time $t = 0.10$ yr for random c_v and for both single and double drainage cases are shown in Fig 5. The different solution trajectories correspond to different c_v values and the mean of these trajectories (obtained by taking the ensemble average of the realizations), represented by a black dotted line, also corresponds to a c_v value that gives the EPWP profile at that particular time.

Fig. 6 exhibits the solution trajectories (obtained from the Feynman-Kac formulation for random c_v) i.e., EPWP variation with normalized depth at different time intervals. Each of the colored solid lines represents EPWP profiles at individual time intervals, where each line is the mean trajectory of all possible solutions for that particular time considering random c_v. Fig. 7 and Fig 8 show the comparison of the Feynman-Kac solution trajectory with the analytical solution and FDM solution of the 1D consolidation equation, respectively, for various time instants for both single and double drainage cases. The mean solution trajectory demonstrates good convergence with both analytical and FDM solutions under single and double drainage scenarios. Moreover, this convergence improves over time as the consolidation process progresses.

The solutions derived from the Feynman-Kac formulations of the 1D consolidation equation are validated against experimental data from a 1D consolidation experiment conducted on saturated natural clay by Mohamedelhassan and Shang (Mohamedelhassan and Shang 2002). These experiments involved surcharge and vacuum loading conditions with single drainage boundaries where the top boundary is permeable and the bottom boundary is impermeable. Pore water pressures were monitored using a pore-water pressure transducer installed at the base of the soil sample. The experiments reported coefficients of consolidation (c_v) ranging from 0.88 to 4.10 m^2/yr. By incorporating these c_v values into the Feynman-Kac formulation, mean solution trajectories are computed and compared with experimental values under surcharge conditions at various time points, as depicted in Fig. 9. Notably, the EPWP profiles derived from the Feynman-Kac formulation exhibit close similarity to those obtained from the 1D consolidation experiment, suggesting a high degree of alignment between the theoretical model and empirical observations. Furthermore, it is worth noting that this alignment remains consistent over time, underscoring the robustness and accuracy of the Feynman-Kac formulation in capturing the dynamics of the consolidation process as observed in the experimental data.

The proposed framework has the capability to incorporate both random and constant values of c_v, enabling the generation of solutions not only for Mexico City clay but also for a wide range of other soil types. Considering a range of soil types such as Swedish medium-sensitive clays (c_v = 0.1-0.2 m^2/yr), San Francisco Bay mud (c_v = 0.6-1.2 m^2/yr), organic silt (c_v = 0.6-3 m^2/yr), glacial lake clays (c_v = 2.0-2.7 m^2/yr), Chicago silt clay (c_v = 2.70 m^2/yr), stiff red clay (c_v = 3.17 m^2/yr), London clay (c_v = 1.90-6.34 m^2/yr), Maine clay (c_v = 6.3-13 m^2/yr), and Boston blue clay (c_v = 6-18 m^2/yr) as documented in (Bardet 1997) along with Mexico City clay, the mean solution trajectories are plotted in Fig 10 depicting variations with depth. It is observed that as c_v increases, the EPWP decreases, resulting in accurate predictions. This observation is consistent with the analytical solutions provided in Eq. 23 and Eq. 26.

The Feynman-Kac solution for the 1D consolidation equation is derived by simulating the zero drift SDE represented in Eq. 28. Considering small drift values of 0.01 and 0.02 in Eq. 28, the solution trajectories of the 1D consolidation equation are obtained. These trajectories are then compared with the mean solution trajectory for the ideal zero drift scenario, depicted in Fig. 11. The black solid curve represents the ideal zero-drift Feynman-Kac mean solution trajectory of the 1D consolidation equation. The comparison highlights the sensitivity of Feynman-Kac solutions to small drifts. Specifically, in the single drainage case, increasing drift values result in a shift in the mean solution trajectory, leading to higher EPWP values. Conversely, in the double drainage case, increasing drift values cause the upper half of the EPWP profile to exhibit higher values, while the lower half shows decreased EPWP values. The EPWP profile slightly shifts to the upper half of the domain, with the maximum EPWP occurring slightly above the mid-depth of the domain. For drift values up to 0.02, deviations of a maximum of 5 percent are observed in both drainage cases, which are within acceptable limits.

The plots illustrating the percentage error between the results of the Feynman-Kac solution and both the FDM and analytical solution for both single and double drainage conditions are presented in Fig. 12 and Fig. 13. Notably, it is observed that the percentage error is lower in scenarios involving random c_v values compared to those with constant c_v, as evident in both single and double drainage conditions. This observation underscores the advantages of incorporating random variability in the coefficient of consolidation (c_v) when utilizing the Feynman-Kac solution method. It suggests that allowing c_v to vary randomly leads to more accurate results compared to scenarios where c_v is held constant.

The robustness of the Feynman-Kac framework in handling randomness in the coefficient of consolidation (c_v) is evident, as indicated by significantly lower MSE and RMSE values observed for both single drainage (0.0180) and double drainage (0.0165) conditions. Correspondingly, the MSE values under single drainage and double drainage cases are 3.25×10^{-4} and 2.71×10^{-4} respectively. Notably, the MSE and RMSE are lower when considering random c_v compared to constant c_v. When comparing the solutions of the proposed method with the analytical solution under both single and double drainage conditions and incorporating random c_v, the RMSE between the analytical solution and Feynman-Kac solution is 0.0236 for single drainage and 0.0220 for double drainage. The corresponding MSE values under single and double drainage cases are 5.55×10^{-4} and, 4.82×10^{-4} respectively. In these cases as well, it is observed that these MSE and RMSE values are lower for random c_v compared to constant c_v.

While the proposed Feynman-Kac framework offers a novel approach to solving the 1D consolidation problem, the current study has some limitations. It is limited to the simplified case of 1D consolidation, as the primary focus is on the foundational aspects of the Feynman-Kac framework. c_v is considered both a constant and a random variable capturing the overall uncertainty in the soil domain for simplicity in presenting the framework. The framework currently focuses on standard boundary conditions (single drainage and double drainage conditions) commonly used in theoretical studies. Despite these limitations, the proposed framework can indeed be extended to solve the two-dimensional (2D) and three-dimensional (3D) consolidation problems. The extension involves solving a number of stochastic differential equations corresponding to the 2D and 3D governing PDEs. In the Feynman-Kac framework, this would involve simulating EPWP trajectories using higher-dimensional Brownian motions, which is a possible direction of future work. Furthermore, the framework can be easily adapted to incorporate random fields to model c_v which would give a better representation of the randomness in c_v. This can be achieved using the Karhunen-Loeve (KL) expansion technique to model c_v as a spatially correlated random field, which will be explored in future studies.

The Feynman-Kac framework does not depend on a grid structure for deriving solutions to the 1D consolidation PDE, making it inherently meshless. For a grid discretization of 100×1000, $\Delta t = 0.01$, $\Delta z = 0.01$, and $c_v = 1.8$ m^2/yr, the FDM solution became unstable, leading to numerical dispersion due to the violation of the stability criteria. In contrast, the Feynman-Kac framework produces stable and accurate solutions. One of the key advantages of the Feynman-Kac approach is its computational efficiency, especially in solving higher-dimensional problems. For solving the 2D consolidation PDE employing a 200×200 spatial grid and the same set of parameters, the Feynman-Kac framework requires 10.95 seconds, whereas FDM requires 18.01 seconds, demonstrating a notable reduction in computational time. This efficiency becomes even more pronounced when random fields are employed to incorporate randomness in higher-dimensional problems, which is a key direction for future work.

CONCLUSIONS

In this paper, an accurate and stable probabilistic framework for solving the 1D consolidation equation for both single and double drainage boundary conditions is developed using the Feynman-Kac formula, and the solution is compared with the existing closed-form analytical solutions, FDM solutions, and experimental results. The proposed framework uses the concept of conditional probabilities at the exit point using the properties of martingales and Markov processes to obtain a corresponding SDE from a deterministic PDE. The background of the proposed framework is that the Feynman-Kac formula links the expected value of SDE simulated till the exit time with the solution of the backward Kolmogorov equation representing the 1D consolidation equation. The 1D consolidation equation is treated as a backward Kolmogorov equation for which the corresponding SDE acts as a generator process.

As compared with the FDM solutions, the results obtained from the Feynman-Kac formulation are stable for any time discretization (Δt). This makes the proposed framework more favorable than the FDM as it uses a meshless path-dependent mechanism. Unlike the closed-form analytical solutions for both single and double drainage conditions, which remain valid only for restrictive standard domains (thin layer systems), the Feynman-Kac framework is more robust and efficient in handling large domain problems (thick layer systems) as the range of time and space variable increases. The Feynman-Kac formulation for the 1D consolidation equation uses MCS to simulate the SDE for the corresponding PDE. The proposed framework does not yield large RMSE for a comparatively lesser number of MCS, thus making it computationally efficient as the convergence rate for the MCS is $O\left(1\middle/\sqrt{M}\right)$, where M is the number of samples.

To accommodate the inherent variability in the coefficient of consolidation (c_v), which is influenced by the type of soil, permeability, and compressibility, normally distributed samples within the range of 0.90-1.5 m^2/yr (as observed in Mexico City Clay) (Bardet 1997) are considered. Each random sampling generates realizations using the proposed framework, and their ensemble average produces the EPWP solution trajectory. The framework also demonstrates robustness across diverse c_v ranges observed for different soil types. The MSE and RMSE for random c_v between the Feynman-Kac solution and FDM/Analytical solution, for both single and double drainage conditions, are found to be very less. The second-order statistics derived from random c_v through the proposed Feynman-Kac formulation are in close agreement with the analytical and FDM results of the 1D consolidation equation. Additionally, the framework exhibits excellent agreement with experimental data. Remarkably, this approach, utilizing 1D Brownian motion, uniquely addresses the 1D consolidation equation for both constant and random c_v, marking a pioneering advancement in the field.

DATA AVAILABILITY STATEMENT

Some or all data, models, or code that support the findings of this study are available from the corresponding author upon reasonable request.

DECLARATION OF CONFLICTING INTERESTS

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

APPENDIX A. ITO'S LEMMA

Let $p : \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable continuous function and $B(t)$ be a Weiner process. Then it holds that

$$dp\left(B_t\right) = p'\left(B_t\right) dB_t + \frac{1}{2} p''\left(B_t\right) dt$$

The integral form that corresponds to Ito's lemma can be written as:

$$p\left(B_t\right) = p\left(B_0\right) + \int_0^t p'\left(B_s\right) dB_s + \frac{1}{2} \int_0^t p''\left(B_s\right)$$

Consider $Z(t)$ as a diffusion process on $[0, T]$ with drift term, $f(t)$ and diffusion term, $L(t)$ given by:

$$dZ_t = f(t)dt + L(t)dB_t$$

Then it holds that

$$dp\left(Z_t\right) = p'\left(Z_t\right) dZ_t + \frac{1}{2} p''\left(Z_t\right) L^2\left(t\right) dt$$

APPENDIX B. ITO TAYLOR SERIES

Let us consider $p : [0,T] \times \mathbb{R}\mathbb{R}$ as a twice differentiable continuous function with respect to both arguments time (t) and space (z) and let $Z(t)$ as a diffusion process on $[0,T]$ given by:

$$dZ_t = f(t)dt + L(t)dB_t$$

The Taylor series expansion for $p(t, Z(t))$ is given as:

$$dp\left(t, Z_t\right) = \frac{\partial p\left(t, Z_t\right)}{\partial t} dt + \frac{\partial p\left(t, Z_t\right)}{\partial Z_t} dZ_t + \frac{1}{2}\frac{\partial^2 p\left(t, Z_t\right)}{\partial t^2}(dt)^2 + \frac{1}{2}\frac{\partial^2 p\left(t, Z\left(t\right)\right)}{\partial Z^2}(dZ_t)^2 + \frac{1}{2}\frac{\partial^2 p\left(t, Z_t\right)}{\partial t \partial Z_t}\left(dZ_t\right)\left(dt\right)$$

Using the multiplication rules of stochastic calculus

$$(dt)^2 = 0$$

$$(dB_t)(dt) = 0$$

$$(dB_t)^2 = dt$$

and expanding Z_t we get

$$(dZ_t)^2 = f\left(t\right)\left(dt\right)^2 + 2f\left(t\right) L\left(t\right)\left(dt\right)\left(dB_t\right) + L^2\left(t\right)\left(dB_t\right)^2 = L^2\left(t\right) dt$$

$$\left(dZ_t\right)\left(dt\right) = f\left(t\right)\left(dt\right)^2 + L\left(t\right)\left(dB_t\right)\left(dt\right) = 0$$

and so

$$dp\left(t, Z_t\right) = \frac{\partial p\left(t, Z_t\right)}{\partial t} dt + \frac{\partial p\left(t, Z_t\right)}{\partial Z_t} dZ_t + \frac{1}{2}\frac{\partial^2 p\left(t, Z_t\right)}{\partial {Z_t}^2} L^2\left(t\right) dt$$

APPENDIX C. QUADRATIC VARIATION

Consider z as a continuous function defined on the interval $[0, t]$. The Quadratic Variation

$Q(V_n)$ (Hassler 2016) of z can be expressed as the limit of a sum given as:

$$QV_n(z,t) = \sum_{i=1}^{n} \left(z(s_i) - z(s_{i-1})\right)^2$$

The interval $[0,t]$ is divided into n partitions (P_n) such that $P_n([0,t]) : 0 = s_0 < s_1 < \cdots < s_n = t$.

If $z = B(t)$, where $B(t)$ is a Weiner process, and $n \to \infty$, it holds that:

a. $\sum\limits_{i=1}^{n} (s_i - s_{i-1})^2 \ \to \int\limits_0^t (ds)^2 = 0$

b. $\sum\limits_{i=1}^{n} \left(B(s_i) - B(s_{i-1})\right)(s_i - s_{i-1}) \xrightarrow{2} \int\limits_0^t dB(s)\, ds = 0$

c. $\sum\limits_{i=1}^{n} \left(B(s_i) - B(s_{i-1})\right)^2 \xrightarrow{2} \int\limits_0^t (dB(s))^2 = t$

where, $\xrightarrow{2}$ represents the convergence in mean square.

Proof. a. $\sum\limits_{i=1}^{n} (s_i - s_{i-1})^2 \ \to \int\limits_0^t (ds)^2 = 0$

Let id be the identity function on $[0,t]$ such that $id(s) = s$. The function is of infinite variation as it increases monotonically given by:

$$V(id,t) = id(t) - id(0) = t$$

If n is finite, then it holds that:

$$Q_n(id,t) = \sum_{i=1}^{n} \left(id(s_i) - id(s_{i-1})\right)^2 = \sum_{i=1}^{n} (s_i - s_{i-1})^2 > 0$$

For n terms of Q_n, the lengths $s_i - s_{i-1}$ are of magnitude $1/n$. But due to squaring, the n terms of

Q_n are having the magnitude $\frac{1}{n^2}$. For $n \rightarrow \infty$, the sum converges to zero as follows:

$$
\begin{aligned}
Q_n\left(id, t\right) &= \sum_{i=1}^{n} \left(s_i - s_{i-1}\right)^2 \\
&\leq \max_{1 \leq i \leq n} \left(s_i - s_{i-1}\right) \sum_{i=1}^{n} \left(s_i - s_{i-1}\right) \\
&= \max_{1 \leq i \leq n} \left(s_i - s_{i-1}\right) V_n\left(id, t\right) \\
&= \max_{1 \leq i \leq n} \left(s_i - s_{i-1}\right) t \rightarrow 0
\end{aligned}
$$

since $\max(s_i - s_{i-1} \rightarrow 0)$ for $n \rightarrow \infty$.

b. $\sum\limits_{i=1}^{n} \left(B\left(s_i\right) - B\left(s_{i-1}\right)\right)\left(s_i - s_{i-1}\right) \xrightarrow{2} \int\limits_{0}^{t} dB\left(s\right) ds = 0$

The covariation CV_n for the above expression is as follows:

$$CV_n = \sum_{i=1}^{n} (B(s_i) - B(s_{i-1}))(s_i - s_{i-1})$$

As, $\mathrm{MSE}(CV_n, 0) \rightarrow 0$, the claim holds for $\mathrm{E}(CV_n) = 0$. Hence, one can obtain:

$$\mathrm{MSE}(CV_n, 0) = \mathrm{Var}(CV_n)$$

Hence, to prove the above condition, it can be shown that the variance tends to zero. Now, due to the independence increment of the Brownian motion, one can write:

$$
\begin{aligned}
\mathrm{Var}(CV_n) &= \sum_{i=1}^{n} \mathrm{Var}\left(B\left(s_i\right) - B\left(s_{i-1}\right)\right)\left(s_i - s_{i-1}\right)^2 \\
&= \sum_{i=1}^{n} \left(s_i - s_{i-1}\right)^3 \leq \max_{1 \leq i \leq n} \left(s_i - s_{i-1}\right) \sum_{i=1}^{n} \left(s_i - s_{i-1}\right)^2 \\
&= \max_{1 \leq i \leq n} \left(s_i - s_{i-1}\right) Q_n\left(id, t\right) \\
&\rightarrow 0
\end{aligned}
$$

Here, $Q_n(id, t)$ is a quadratic variation of the identity function, which is shown in the proof. a as

$Q_n(id,t) \rightarrow 0$. Hence, one can claim that the above identity exists.

c. $\sum\limits_{i=1}^{n} (B(s_i) - B(s_{i-1}))^2 \xrightarrow{2} \int\limits_0^t (dB(s))^2 = t$

To prove the above statement, one has to show that the mean square error,

$$\mathrm{MSE}\,(Q_n(B,t),t) = \mathrm{E}\left[(Q_n(B,t) - t)^2\right] \rightarrow 0$$

For this proof, two steps will be required. One has to show $\mathrm{E}(Q_n(B,t)) = t$ and $\mathrm{Var}(Q_n(B,t)) \rightarrow 0$ as $n \rightarrow \infty$

(1) In the first step, one needs to show $\mathrm{E}(Q_n(B,t)) = t$

$$\begin{aligned} \mathrm{E}(Q_n(B,t)) &= \sum_{i=1}^{n} \mathrm{Var}(B(s_i) - B(s_{i-1})) \\ &= \sum_{i=1}^{n} (s_i - s_{i-1}) \\ &= s_n - s_0 \\ &= t - 0 = t \end{aligned}$$

(2) In the second step, one needs to show that the $\mathrm{Var}(Q_n(B,t)) \rightarrow 0$ is $n \rightarrow \infty$. As the Brownian motion possesses independent increment, one can write:

$$\mathrm{Var}(Q_n(B,t)) = \sum_{i=1}^{n} \mathrm{Var}\left[(B(s_i) - B(s_{i-1}))^2\right] \tag{37}$$

Now, further solving for $\mathrm{Var}\left[(B(s_i) - B(s_{i-1}))^2\right]$ using $B(s_i) - B(s_{i-1}) \sim \mathcal{N}(0, s_i - s_{i-1})$ and applying properties of kurtosis of 3 for Gaussian random variables gives:

$$\begin{aligned} \mathrm{Var}\left[(B(s_i) - B(s_{i-1}))^2\right] &= \mathrm{E}\left[(B(s_i) - B(s_{i-1}))^4\right] - \left\{\mathrm{E}\left[(B(s_i) - B(s_{i-1}))^2\right]\right\}^2 \\ &= 3[\mathrm{Var}(B(s_i) - B(s_{i-1}))]^2 - (s_i - s_{i-1})^2 \\ &= 3(s_i - s_{i-1})^2 - (s_i - s_{i-1})^2 = 2(s_i - s_{i-1})^2 \end{aligned}$$

Now, substituting the value of $\mathrm{Var}\left[(B(s_i) - B(s_{i-1}))^2\right]$ in Eq. 37 and further evaluating the summation gives:

$$\begin{aligned} \mathrm{Var}\left(Q_n\left(B,t\right)\right) &= 2\sum_{i=1}^{n}\left(s_i - s_{i-1}\right)^2 \\ &\leq 2\max_{1\leq i\leq n}\left(s_i - s_{i-1}\right)\sum_{i=1}^{n}\left(s_i - s_{i-1}\right) \\ &= 2\max_{1\leq i\leq n}\left(s_i - s_{i-1}\right)\left(s_n - s_0\right) \\ &\to 0,\quad n\to\infty. \end{aligned}$$

Therefore, $\sum_{i=1}^{n}\left(B\left(s_i\right) - B\left(s_{i-1}\right)\right)^2 \xrightarrow{2} \int_0^t \left(dB\left(s\right)\right)^2 = t$

APPENDIX D. FEYNMAN-KAC FORMULA FOR HOMOGENEOUS EQUATION

In the case of a homogeneous PDE ($rp = 0$ in Eq. 5), the corresponding SDE will be the same, but now one has to apply Ito's formula to $p(t,z)$ instead of $\exp(-rt)p(t,z)$. Substituting the corresponding SDE in Eq. (7) and further simplifying gives

$$dp = \frac{\partial p}{\partial z} L\left(t,z\right) dB_t$$

Again, integrating from any arbitrary point t' to the exit time T_e yields

$$p\left(T_e, z\left(T_e\right)\right) - p\left(t', z\left(t'\right)\right) = \int_{t'}^{T_e} \frac{\partial p}{\partial z} L\left(t,z\right) dB_t$$

Substituting the value at the terminal point gives

$$\Phi\left(z\left(T_e\right)\right) - p\left(t', z\left(t'\right)\right) = \int_{t'}^{T_e} \frac{\partial p}{\partial z} L\left(t,z\right) dB_t$$

Taking the expectation operator on both sides and recalling the properties of Ito's integral, one can derive the Feynman-Kac solution for a homogeneous equation as

$$p\left(t', z\left(t'\right)\right) = E\left[\Phi\left(z\left(T_e\right)\right)\right]$$

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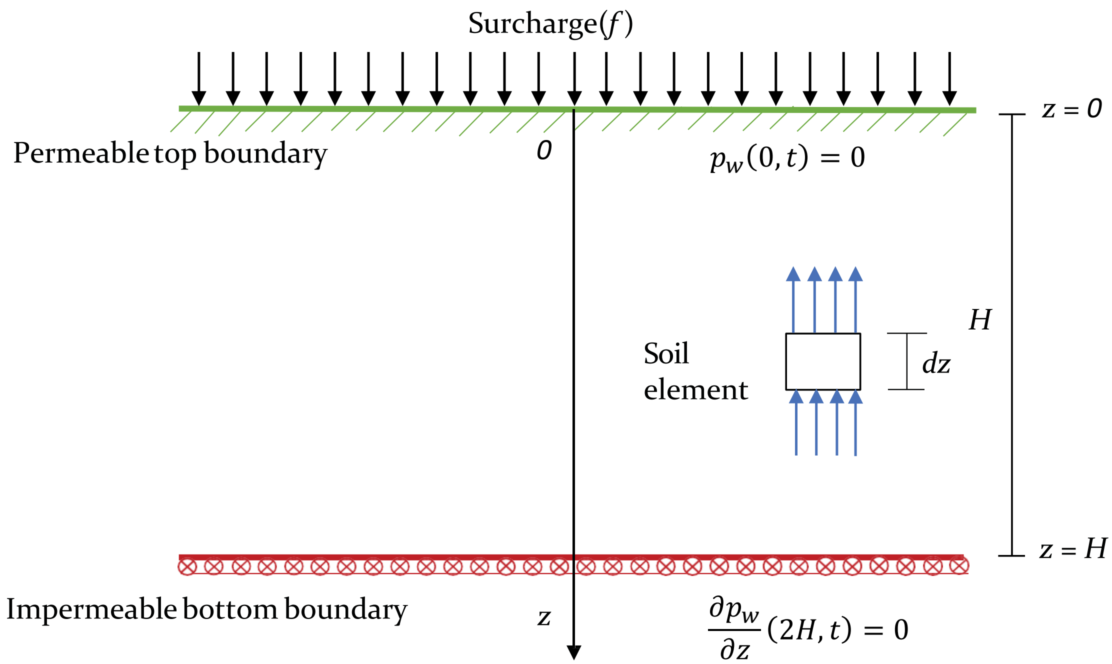


Fig. 1. Single drainage condition for 1D consolidation problem. The top boundary is permeable and the bottom boundary is impermeable.

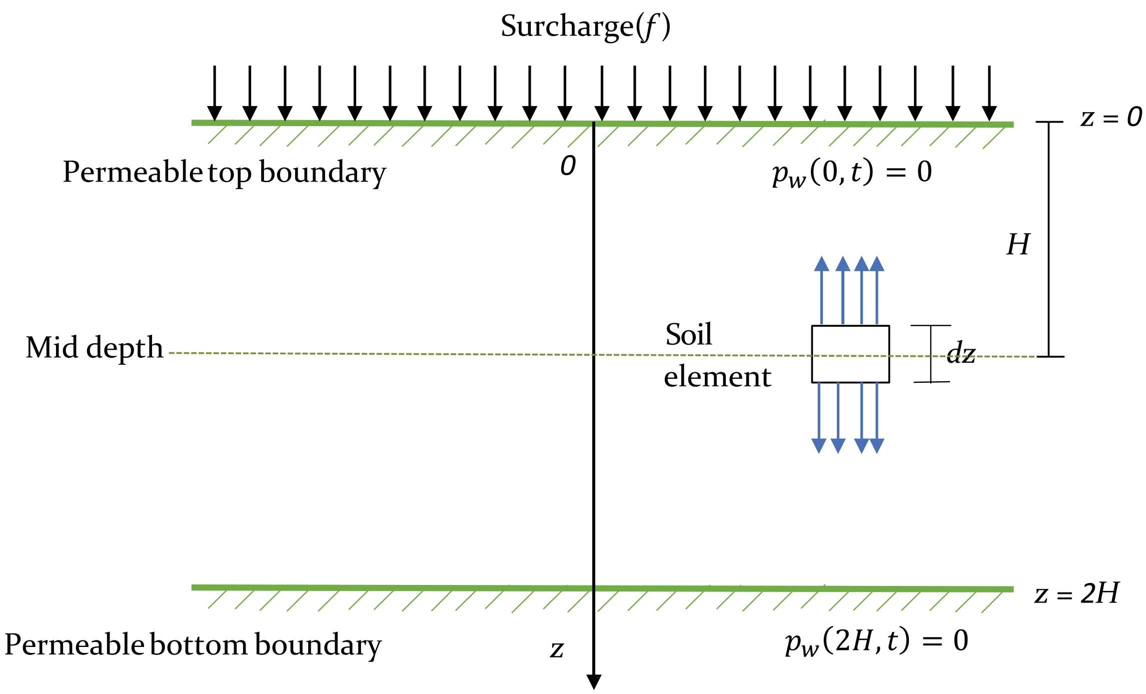
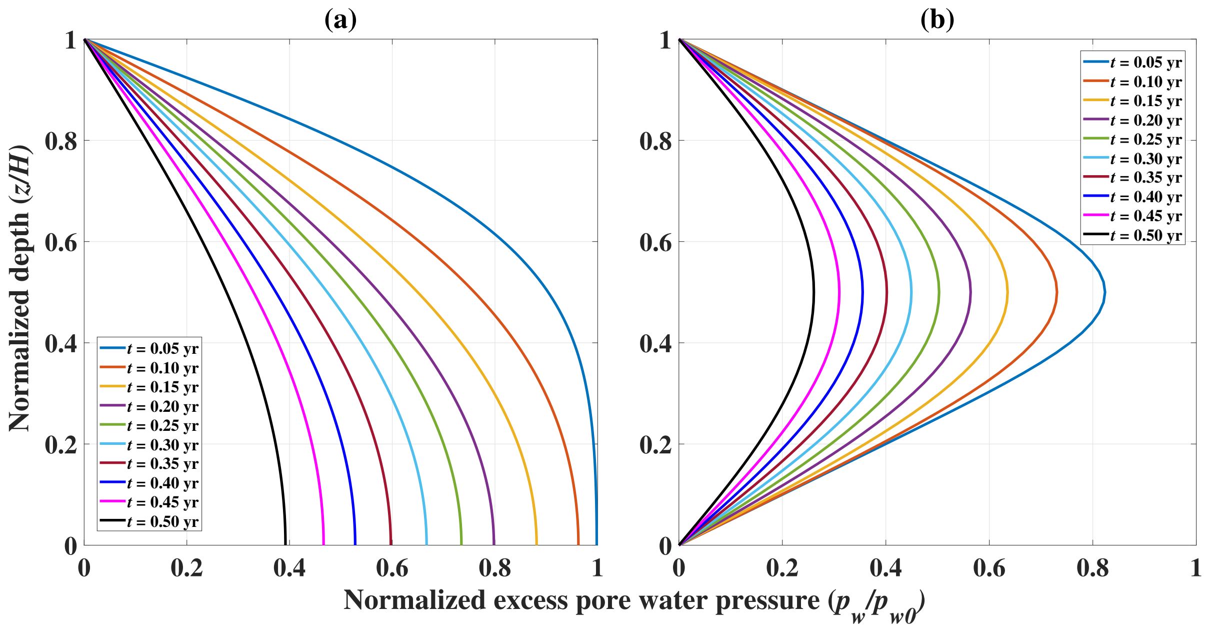


Fig. 2. Double drainage condition for 1D consolidation problem. Both the top and bottom boundaries are permeable.



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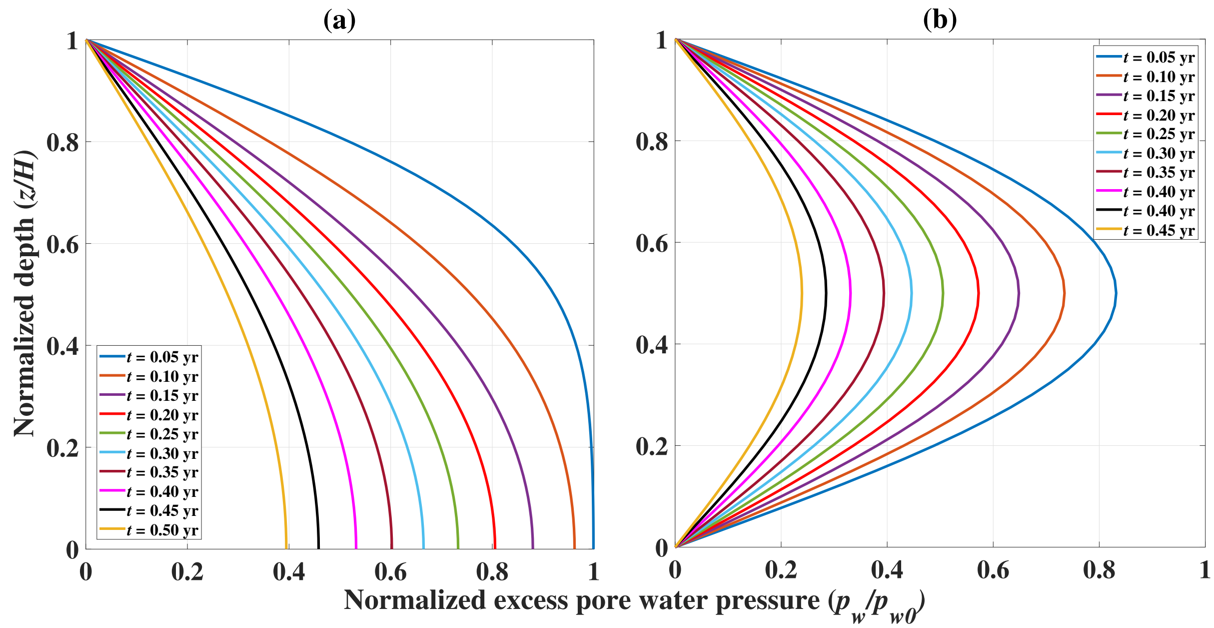

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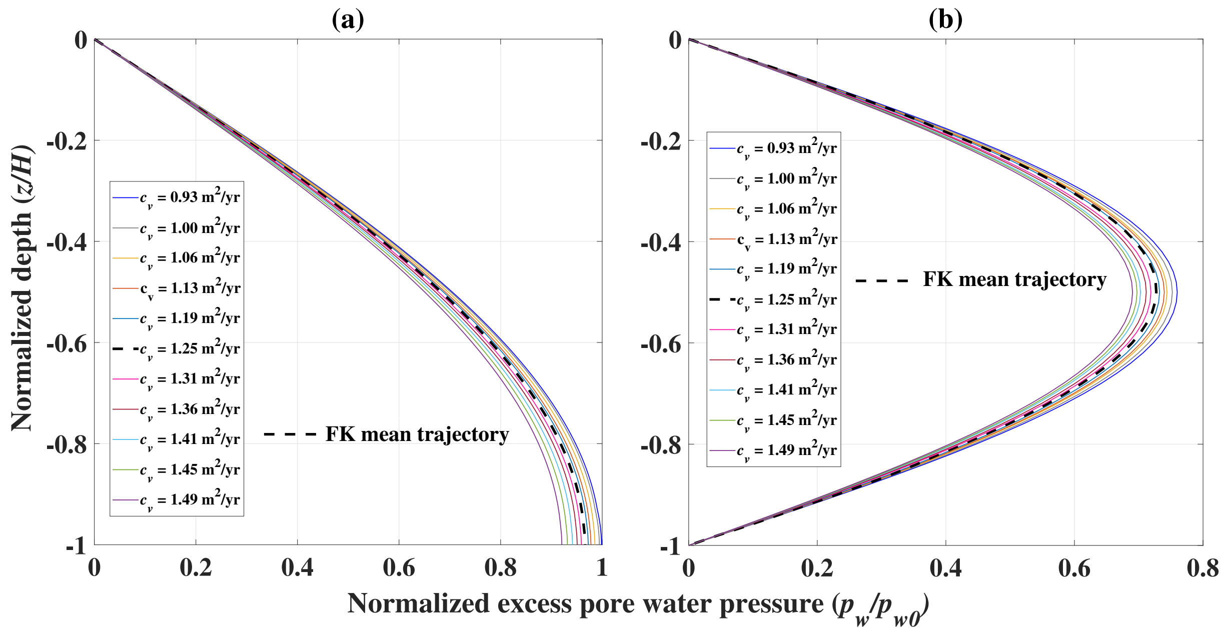

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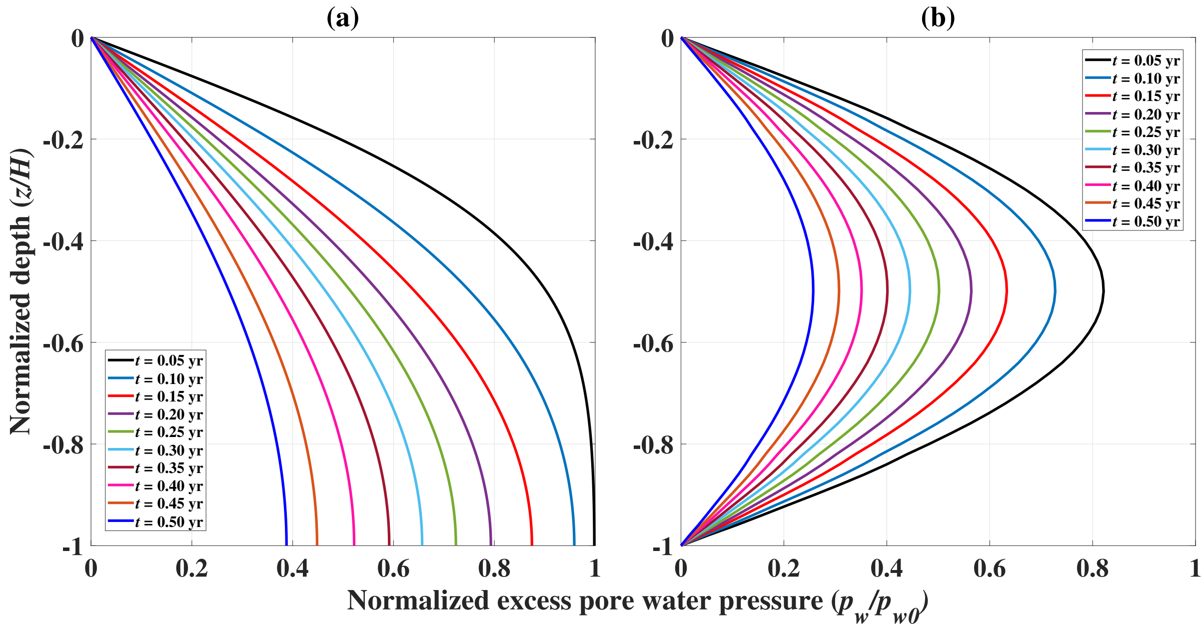

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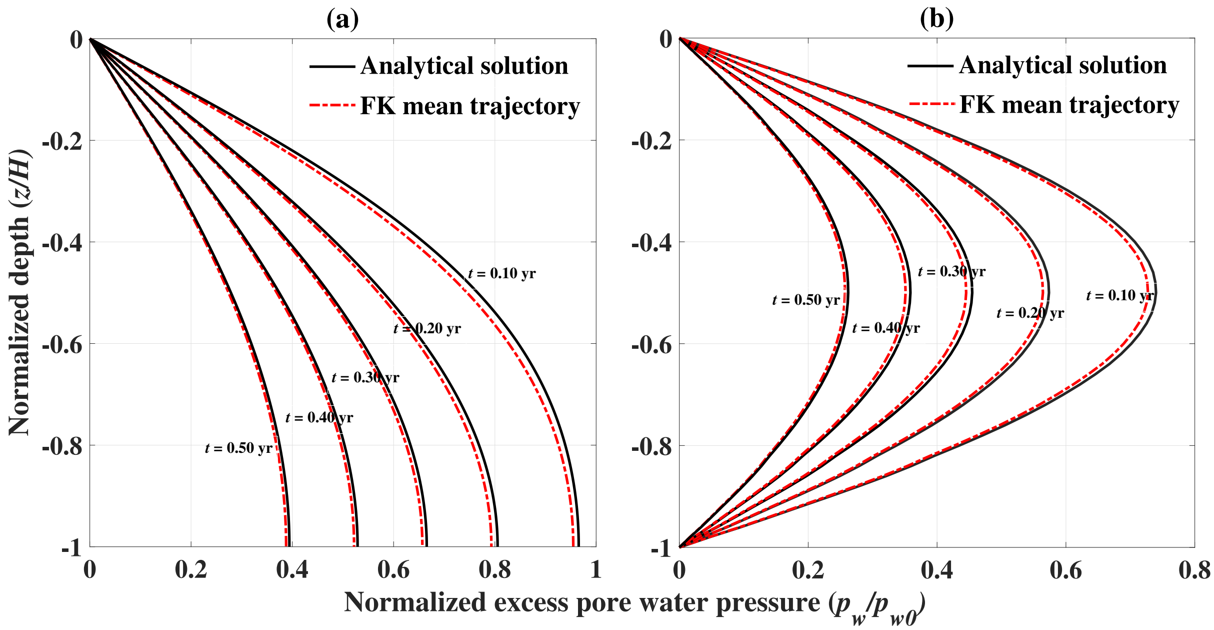

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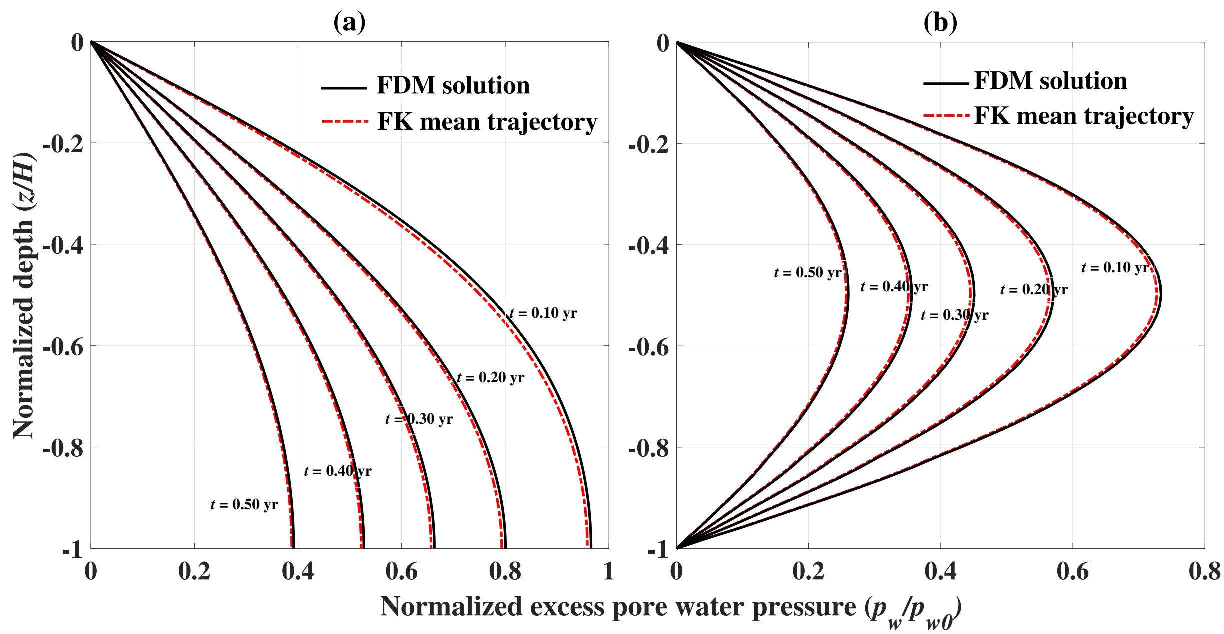

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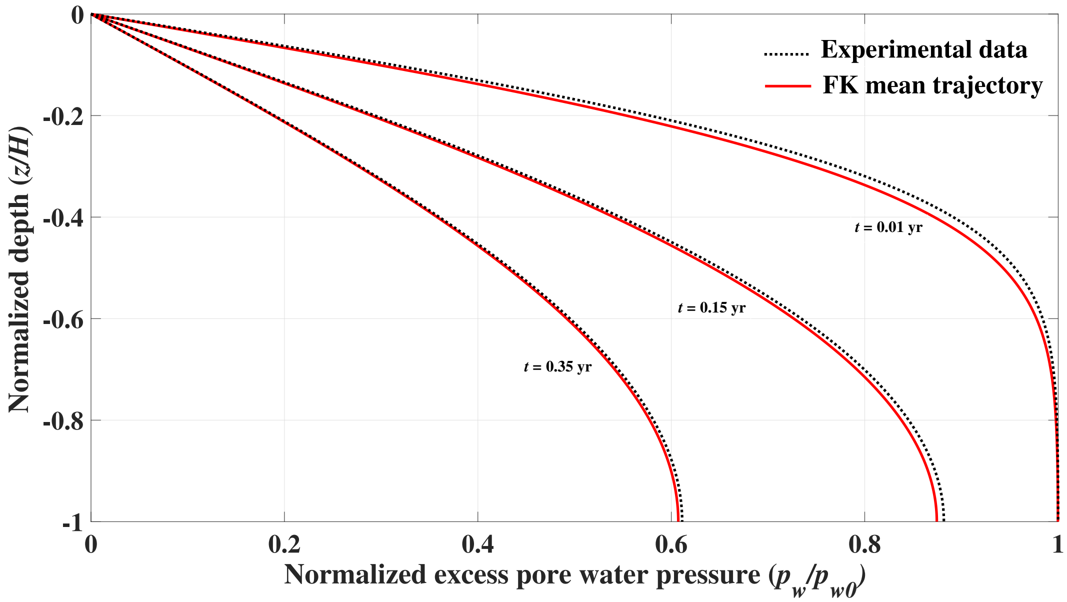

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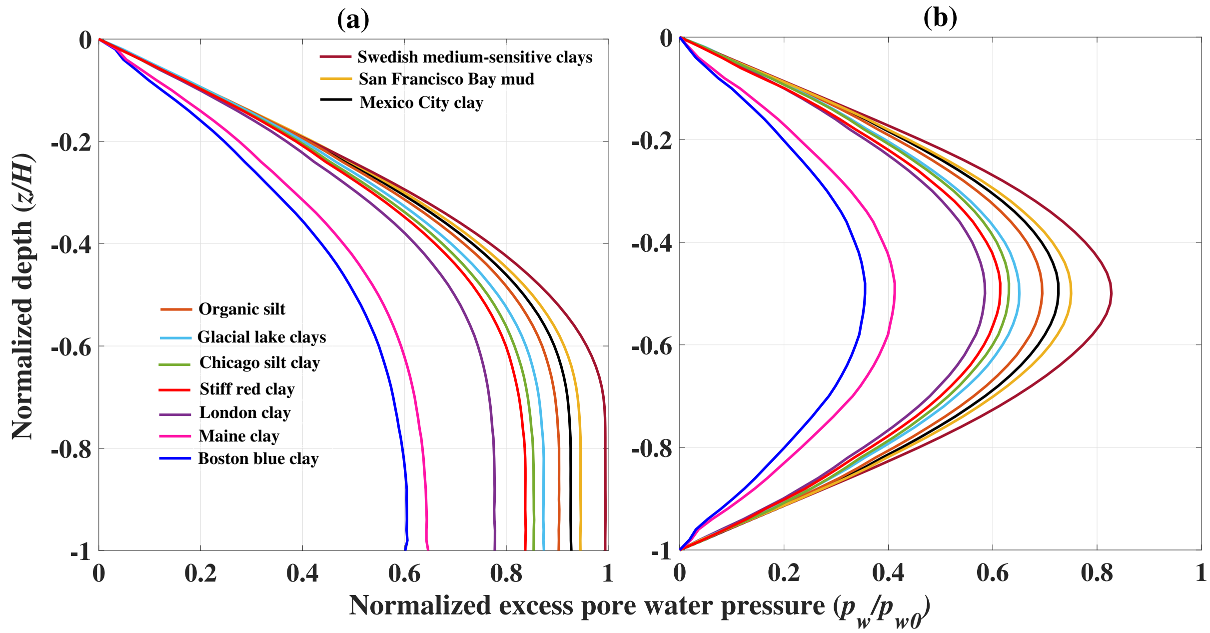

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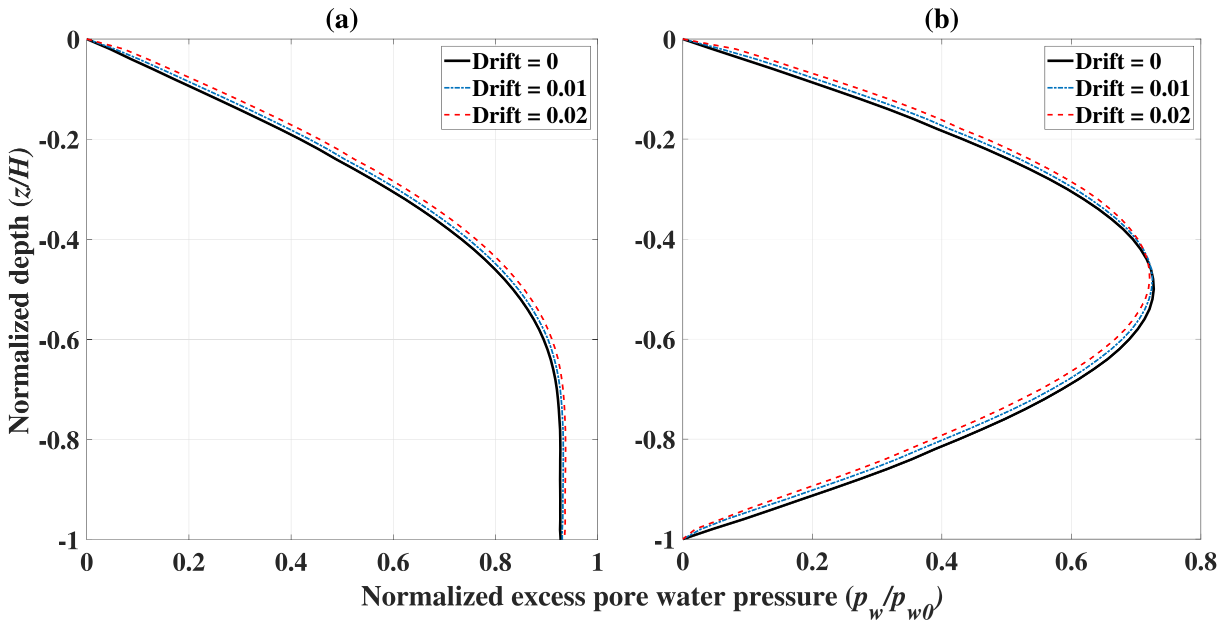

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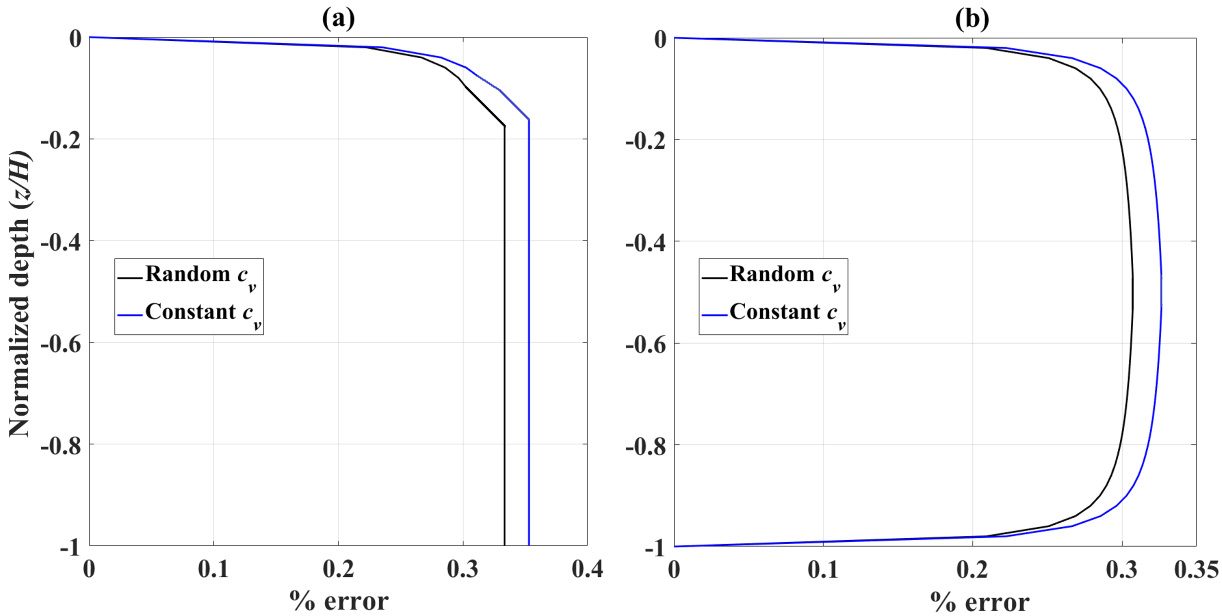

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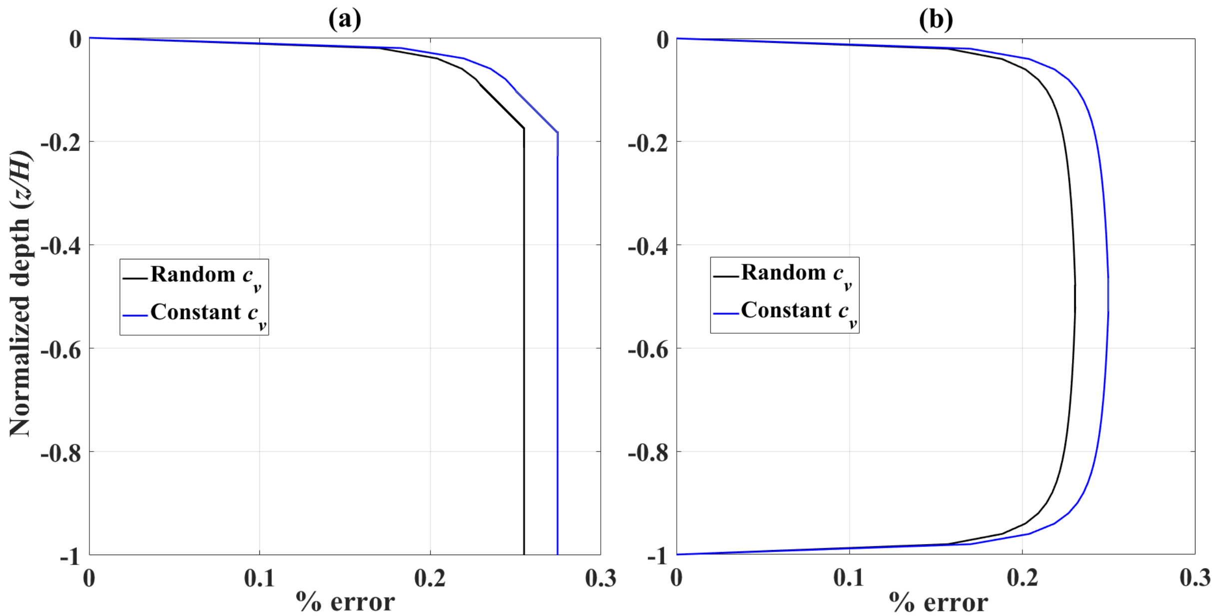

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