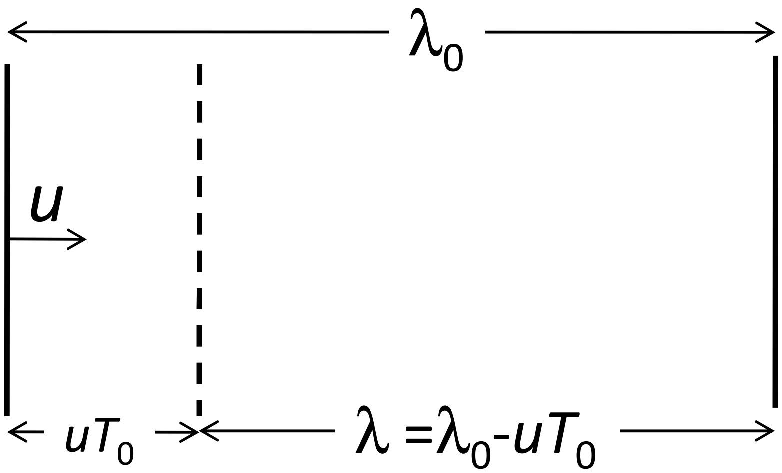
Lecture-XXI

Special Theory of Relativity
Doppler effect & Relativistic dynamics

The Doppler Effect:

The Doppler effect of sound (in introductory physics) is represented by an *increased frequency* of sound as a source such as a train (with whistle blowing) approaches a receiver (our eardrum) and a *decreased frequency* as the source recedes.



Let T_0 be the time it takes a wave to move one wavelength λ_0. Then $\lambda_0 = vT_0$.

$$\lambda = \lambda_0 - uT_0 = (v-u)\, T_0$$

$$\frac{\mathrm{v}}{\nu} = (\mathrm{v}-u)\frac{1}{\nu_0} \quad \text{or} \quad \nu = \frac{\mathrm{v}}{(\mathrm{v}-u)}\nu_0$$

$$\text{Observer moving:} \quad \nu = \frac{u+\mathrm{v}}{u}\nu_0$$

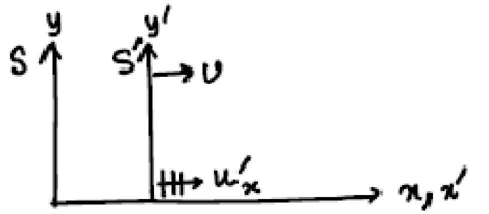
Relativistic Doppler Effect:

A light source is moving at a constant speed *v* toward a stationary observer. The source emits EM waves with $\nu_0 = 1/T_0$ in its rest frame. What is the frequency measured by a stationary observer?

$$\lambda = cT - \mathrm{v}T = (c-\mathrm{v})T, \nu = \frac{c}{(c-\mathrm{v})}\frac{1}{T}, T = \gamma T_0, \nu = \frac{c}{(c-\mathrm{v})}\frac{1}{\gamma T_0} = \sqrt{\frac{c+\mathrm{v}}{c-\mathrm{v}}}\nu_0$$

Another approach: Consider a plane monochromatic light waves of unit amplitude emitted from a source at the origin of the S'-frame propagating along x' axis.

$$\cos\left(k'_x x' - \omega' t'\right)$$



$k' = \frac{2\pi}{\lambda'}$; Wave vector

$\omega' = 2\pi\nu'$; angular frequency.

$$\lambda'\nu' = c = \frac{\omega'}{k'}$$

In the S-frame these wavefronts will still be planes, for the Lorentz transformation is linear and a plane transforms into a plane. Hence,

$$\cos\left(k_x x - \omega t\right)$$
$$= \cos\left[k_x \gamma\left(x' + vt'\right) - \omega\gamma\left(t' + v\frac{x'}{c^2}\right)\right]$$
$$= \cos\left[\gamma\left(k_x - \frac{\omega v}{c^2}\right)x' - \gamma\left(\omega - k_x v\right)t'\right]$$

$k = \frac{2\pi}{\lambda}$

$\omega = 2\pi\nu$

$\lambda\nu = c = \frac{\omega}{k}.$

When the amplitude of the wave vanishes at a certain spacetime point, it must vanish in both systems which requires that the phase in primed and unprimed frames must be equal:

$$k'_x = \gamma\left(k_x - \frac{v}{c^2}\omega\right) \quad ; \quad \omega' = \gamma\left(\omega - k_x v\right)$$

$$\left(k'_y = k_y,\ k'_z = k_z\right)$$

Inverse transformations:

$$k_x = \gamma\left(k'_x + \frac{v}{c^2}\omega'\right) \quad ; \quad \omega = \gamma\left(\omega' + k'_x v\right)$$

See The Feynman Lectures on Physics, Vol.1, Ch.34, Page-416 (Narosa Publication, New Delhi

Case-I: when observer and source receding

$$k'_x = -k'_x = -\omega'/c, \quad \mathrm{v} = \mathrm{v}$$

$$\omega = \gamma\left(\omega' + k'_x \mathrm{v}\right) = \gamma\left(\omega' - \mathrm{v}\,\omega'/c\right) = \gamma\left(1 - \mathrm{v}/c\right)\omega'$$

$$\nu = \nu_0 \sqrt{\frac{c-\mathrm{v}}{c+\mathrm{v}}} \qquad \text{Classically: } \nu = \nu_0\left(1 - \mathrm{v}/c\right)$$

Case-II: when observer and source approaching

$$k'_x = -k'_x = -\omega'/c, \quad \mathrm{v} = -\mathrm{v}$$

$$\omega = \gamma\left(\omega' + k'_x \mathrm{v}\right) = \gamma\left(\omega' + \mathrm{v}\,\omega'/c\right) = \gamma\left(1 + \mathrm{v}/c\right)\omega'$$

$$\nu = \nu_0 \sqrt{\frac{c+\mathrm{v}}{c-\mathrm{v}}} \qquad \text{Classically: } \nu = \nu_0\left(1 + \mathrm{v}/c\right)$$

Case-III: when observer and source are in perpendicular directions

Observer: in S at (0,0,0); Source: in S' at (0,y,0). They pass each other at t=t'=0,

$$k'_x = 0, k'_y = -k, k'_z = 0$$

$$\omega = \gamma\left(\omega' + k'_x \mathrm{v}\right) = \gamma\omega', \quad \nu = \frac{\nu_0}{\sqrt{1 - \mathrm{v}^2/c^2}} \qquad \text{Classically: No such effect.}$$

Example 1: A rocket ship is receding from earth at a speed of 0.2c. A light in the rocket ship of wavelength λ=4500A appears blue to a passenger on the ship. What colour would it appear to an observer on earth?

$$\nu = \nu_0 \sqrt{\frac{1-u/c}{1+u/c}}, \quad \nu\lambda = c$$

$$\lambda = \lambda_0 \sqrt{\frac{1+u/c}{1-u/c}} = 4500 \times \sqrt{\frac{1+0.2}{1-0.2}} \approx 5511\text{A:} \quad \text{Green}$$

Example 2: A source of sodium D2 line of wavelength λ=5890A is moving on a circle with a constant speed 0.1c. Find the change in wavelength for an observer at the centre.

$$\nu = \frac{\nu_0}{\sqrt{1-u^2/c^2}} \Rightarrow \lambda = \lambda_0 \sqrt{1-u^2/c^2} = 5890\sqrt{1-0.1^2}$$

$$\Delta\lambda = 5890\left(1-\sqrt{0.99}\right) \approx 30\text{A}$$

The Doppler shift in the frequency of light waves arriving from distant galaxies is one of the main sources of our knowledge of the Universe.

The light arriving from distant galaxies is shifted toward lower frequencies. This is called "the reddening of galaxies".

How do we know that the frequency is lower? Well, all stars emit certain characteristic "spectral lines", the frequency of which is well known. One of such lines is "the blue line of hydrogen", with wavelength λ= 434 nm. Suppose that in the light from a distant galaxy the same line has a wavelength of λ'= 600 nm – such light is no longer blue, but red (therefore, the term "reddening").

Question: what is the "receding speed" *u* of that galaxy?

$$\text{We use}: \ \nu = \frac{c}{\lambda}; \quad \text{hence,} \ \frac{c}{600\,\text{nm}} = \frac{c}{434\,\text{nm}}\sqrt{\frac{1-u/c}{1+u/c}}$$

$$434^2 \times (1+u/c) = 600^2 \times (1-u/c)$$

$$600^2 - 434^2 = \left(600^2 + 434^2\right) \times \frac{u}{c} \quad \Rightarrow \ u = 0.31c$$

Relativistic Dynamics

- **Momentum:**

There is one problem with $\vec{f} = m\,\vec{a}\,,\qquad \vec{a} = \frac{d\vec{v}}{dt}$

…these relations are inconsistent with our relativity postulates. It predict that a particle subject to a constant force (and initially at rest) will acquire a velocity which can become arbitrarily large,

$$\vec{v}(t) = \int_0^t \frac{d\vec{v}}{dt'}\,dt' = \frac{\vec{f}\,t}{m} \longrightarrow \infty \quad \text{as } t \to \infty\,.$$

This flatly contradicts the prediction of special relativity (and causality) that no signal can propagate faster than c. These expressions must be modified unless v/c << 1.

What is the correct relativistic definition of momentum? We would like to define it such that total momentum is conserved in all inertial frames.

We have already seen a Lorentz Invariant or scalar, Δs^2

We define a relativistic" momentum: $\vec{p} = m_0\frac{d\vec{r}}{d\tau} = m_0\frac{d\vec{r}}{dt/\gamma} = m_0\gamma\frac{d\vec{r}}{dt}$

$$\frac{(ds)^2}{c^2} = (dt)^2\left[1 - \frac{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}{c^2}\right]$$

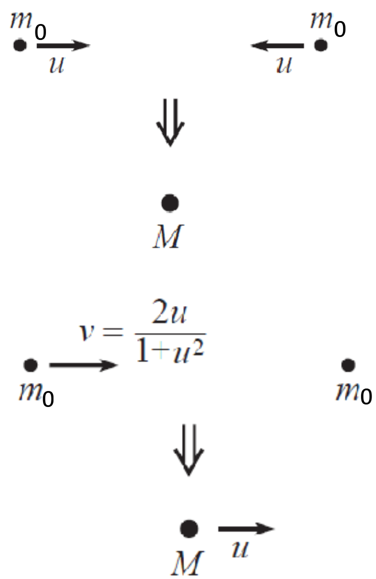
$$= (dt)^2\left[1 - \frac{v^2}{c^2}\right].$$

$$\boxed{\vec{p} = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\vec{v} = \gamma m_0\vec{v}.}$$

The quantity *ds/c* is called "**proper**" time *dτ* and is Lorentz invariant

$$d\tau = dt\sqrt{1 - \frac{v^2}{c^2}} = \frac{dt}{\gamma}$$

Exercise: Consider the following system. Two identical particles of mass *m head toward* each other, both with speed *u. They stick together and* form a particle of mass *M. M is at rest, due to the symmetry of the situation.*



Let's consider it in the frame moving to the left at speed *u. This situation is* shown in Figure. The right mass is at rest, the left mass moves to the right at speed $v = \frac{2u}{1+u^2/c^2}$ *from the velocity-addition formula, and the final mass M* moves to the right at speed *u.*

$$\gamma = \frac{1}{\sqrt{1-v^2}} = \frac{1}{\sqrt{1-\left(\frac{2u}{1+u^2/c^2}\right)^2}} = \frac{1+u^2/c^2}{1-u^2/c^2}$$

Conservation of momentum in this collision then gives

$$\gamma m_0 v + 0 = \gamma M u$$

$$\Rightarrow m_0\left(\frac{1+u^2/c^2}{1-u^2/c^2}\right)\left(\frac{2u}{1+u^2}\right) = \frac{Mu}{\sqrt{1-u^2/c^2}}$$

$$\Rightarrow \boxed{M = \frac{2m_0}{\sqrt{1-u^2/c^2}}}$$

Conservation of momentum therefore tells us that *M does not equal 2m* $_0$*. But if u* is very small, then *M is approximately equal to 2m* $_0$*, as we know from everyday* experience.

- **Energy:**

In relativistic mechanics, it proves useful to use the definition for Kinetic Energy in Newtonian Mechanics (for simplicity, we restrict the motion only in one dimension along x),

$$K = \int_{u=0}^{u} F\,dx = \int \frac{d}{dt}(mu)\,dx = \int d(mu)\frac{dx}{dt}$$

$$= \int (m\,du + u\,dm)\,u = \int_0^u (mu\,du + u^2 dm)$$

in which both u and m are variables. Since $m = \dfrac{m_0}{\sqrt{1-u^2/c^2}} \Rightarrow m^2c^2 - m^2u^2 = m_0^2c^2$

Taking differentials, $\boxed{mu\,du + u^2 dm = c^2 dm}$

$$\text{Therefore} \quad K = \int_{u=0}^{u} c^2 dm = c^2 \int_{m_0}^{m} dm = mc^2 - m_0c^2$$

$$\text{or equivalently,} \quad K = m_0c^2\left[\frac{1}{\sqrt{1-u^2/c^2}} - 1\right]$$

In the limit $\frac{u}{c} \ll 1$, $\longrightarrow K \simeq m_0c^2\left(1 + \frac{1}{2}\frac{u^2}{c^2} - 1\right) = \frac{1}{2}m_0u^2$

$$mc^2 = K + m_0 c^2$$

K → Work done on the particle to bring it from rest to speed u.

Einstein proposed the relation as $\boxed{E = mc^2}$

- The first term arises from external work and the second one represents the **rest energy** the particle possesses by virtue of its mass.

Energy momentum relation:

Using the definition of relativistic momentum, $\mathbf{p} = m\mathbf{u} = \dfrac{m_0\mathbf{u}}{\sqrt{1 - u^2/c^2}} = m_0\mathbf{u}\gamma$

$$\begin{aligned} E^2 - |\mathbf{p}|^2c^2 &= \gamma^2 m_0^2 c^4 - \gamma^2 m_0^2 |\mathbf{v}|^2 c^2 \\ &= \gamma^2 m_0^2 c^4 \left(1 - \frac{v^2}{c^2}\right) \\ &= m_0^2 c^4. \end{aligned}$$

$$\boxed{E^2 = p^2c^2 + m_0^2 c^4}$$

- Massless particle: If we take $m_0 = 0$, then the energy momentum relation yields,

$$E = pc.$$

In order to have nonzero momentum we must have a finite value for

$$\mathbf{p} = m_0\mathbf{u} \Big/ \sqrt{1 - \frac{u^2}{c^2}}.$$

in the limit $m_0 \longrightarrow 0$. This is only possible if $u \longrightarrow c$ as $m_0 \longrightarrow 0$;
Therefore the conclusion is: *massless particles must travel at the speed of light.*

Example: An electron is accelerated in a potential difference of 10^4 Volts. Find its kinetic energy *K*, velocity *u* and mass *m*.

$$K = eV = 1.602\times10^{-15}\,\mathrm{J} = mc^2 - m_0c^2$$

$$m_0 = 9.109\times10^{-31}\,\mathrm{Kg},\ m = \frac{K}{c^2} + m_0 = 9.287\times10^{-31}\,\mathrm{Kg}$$

$$m = \frac{m_0}{\sqrt{1-u^2/c^2}},\ \ u = c\sqrt{1-m_0^2/m^2} = 5.85\times10^7\,\mathrm{m/s}$$