Notes on video lecture 1

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We are going to do "Multivariable Calculus" in this part of the course MA 101. In this lecture, we will start with "vector calculus".

The word "Multivariable" means more than one variables. In this course, we are going to deal with functions of more than one variables. The very first topic in the syllabus is "vector calculus". Vector calculus is going to introduce us to geometry in 2 and 3 dimensions. There are many geometric sets that we are going to deal with in this course.

1 Review of vectors

Question: What is a vector?
Answer: A vector is a physical quantity which has both magnitude and direction.

Example 1.0.1. Suppose we consider the motion of a particle, which is moving from one point A to another point B. This is designated by an arrow joining A and B (with the arrow head pointing towards the point B). This arrow is called the **displacement vector**. The point A (from where the particle started with) is called the **initial point** of the vector. And the point B is called the **terminal point** of the vector. The distance covered from A to B is called the **magnitude of the vector**.
Generally, this vector is denoted by $\overrightarrow{AB}$ and its magnitude is denoted by $|\overrightarrow{AB}|$. Once the particle moves from the point A to B, that is going to give us the direction also. □

In this course, we will indicate a vector by putting an arrow over some mathematical symbol.

Definition 1.0.2. Let us consider two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ such that their lengths (or magnitudes) are the same and they are parallel. That means, the directions of $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are the same and their magnitudes are also the same.
Then we say that the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are **equivalent** and denote it by $\overrightarrow{AB} = \overrightarrow{CD}$.
Note: The initial and terminal points of $\overrightarrow{AB}$ and $\overrightarrow{CD}$ need not be the same!

Figure 1.0.1 below illustrates the above definition of equivalent vectors.

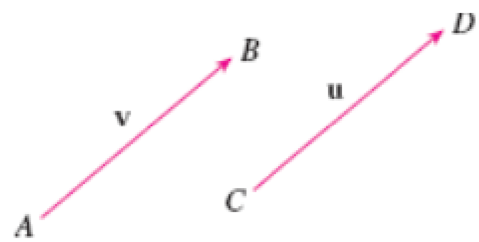

Figure 1.0.1: Equivalent vectors

1.1 Combining vectors

Definition 1.1.1. Consider 2 vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ such that the terminal point of $\overrightarrow{AB}$ is the same as the initial point of $\overrightarrow{BC}$. If we now join the initial point A of $\overrightarrow{AB}$ and the terminal point C of $\overrightarrow{BC}$, then we get a new vector $\overrightarrow{AC}$, which is called the **sum** of the two vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$. We write

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}.$$

Figure 1.1.1 below illustrates the above concept of the sum of two vectors. The above definition gives one way (namely, **addition**) of combining 2 vectors. Now, a natural question arises.
Question: In the above definition, we had taken two vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ such that the initial point of one vector equals the terminal point of the other vector. What happens when the two vectors that we consider donot have this property?
Answer: Let us take two vectors $\vec{a}$ and $\vec{b}$ which do not have this property. But we have learnt the concept of equivalent vectors in Definition 1.0.2 above.

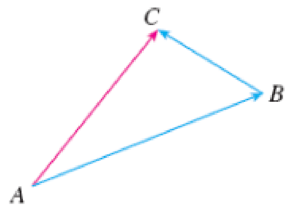

Figure 1.1.1: Sum of two vectors

Therefore, we can do some kind of a "parallel displacement" of one of the vectors (say, $\vec{b}$) and bring it to a position in such a way that the terminal point of $\vec{a}$ equals the initial point of $\vec{b}$. Then we can take their sum and call it $\vec{a}+\vec{b}$.

Definition 1.1.2. If $\vec{b}$ is a given vector, then $-\vec{b}$ is going to be the vector having the same magnitude as $\vec{b}$ but having opposite direction.
If we consider another vector $\vec{a}$ and join the initial point of $\vec{a}$ with the terminal point of $-\vec{b}$, we get a new vector which is denoted by $\vec{a}-\vec{b}$. This new vector so obtained is called the **difference** or the **subtraction** of the two vectors $\vec{a}$ and $\vec{b}$.

Figure 1.1.2 below illustrates the above concept of the subtraction of two vectors.

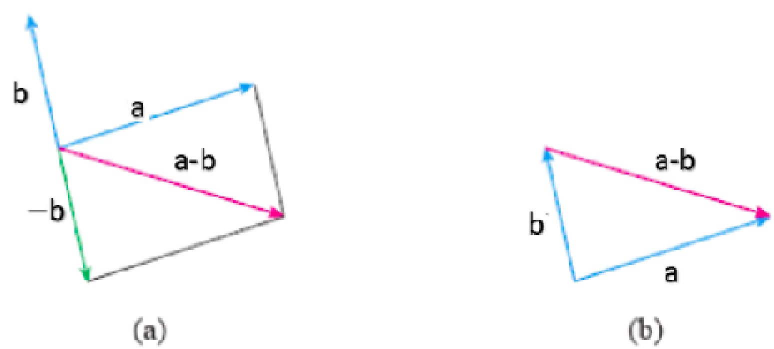

Figure 1.1.2: Subtraction of two vectors

Definition 1.1.3. Consider a vector $\vec{a}$. We can multiply the vector $\vec{a}$ by a nonzero scalar λ. Then $\overrightarrow{\lambda a}$ is a new vector whose magnitude is equal to $|\lambda|$

times the magnitude of $\vec{a}$. And the direction of the new vector $\overrightarrow{\lambda a}$ equals that of $\vec{a}$ if $\lambda > 0$ and opposite to that of $\vec{a}$ if $\lambda < 0$. This is called **scalar multiplication** of a vector $\vec{a}$ by a nonzero scalar λ.

Figure 1.1.3 below illustrates the above concept of multiplication of a vector by a scalar.

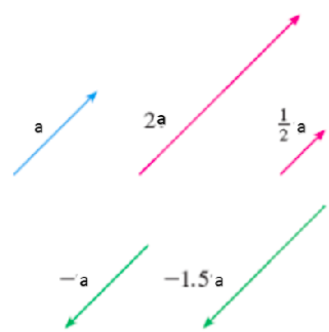

Figure 1.1.3: Scalar multiplication

1.2 The triangle law

Consider 2 vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ such that the initial point of $\overrightarrow{BC}$ equals the terminal point of $\overrightarrow{AB}$. Then the sum of the two vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ is equal to the third side of the triangle formed by the vectors $\overrightarrow{AB}, \overrightarrow{BC}$ and $\overrightarrow{AC}$. Figure 1.2.1 below illustrates the above concept of the triangle law.

1.3 The parallelogram law

Suppose $\vec{a}$ and $\vec{b}$ are two arbitrary vectors in a plane. By doing a parallel displacement, we can bring the vector $\vec{b}$ to a place such that the initial points of $\vec{a}$ and $\vec{b}$ are the same. Then we can construct a parallelogram whose sides are given by $\vec{a}$ and $\vec{b}$. Then the diagonal of this parallelogram is going to designate the sum of the two vectors $\vec{a}$ and $\vec{b}$.
Note: The triangle law and the parallelogram law can be realized for subtraction $(\vec{a} - \vec{b})$ of two vectors as well. The only point is that, one has to take a vector in the direction opposite to that of $\vec{b}$.
Figure 1.3.1 below illustrates the above concept of the parallelogram law.

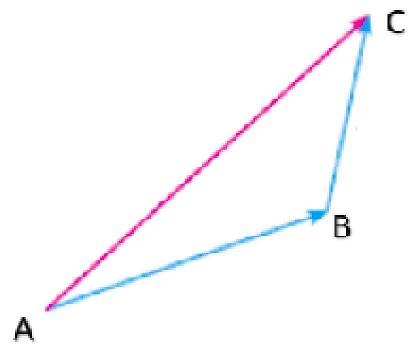

Figure 1.2.1: The triangle law

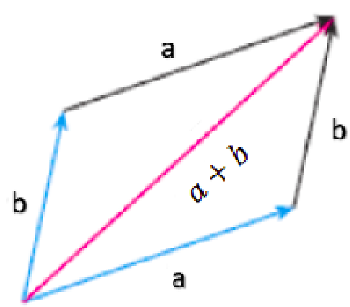

Figure 1.3.1: The parallelogram law

1.4 Components of a vector and the position vector

Consider the xy-plane or $\mathbb{R}^2$. Let P be a point in the xy-plane. Suppose the coordinates of P are (a_1, a_2). Let O denote the origin of the xy-plane. If we drop a perpendicular from the point P to the x-axis (say, PL), then the length of OL is a_1 and the length of PL is a_2. See Figure 1.4.1 below for an illustration. Let $\vec{a}$ be the vector $\overrightarrow{OP}$. Then we write

$$\vec{a} = < a_1, a_2 >,$$

where a_1 and a_2 are called the **components of** the vector $\vec{a}$. We refer to a_1 as the component along the x-axis and similarly for a_2.
Similarly, let us consider the xyz-space, and consider a point P there, whose

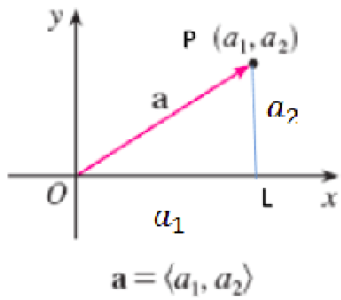

Figure 1.4.1: Components of a vector in a plane

coordinates are (a_1, a_2, a_3). Let PL denote the perpendicular from the point P on the xy-plane. Let M denote the foot of the perpendicular from the point L on the x-axis and N denote the foot of the perpendicular from the point L on the y-axis. Then we have $PL = a_3, OM = a_1, ON = a_2$. If $\overrightarrow{OP} = \vec{a}$, then we write

$$\vec{a} =< a_1, a_2, a_3 >,$$

where a_1, a_2, a_3 are called the **components of** the vector $\vec{a}$. See Figure 1.4.2 below for an illustration.

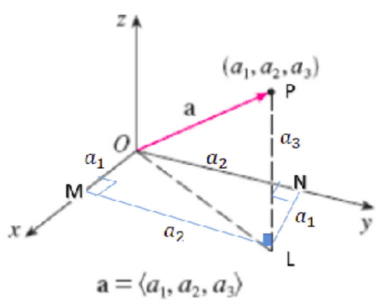

Figure 1.4.2: Components of a vector in space

For any positive integer $n > 3$, one can similarly define $a_1, \ldots, a_n$ to be the **components of** an n-dimensional vector $\vec{a} =< a_1, \ldots, a_n >$.

Definition 1.4.1. If we consider a point P in the xy-plane (or in the xyz-space) and if O denotes the origin, then the vector $\overrightarrow{OP}$ is called the **position vector** of the point P.

1.5 Geometrical interpretation of addition of two vectors

Let $\vec{a}$ and $\vec{b}$ be the vectors $< a_1, a_2 >$ and $< b_1, b_2 >$ respectively in the xy-plane. Then we **claim** that:

$$\vec{a} + \vec{b} =< a_1 + b_1, a_2 + b_2 > .$$

That is, we have to prove that the components of the vector $\vec{a} + \vec{b}$ are given by $a_1 + b_1$ and $a_2 + b_2$ respectively.
Proof of claim: We can make a parallel displacement of the vector $\vec{b}$ and bring it to a place such that the terminal point of $\vec{a}$ equals the initial point of $\vec{b}$. Then we know how to draw the vector $\vec{a} + \vec{b}$. The following figure (Figure 1.5.1) illustrates the entire procedure and also explains what are the components of the vector $\vec{a} + \vec{b}$, completing the proof of the claim.

This whole concept can be easily extended to 3 dimensions. Also, we can

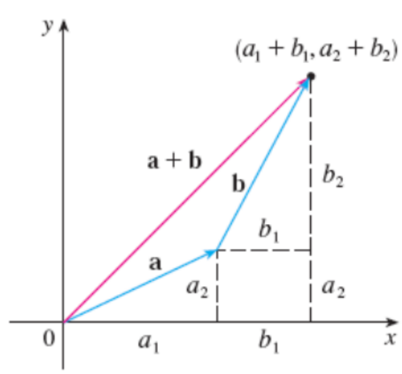

Figure 1.5.1: Components of $\vec{a} + \vec{b}$

similarly have a geometrical interpretation of the following **claim** in a similar way:
Claim: $\vec{a} - \vec{b} =< a_1 - b_1, a_2 - b_2 >$.

1.5.1 An application

Now suppose we consider two points P and Q in the xyz-space. Say, the coordinates of P are (x_1, y_1, z_1) and that of Q are (x_2, y_2, z_2). Let O denote the origin of the xyz-space. Then $\overrightarrow{OP}$ is the position vector of P and $\overrightarrow{OQ}$ is the position vector of Q. If P and Q are joined, then from the definition of addition of vectors, we can see that

$$\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ}.$$

In other words,

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}.$$

In terms of components, we therefore have:

$$\overrightarrow{PQ} =< x_2, y_2, z_2 > - < x_1, y_1, z_1 > .$$

But then, using one of the above claims, we have that:

$$\overrightarrow{PQ} =< x_2 - x_1, y_2 - y_1, z_2 - z_1 > .$$

So, to conclude, if we want to express a vector $\overrightarrow{PQ}$ where P and Q are two points lying in the xyz-space, then if we know the coordinates of the points P and Q, then $\overrightarrow{PQ}$ can be written as above ($\overrightarrow{PQ} =< x_2 - x_1, y_2 - y_1, z_2 - z_1 >$) in terms of components.

1.6 Magnitude or length of a vector

Let P be a point in the xy-plane, whose coordinates are (a_1, a_2). Let O denote the origin of the xy-plane. Then $\overrightarrow{OP} =< a_1, a_2 >$. Let L denote the foot of the perpendicular from the point P on the x-axis. Then we know that the length of OL equals a_1 and the length of PL equals a_2. Consider the right angled triangle OPL. By Pythagoras theorem, the length of OP equals $\sqrt{a_1^2 + a_2^2}$. This length is denoted by $|\overrightarrow{OP}|$ and is called the **magnitude** of the position vector $\overrightarrow{OP}$.

Similarly, in 3 dimensions, if the coordinates of a point P are (a_1, a_2, a_3), then the magnitude of the position vector $\overrightarrow{OP}$ is given by

$$|\overrightarrow{OP}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

For a quick proof of this, let L denote the foot of the perpendicular from the point P on the xy-plane. Then the coordinates of L are $(a_1, a_2, 0)$ and the length of $\overrightarrow{PL}$ equals a_3. Let A denote the foot of the perpendicular from L on the x-axis and B denote the foot of the perpendicular from L on the y-axis. Then the length of $\overrightarrow{OA}$ equals a_1 and the length of $\overrightarrow{OB}$ equals a_2 and $|\overrightarrow{OL}| = \sqrt{a_1^2 + a_2^2}$. Now, consider the right angled triangle OLP. Then $|\overrightarrow{OP}| = \sqrt{|\overrightarrow{OL}|^2 + |\overrightarrow{PL}|^2}$. That is,

$$|\overrightarrow{OP}| = \sqrt{a_1^2 + a_2^2 + a_3^2}.$$

1.6.1 A concluding remark

Till now, we have considered the length or magnitude of only position vectors (which start at the origin and end at some point). Recall from our earlier discussion that

$$\overrightarrow{PQ} = < x_2 - x_1, y_2 - y_1, z_2 - z_1 >,$$

where $\overrightarrow{PQ}$ was an arbitrary vector lying in the 3 dimensional space, which was obtained by joining the two points P and Q. What about the magnitude of $\overrightarrow{PQ}$?

To answer this question, recall that we can bring the vector $\overrightarrow{PQ}$ to a place such that it starts at the origin and ends at some point, using a parallel displacement.

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