DEPARTMENT OF MATHEMATICS, IIT - GUWAHATI
Odd Semester of the Academic Year 2021-2022
MA 101 Mathematics I
Problem Sheet 5: Line integrals, Multiple integrals and applications, Green's Theorem, Stokes Theorem and Divergence Theorem.
Instructors: Dr. J. C. Kalita and Dr. S. Upadhyay

---

1. Calculate the line integral of the vector field along the path described:
   (a) $f(x,y) = (x^2+y^2)\mathbf{i} + (x^2-y^2)\mathbf{j}$ from $(0,0)$ to $(2,0)$ along the curve $y = 1-|1-x|$
   (b) $f(x,y,z) = 2xy\mathbf{i} + (x^2+z)\mathbf{j} + (y+z)\mathbf{k}$ from $(1,0,2)$ to $(3,4,1)$ along a line segment
   (c) $f(x,y,z) = x\mathbf{i} + y\mathbf{j} + (xz-y)\mathbf{k}$, along the path described by $\alpha(t) = t^2\mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k}$, $0 \le t \le 1$.

2. Find the line integral of $f(x,y,z) = z$ with respect to arc length of the curve given by $\mathbf{r}(t) = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} + (t)\mathbf{k}$, $0 \le t \le 1$.

3. For each of the following vector fields show that f is not a gradient vector in $\mathbf{R}^2$. Then for each of the following find a closed path C such that $\oint_C f \neq 0$ and if possible find a closed path C such that $\oint_C f = 0$.
   (a) $f(x,y) = y\mathbf{i} - x\mathbf{j}$
   (b) $f(x,y) = \frac{y}{(x^2+y^2)}\mathbf{i} - \frac{x}{(x^2+y^2)}\mathbf{j}$, for $(x,y) \neq (0,0)$.

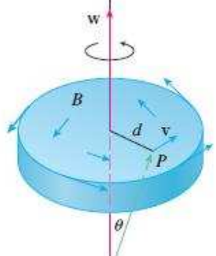
4. Show that each of the following functions F is a gradient vector and find an f for each F such that $F = \nabla f$.
   (a) $F(x,y) = 3x^2y\mathbf{i} + x^3\mathbf{j}$
   (b) $F(x,y) = (\sin y - y\sin x + x)\mathbf{i} + (\cos x + x\cos y + y)\mathbf{j}$.

5. Use Green's theorem to evaluate the line integral along the given positively oriented curve:
   (a) $\displaystyle\int_C (y + e^{\sqrt{x}})dx + (2x + \cos y^2)dy$, C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$
   (b) $\displaystyle\int_C xydx + 2x^2dy$, C consists of the line segment from $(-2,0)$ to $(2,0)$ and top half of the circle $x^2+y^2 = 4$.

6. Use Green's theorem to find out the work done by the force $\mathbf{F}(x,y) = x(x+y)\mathbf{i} + xy^2\mathbf{j}$ in moving a particle from the origin along x-axis to $(1,0)$ and then along the line segment to $(0,1)$, and then back to the origin along y-axis.

7. Let D be a region bounded by a simple closed path C in the xy-plane. Use Green's theorem to prove that the coordinates of the centroid $(\bar{x},\bar{y})$ of D are
$$\bar{x} = \frac{1}{2A}\oint x^2dy, \quad \bar{y} = -\frac{1}{2A}\oint y^2dx \quad \text{where } A \text{ is the area of } D.$$

8. If $\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j}$ and $r = |\mathbf{r}|$, let
$$f(x,y) = \frac{\partial(logr)}{\partial y}\mathbf{i} - \frac{\partial(logr)}{\partial x}\mathbf{j}$$
for $r > 0$.
Let C be a smooth simple closed curve in the annulus $1 < x^2+y^2 < 25$, then find all possible values of the line integral of f along C.



9. The exercise demonstrates a connection between curl vector and rotations. Let $\mathbf{B}$ be a rigid body rotating about z-axis. The rotation can be described by the vector $\mathbf{w} = \omega\mathbf{k}$, where ω is the angular speed of $\mathbf{B}$, that is, the tangential speed at any point P in B divided by the distance d from the axis of rotation. Let $\mathbf{r} =< x, y, z >$ be the position vector of P.

   (a) By considering the angle θ in the figure, show that the velocity field of B is given by $\mathbf{v} = \mathbf{w} \times \mathbf{r}$

   (b) Show that $\mathbf{v} = -\omega y\mathbf{i} + \omega x\mathbf{j}$

   (c) Show that $\nabla \times \mathbf{v} = 2\mathbf{w}$.

10. Use Stokes' Theorem to evaluate

   (a) $\displaystyle\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$ where $\mathbf{F} = xyz\mathbf{i} + xy\mathbf{j} + x^2yz\mathbf{k}$ and S consists of the top and the four sides (but not the bottom) of the cube with vertices $(\pm1, \pm1, \pm1)$, oriented outward

   (b) $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$ where $\mathbf{F} = 2z\mathbf{i} + 4x\mathbf{j} + 5y\mathbf{k}$ and C is the curve of intersection of the plane $z = x + 4$ and the cylinder $x^2 + y^2 = 4$.

11. Calculate the work done by the force field

$$\mathbf{F}(x, y, z) = (x^x + z^2)\mathbf{i} + (y^y + x^2)\mathbf{j} + (z^z + y^2)\mathbf{k}$$

   when a particle moves under its influence around the edge of the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies in the first octant, in a counterclockwise direction as viewed from above.

12. Let S be the surface of the solid cylinder T bounded by the planes $z = 0$ and $z = 3$ and the cylinder $x^2+y^2 = 4$. Calculate the outward flux $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{n}dS$ given $\mathbf{F}(x, y, z) = (x^2+y^2+z^2)(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$.

13. Use Divergence Theorem to evaluate $\displaystyle\iint_S \mathbf{F} \cdot \mathbf{n}dS$ where

$$\mathbf{F}(x, y, z) = z^2x\mathbf{i} + \left(\frac{1}{3}y^3 + \tan z\right)\mathbf{j} + (x^2z + y^2)\mathbf{k}$$

   and S is the top half of the sphere $x^2 + y^2 + z^2 = 1$.