April 21, 2017
Lecture – 40

Non Dimensionalisation of Basic Equation

- You have seen the Buckingham Π method to non-dimensionalise the relations or form non-dimensional relation for various experimental data.
- You also developed or came up with non-dimensional numbers or parameters.
- In today's class, we will see the non-dimensionalisation of the basic equations and see whether we can get some non-dimensional numbers.

Recall for an incompressible fluid (mainly liquid), the continuity equation was:

$\vec{\nabla}.\vec{v} = 0$ (vector form)

$\frac{\partial v_i}{\partial x_i} = 0$ (index form)

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ (expanded form)

Also for the incompressible liquid, the momentum equation was:

$$\rho\left[\frac{\partial \vec{v}}{\partial t} + u\frac{\partial \vec{v}}{\partial x} + v\frac{\partial \vec{v}}{\partial y} + w\frac{\partial \vec{v}}{\partial z}\right] = \rho\vec{g} - \vec{\nabla}p + \mu\nabla^2\vec{v} \qquad \text{(Vector Form)}$$

$$\rho\left[\frac{\partial v_i}{\partial t} + v_j\frac{\partial(\rho v_i)}{\partial x_j}\right] = \rho g_i - \frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 v_i}{\partial x_j \partial x_j}$$

The typical boundary conditions used to solve fluid flow are:

- Fixed solid surface, i.e. $\vec{v} = 0$
- Inlet or outlet flow, known. i.e. given $\vec{v}$, p, etc.
- Free surface, such that $p_{liquid} = p_{atm}$ or water surface.

For incompressible liquids, the variables and parameters are now:

ρ = constant (A parameter)

p, $\vec{v}$, x, y, z, t are the variables present in the equation.

- The basic dimensions are M, L, T
- We need to non-dimensionalise the variables.
- For that, we need to fix some reference values.

Let the reference velocity be v and reference length L (Reference value means they are not variables and they are known values apriori).

The reference values can be eg: Inlet or Upstream velocity U, it can be diameter of immersed body, or diameter of pipe, or length of aerofoil, etc.

Define Non-dimensional variables:

$$\vec{v}* = \frac{\vec{v}}{U}$$

Please note that we will be using * as superfix on each variable to show that it is a non-dimensional variable.

$$u* = \frac{u}{U}, \quad v* = \frac{v}{U}, \quad w* = \frac{w}{U}$$

$$x* = \frac{x}{L}, \quad y* = \frac{y}{L}, \quad z* = \frac{z}{L}$$

$$\nabla * = \hat{i}\frac{\partial}{\partial x *} + \hat{j}\frac{\partial}{\partial y *} + \hat{k}\frac{\partial}{\partial z *}$$

or

$$\nabla * = L\hat{i}\frac{\partial}{\partial x} + L\hat{j}\frac{\partial}{\partial y} + L\hat{k}\frac{\partial}{\partial z}$$

$$= L\left[\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right]$$

or

$$\nabla * = L\nabla$$

$$t* = t\frac{U}{L}$$

$$p* = \frac{p + \rho g z}{\rho u^2}$$

(**Note :** $\nabla *$ is an operator)
$p*$ is computed with respect to piezo metric head ($p + \rho g z$) in the numerator.

Note : ρ , U, L are constant.

Therefore,

$$\frac{\partial u}{\partial x} = \frac{\partial (Uu*)}{\partial (Lx*)} = \frac{U}{L}\frac{\partial u*}{\partial x*}$$

Similarly,

$$\frac{\partial v}{\partial y}=\frac{\partial\left(Uv*\right)}{\partial(Ly*)}=\frac{U}{L}\frac{\partial v*}{\partial y*}$$

$$\frac{\partial w}{\partial z}=\frac{\partial\left(Uw*\right)}{\partial(Lz*)}=\frac{U}{L}\frac{\partial w*}{\partial z*}$$

The continuity equation can be expressed in non-dimensional form as:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

$$\frac{U}{L}\left[\frac{\partial u*}{\partial x*}+\frac{\partial v*}{\partial y*}+\frac{\partial w*}{\partial z*}\right]=0$$

$$\vec{\nabla}*.\vec{v}*=0 \qquad (1)$$

The **momentum equation** can be non dimensionalised:

Note : $\frac{\partial\vec{v}}{\partial t}=\frac{\partial\left(U\vec{v}*\right)}{\partial\left(\frac{Lt*}{U}\right)}=\frac{U^2}{L}\frac{\partial\vec{v}*}{\partial t*}$

The left side of the Naiver – Stokes momentum equation

$$\rho\left[\frac{U^2}{L}\left(\frac{\partial\vec{v}*}{\partial t*}\right)+\left(Uu*\right)\frac{U}{L}\frac{\partial u*}{\partial x*}+\left(Uv*\right)\frac{U}{L}\frac{\partial v*}{\partial y*}+\left(Uw*\right)\frac{U}{L}\frac{\partial w*}{\partial z*}\right]$$

$$=\rho\frac{U^2}{L}\left[\frac{\partial\vec{v}*}{\partial t*}+u*\frac{\partial\vec{v}*}{\partial x*}+v*\frac{\partial\vec{v}*}{\partial y*}+w*\frac{\partial\vec{v}*}{\partial z*}\right]$$

$$=\rho\frac{U^2}{L}\left[\frac{\partial\vec{v}*}{\partial t*}+(\vec{v}*.\nabla*)\vec{v}*\right]$$

$$=\rho\frac{U^2}{L}\left[\frac{d\vec{v}*}{dt*}\right]$$

The non dimensionalisation of RHS of Naiver – Stokes momentum equation

(Recall $p*=\frac{p+\rho gz}{\rho u^2}$)

You can derive yourself and check.

You will get

$$\rho\vec{g}-\vec{\nabla}p+\mu\nabla^2\vec{v}=-\rho\frac{U^2}{L}\nabla*p*+\frac{\mu}{\rho UL}\left(\rho\frac{U^2}{L}\right)\nabla*^2(\vec{v}*)$$

$$\frac{d\vec{v}*}{dt*} = -\nabla * p * + \frac{\mu}{\rho UL} \nabla *^2 (\vec{v}*) \tag{2}$$

Equation (2) is non-dimensional form of linear momentum equation.

You can see from equation (2), there are non-dimensional number like Reynolds number present ij the expression. Therefore,

$$\frac{d\vec{v}*}{dt*} = -\nabla * p * + \frac{1}{R_e} \nabla *^2 (\vec{v}*)$$

For boundary conditions, if $\vec{v} = 0$

Then non-dimensional form, $\vec{v}* = 0$ (For fixed solid surface)

For inlet and outlet known values, you can provide $p*, \vec{v}*$.

For free surface,

$z* = \eta*$ or $w* = \frac{d\eta*}{dt}$

$$p* = \frac{p_{atm}}{\rho u^2} + \frac{gL}{U^2} z*$$

The free surface pressure condition has non-dimensional numbers (or parameter):

Euler Number , Eu = $\frac{p_{atm}}{\rho u^2}$

Froude Number, $Fr^2 = \frac{U^2}{gL}$

Similarly, in fluid mechanics, you will come up with various such types of non-dimensional parameters:

I. Reynolds Number, $R_e = \frac{\rho UL}{\mu}$

II. Mach Number, $Ma = \frac{U}{a}$

III. Froude Number, $Fr = \frac{U}{\sqrt{gL}}$

IV. Weber Number, $We = \frac{\rho U^2 L}{\gamma}$

V. Cavitation Number, $C_a = \frac{p - p_v}{\rho U^2}$