Kinematics of Rigid Bodies :: Relative Acceleration

Relative velocities of two points *A* and *B* in plane motion in terms of nonrotating reference axes:

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$$

Differentiating wrt time:

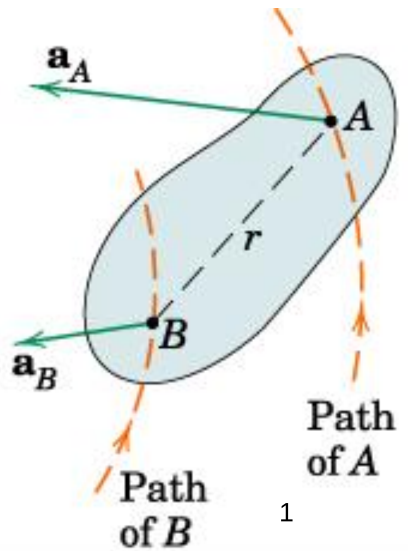
$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B}$$

→**Acceleration of point *A*** is equal to **vector sum** of **acceleration of point *B*** and the **acceleration of *A* appearing to a nonrotating observer moving with *B***

Relative Acceleration due to Rotation

:: Observer moving with *B* perceives *A* to have circular motion about *B*

- **Relative acceleration** term will have both **normal** and **tangential** components
- **Normal component of accln** will be directed from *A* towards B **due to change in direction of $\mathbf{v}_{A/B}$**.
- **Tangential component of accln** will be perpendicular to *AB* **due to the change in the magnitude of $\mathbf{v}_{A/B}$**

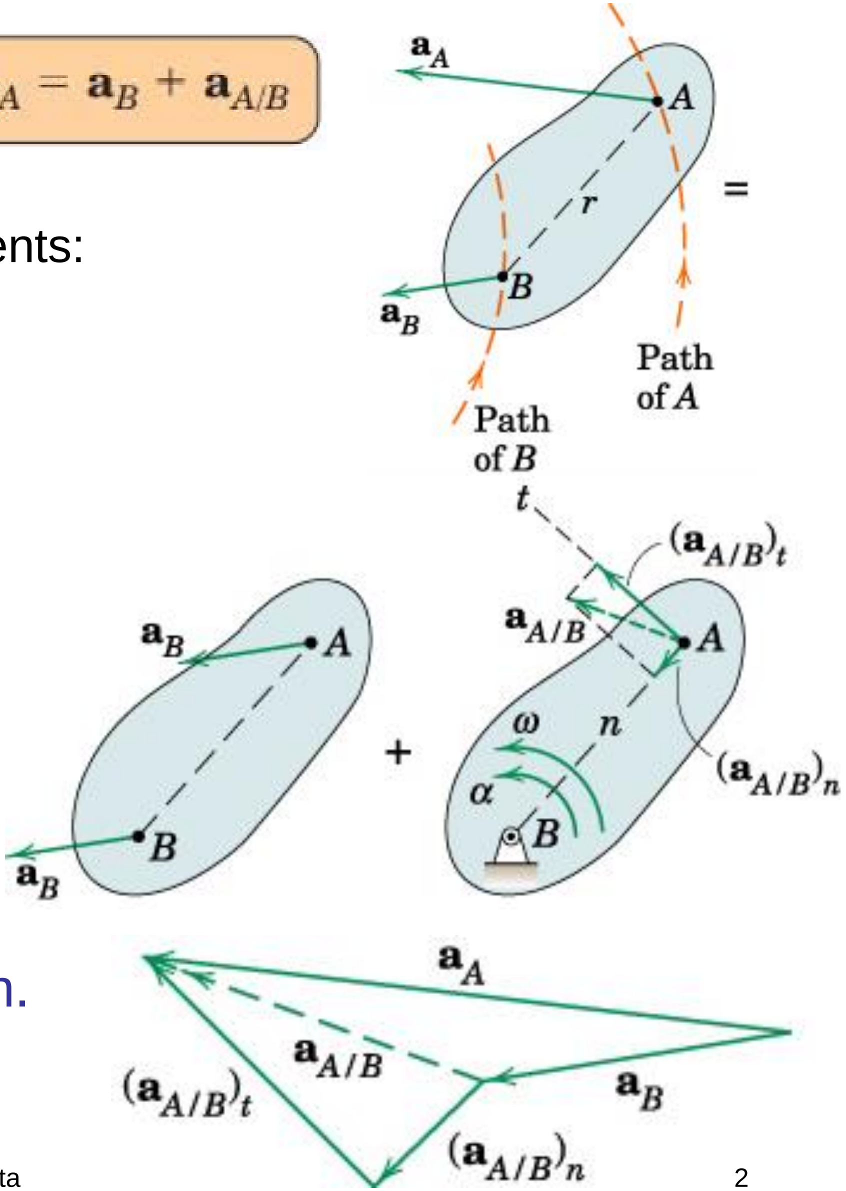


$\mathbf{a}_A$ and $\mathbf{a}_B$ are the absolute accelerations of *A* and *B*.
→ Not tangent to the path of motion when the motion is curvilinear.

Kinematics of Rigid Bodies :: Relative Acceleration

$$\mathbf{a}_A = \mathbf{a}_B + (\mathbf{a}_{A/B})_n + (\mathbf{a}_{A/B})_t$$

$$\mathbf{a}_A = \mathbf{a}_B + \mathbf{a}_{A/B}$$



The magnitudes of the relative accln components:

$$(a_{A/B})_n = {v_{A/B}}^2/r = r\omega^2$$
$$(a_{A/B})_t = \dot{v}_{A/B} = r\alpha$$

Acceleration components in vector notations:

$$(\mathbf{a}_{A/B})_n = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$
$$(\mathbf{a}_{A/B})_t = \boldsymbol{\alpha} \times \mathbf{r}$$

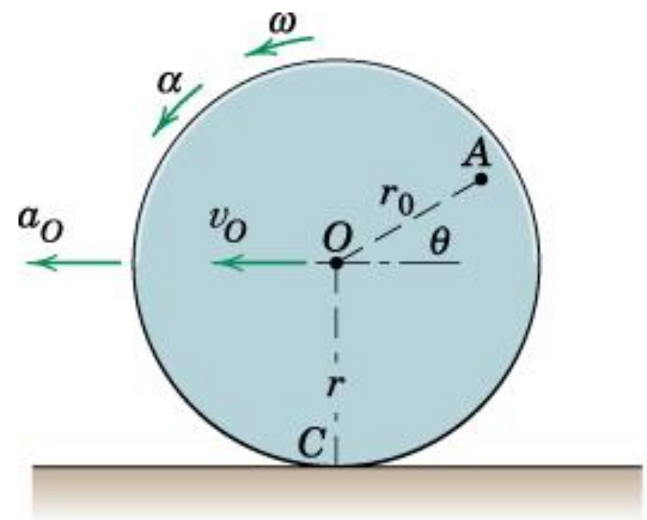
r is the vector locating *A* from *B*

→Relative accln terms depend on the absolute angular vel and angular accln.

Alternatively: $\mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$

Example on Relative Acceleration

The wheel of radius r rolls to the left without slipping and, at the instant considered, the center O has a velocity $\mathbf{v}_O$ and an acceleration $\mathbf{a}_O$ to the left. Determine the acceleration of points A and C on the wheel for the instant considered.



Angular velocity and angular accln of wheel:

$$\omega = v_O/r \qquad \text{and} \qquad \alpha = a_O/r$$

Accln of *A* in terms of given accln of *O*:

$$\mathbf{a}_A = \mathbf{a}_O + \mathbf{a}_{A/O} = \mathbf{a}_O + (\mathbf{a}_{A/O})_n + (\mathbf{a}_{A/O})_t$$

The relative accln terms are viewed as though *O* were fixed. For circular motion of *A* @ *O*, magnitudes of the relative accln terms:

$$(a_{A/O})_n = r_0\omega^2 = r_0\left(\frac{v_O}{r}\right)^2$$

$$(a_{A/O})_t = r_0\alpha = r_0\left(\frac{a_O}{r}\right)$$

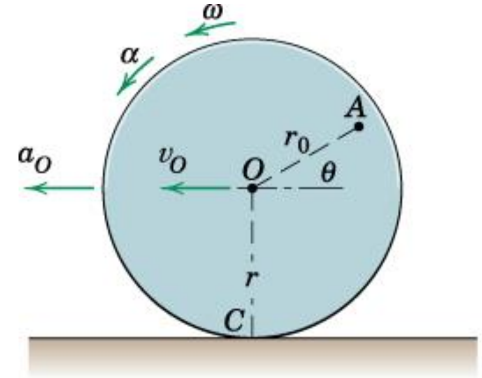
Example on Relative Acceleration

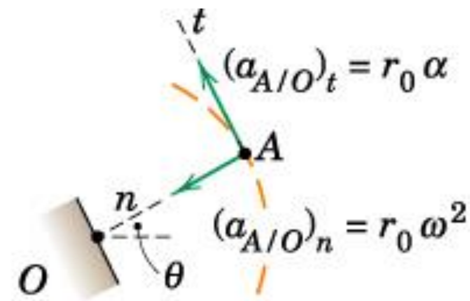
$$\omega = v_O/r \qquad \text{and} \qquad \alpha = a_O/r$$

$$\mathbf{a}_A = \mathbf{a}_O + \mathbf{a}_{A/O} = \mathbf{a}_O + (\mathbf{a}_{A/O})_n + (\mathbf{a}_{A/O})_t$$

$$(a_{A/O})_n = r_0\omega^2 = r_0\left(\frac{v_O}{r}\right)^2$$

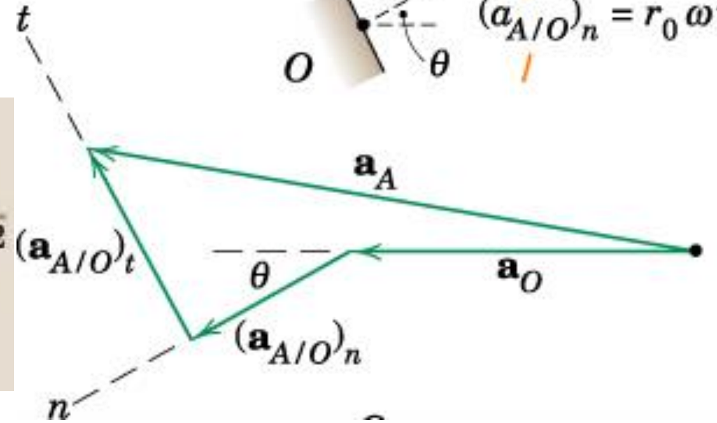
$$(a_{A/O})_t = r_0\alpha = r_0\left(\frac{a_O}{r}\right)$$





Adding the vectors head to tail will give $\mathbf{a}_A$
Magnitude of $\mathbf{a}_A$ is given by:

$$\begin{aligned} a_A &= \sqrt{(a_A)_n{}^2 + (a_A)_t{}^2} \\ &= \sqrt{[a_O \cos\theta + (a_{A/O})_n]^2 + [a_O \sin\theta + (a_{A/O})_t]^2} \\ &= \sqrt{(r\alpha \cos\theta + r_0\omega^2)^2 + (r\alpha \sin\theta + r_0\alpha)^2} \end{aligned}$$



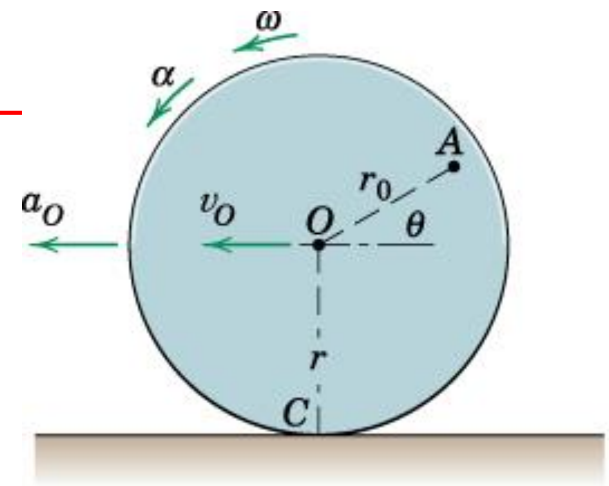
Direction of $\mathbf{a}_A$ can also be computed.

Example on Relative Acceleration

Acceleration for Point C

Point *C* is the instantaneous center of zero velocity

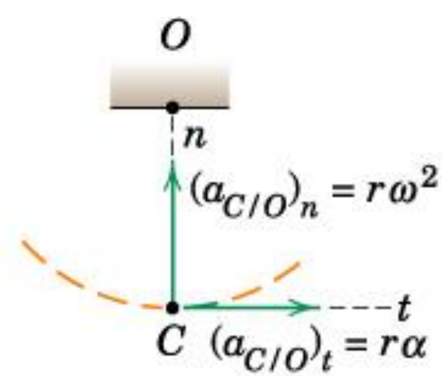
$$\mathbf{a}_C = \mathbf{a}_O + \mathbf{a}_{C/O}$$



The components of the relative acceleration are:

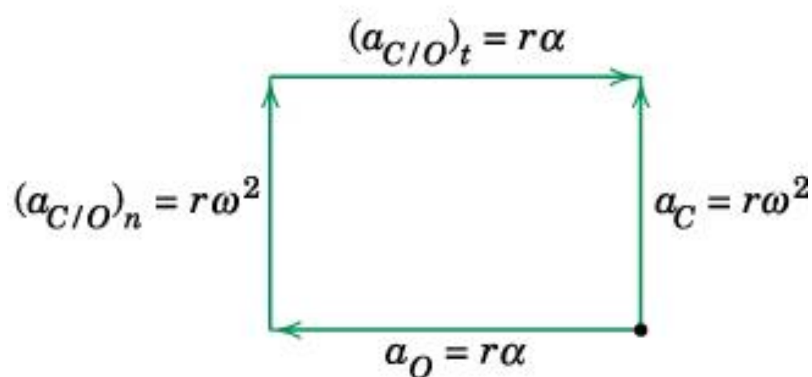
$(a_{C/O})_n = r\omega^2$ directed from *C* to *O*

$(a_{C/O})_t = r\alpha$ directed towards right
due to **anticlockwise angular accln** of *CO @ O*



→ $a_C = r\,\omega^2$

→**Accln** of the **instantaneous center of zero velocity** is **independent of *α*** and is **directed towards the center of the wheel**



Example on Relative Velocity and Acceleration

Crank *CB* oscillates about *C* through a **limited arc**, causing **crank *OA* to oscillate about *O***. When the linkage passes the position shown with ***CB* horizontal and *OA* vertical**, the **angular velocity of *CB* is 2 rad/s counter-clockwise**. For this instance, determine the **angular velocities** and **angular accelerations of *OA* and *AB***.

Solution:

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

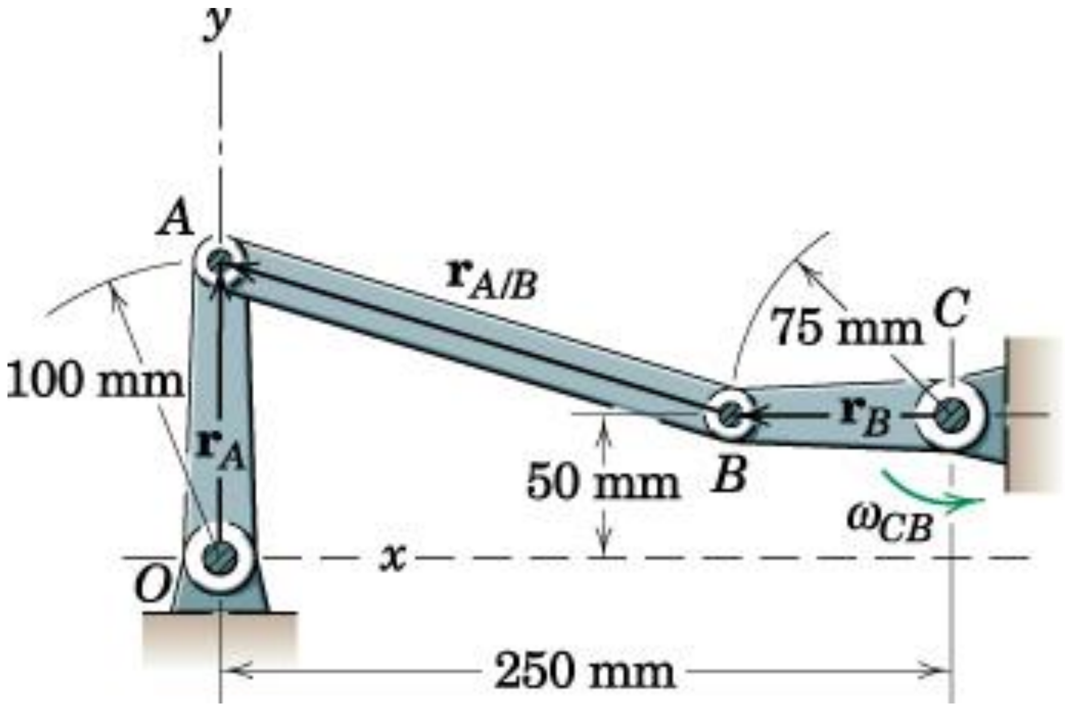
$$\mathbf{a}_n = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

$$\mathbf{a}_t = \boldsymbol{\alpha} \times \mathbf{r}$$

$$\mathbf{v}_{A/B} = \boldsymbol{\omega} \times \mathbf{r}$$

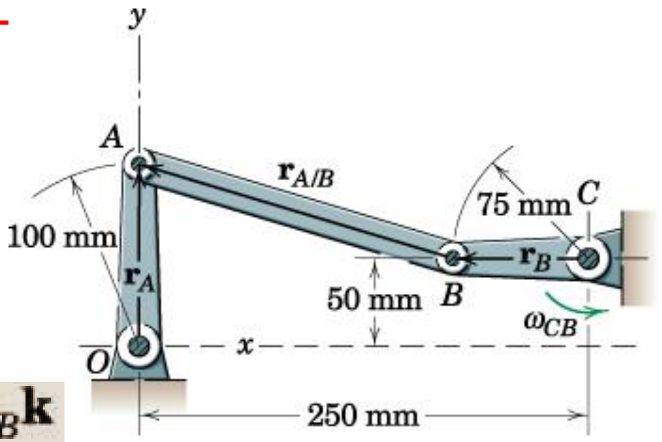
$$(\mathbf{a}_{A/B})_n = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

$$(\mathbf{a}_{A/B})_t = \boldsymbol{\alpha} \times \mathbf{r}$$



Example on Relative Velocity and Acceleration

$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$ $\mathbf{v}_{A/B} = \boldsymbol{\omega} \times \mathbf{r}$



Writing the relative velocity of *A* wrt *B*:

$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$

$$\boldsymbol{\omega}_{OA} \times \mathbf{r}_A = \boldsymbol{\omega}_{CB} \times \mathbf{r}_B + \boldsymbol{\omega}_{AB} \times \mathbf{r}_{A/B}$$

$$\boldsymbol{\omega}_{OA} = \omega_{OA}\mathbf{k} \qquad \boldsymbol{\omega}_{CB} = 2\mathbf{k} \text{ rad/s} \qquad \boldsymbol{\omega}_{AB} = \omega_{AB}\mathbf{k}$$

Substituting:

$$\mathbf{r}_A = 100\mathbf{j} \text{ mm} \qquad \mathbf{r}_B = -75\mathbf{i} \text{ mm} \qquad \mathbf{r}_{A/B} = -175\mathbf{i} + 50\mathbf{j} \text{ mm}$$

$$\omega_{OA}\mathbf{k} \times 100\mathbf{j} = 2\mathbf{k} \times (-75\mathbf{i}) + \omega_{AB}\mathbf{k} \times (-175\mathbf{i} + 50\mathbf{j})$$

$$-100\omega_{OA}\mathbf{i} = -150\mathbf{j} - 175\omega_{AB}\mathbf{j} - 50\omega_{AB}\mathbf{i}$$

Matching coefficients of respective **i**- and **j**-terms

$-100\omega_{OA} + 50\omega_{AB} = 0$ and $25(6 + 7\omega_{AB}) = 0$

→ $\omega_{AB} = -6/7 \text{ rad/s}$ $\omega_{OA} = -3/7 \text{ rad/s}$

Both angular velocities are acting clockwise (in the –ve **k** direction since counter-clockwise direction was taken positive (+ve **k**) for angular velocities).

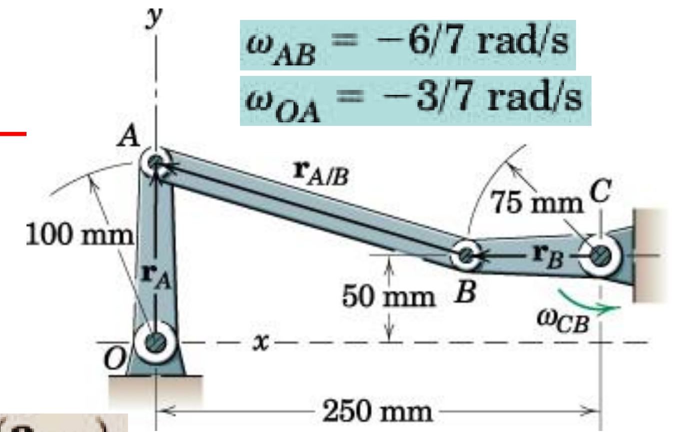
Example on Rel Vel and Accln

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$
$$\mathbf{a}_n = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$
$$\mathbf{a}_t = \boldsymbol{\alpha} \times \mathbf{r}$$

$$(\mathbf{a}_{A/B})_n = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$
$$(\mathbf{a}_{A/B})_t = \boldsymbol{\alpha} \times \mathbf{r}$$

$\omega_{AB} = -6/7$ rad/s

$\omega_{OA} = -3/7$ rad/s



Writing relative accln of *A* wrt *B*: $\mathbf{a}_A = \mathbf{a}_B + (\mathbf{a}_{A/B})_n + (\mathbf{a}_{A/B})_t$

Writing absolute accln of A and B in their *n-t* comp:

$$\mathbf{a}_A = \boldsymbol{\alpha}_{OA} \times \mathbf{r}_A + \boldsymbol{\omega}_{OA} \times (\boldsymbol{\omega}_{OA} \times \mathbf{r}_A)$$
$$= \alpha_{OA}\mathbf{k} \times 100\mathbf{j} + (-\tfrac{3}{7}\mathbf{k}) \times (-\tfrac{3}{7}\mathbf{k} \times 100\mathbf{j})$$
$$= -100\alpha_{OA}\mathbf{i} - 100(\tfrac{3}{7})^2\mathbf{j} \text{ mm/s}^2$$

$$\mathbf{a}_B = \boldsymbol{\alpha}_{CB} \times \mathbf{r}_B + \boldsymbol{\omega}_{CB} \times (\boldsymbol{\omega}_{CB} \times \mathbf{r}_B)$$
$$= \mathbf{0} + 2\mathbf{k} \times (2\mathbf{k} \times [-75\mathbf{i}])$$
$$= 300\mathbf{i} \text{ mm/s}^2$$

$\boldsymbol{\alpha}_{CB} = 0$, since $\boldsymbol{\omega}_{CB}$ is constant

The relative acclns:

$$(\mathbf{a}_{A/B})_n = \boldsymbol{\omega}_{AB} \times (\boldsymbol{\omega}_{AB} \times \mathbf{r}_{A/B})$$
$$= -\tfrac{6}{7}\mathbf{k} \times [(-\tfrac{6}{7}\mathbf{k}) \times (-175\mathbf{i} + 50\mathbf{j})]$$
$$= (\tfrac{6}{7})^2(175\mathbf{i} - 50\mathbf{j}) \text{ mm/s}^2$$

$$(\mathbf{a}_{A/B})_t = \boldsymbol{\alpha}_{AB} \times \mathbf{r}_{A/B}$$
$$= \alpha_{AB}\mathbf{k} \times (-175\mathbf{i} + 50\mathbf{j})$$
$$= -50\alpha_{AB}\mathbf{i} - 175\alpha_{AB}\mathbf{j} \text{ mm/s}^2$$

$\mathbf{r}_{A/B}$: vector from *B* to *A*

Substituting and equating the coefficients:

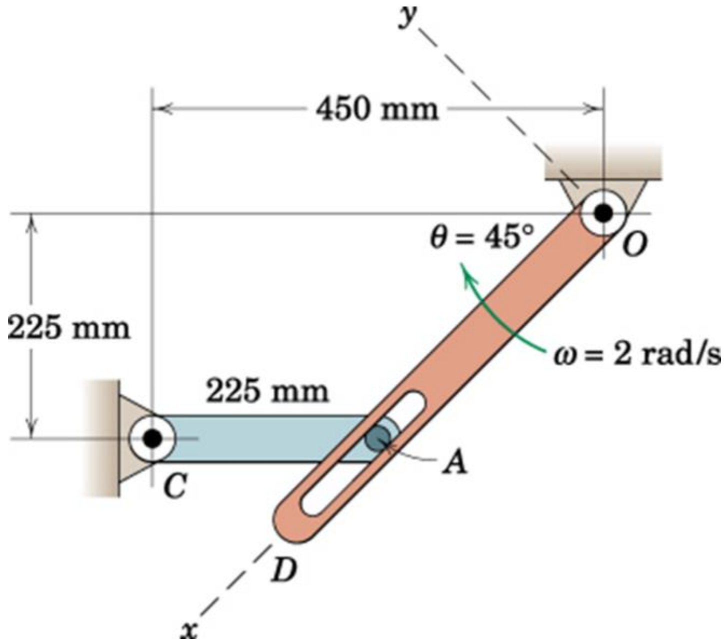
$$-100\alpha_{OA} = 429 - 50\alpha_{AB}$$
$$-18.37 = -36.7 - 175\alpha_{AB}$$

$\alpha_{AB} = -0.1050 \text{ rad/s}^2$

$\alpha_{OA} = -4.34 \text{ rad/s}^2$

Plane Kinematics of Rigid Bodies

Motion Relative to Rotating Axes



- Rigid body mechanisms constructed such that sliding occur at their connections
- Analyzing motion of two points on a mechanism that are not located on the same rigid body

Plane Kinematics of Rigid Bodies

Motion Relative to Rotating Axes

Consider plane motion of two particles *A* and *B* (moving independently of each other) in fixed *X*-*Y* plane.

•Observing motion of point *A* from a moving reference frame *x*-*y* (origin attached to *B*) that rotates with ω

$$\boldsymbol{\omega} = \omega\mathbf{k} = \dot{\theta}\mathbf{k}$$

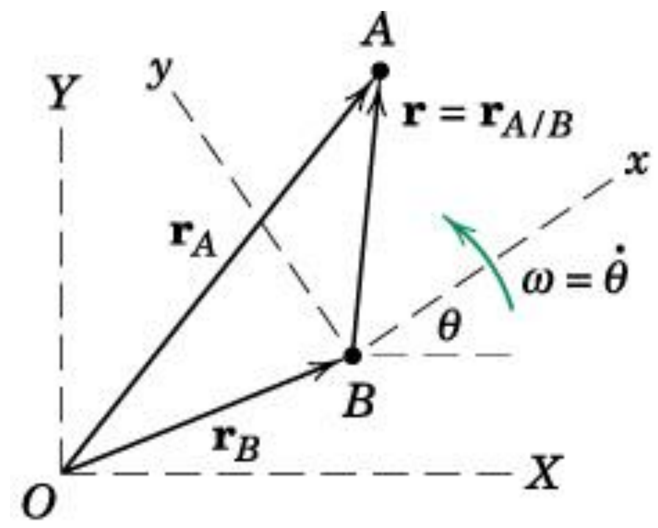
the vector is normal to the plane of the motion (@ +ve *z*-direction using right hand rule)

The absolute position vector of *A*:

$\mathbf{r}_A = \mathbf{r}_B + \mathbf{r} \quad = \mathbf{r}_B + (x\mathbf{i} + y\mathbf{j})$

i and **j** are the unit vectors attached to the *x*-*y* frame

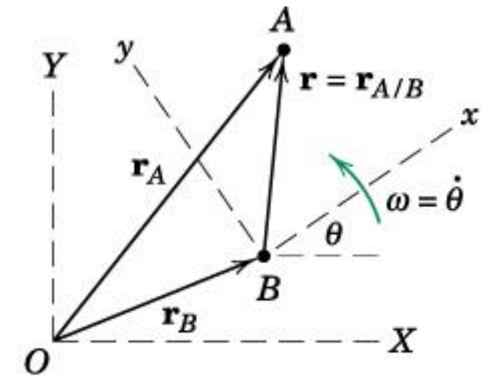
$\mathbf{r} = \mathbf{r}_{A/B} = x\mathbf{i} + y\mathbf{j}$ :: the position vector of *A* wrt *B*



Reference Frame rotating with some accln is known as non-inertial or non-Newtonian reference frame

Plane Kinematics of Rigid Bodies

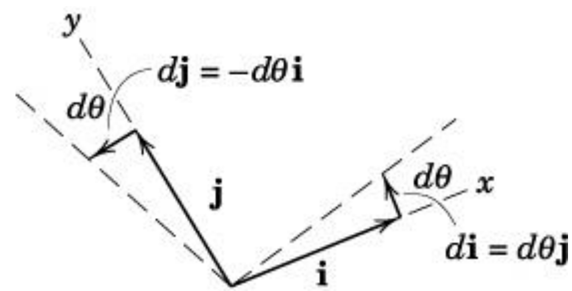
Motion Relative to Rotating Axes



Differentiating the posn vector eqn to obtain vel & accl eqn:

- The unit vectors are rotating with the x-y axes
  → time derivatives must be evaluated.

When *x-y* axes rotate during *dt* through an angle $d\theta = \omega dt$:

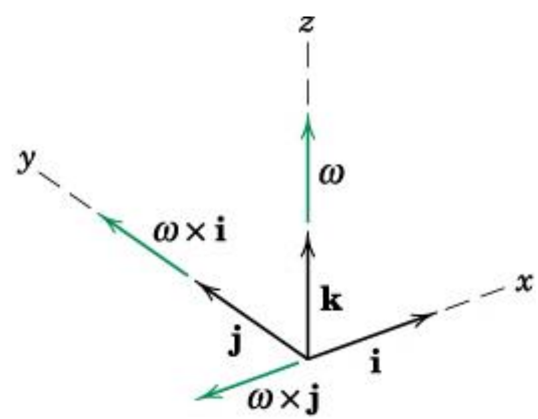


- Differential change in **i** → *d***i**
  - *d***i** has direction of **j**
  - magnitude of *d***i** = *dθ* x magnitude of **i** = *dθ*
  - Therefore, *d***i** = *dθ* **j**
- Differential change in **j** → *d***j**
  - *d***j** has negative *x*-direction
  - Therefore, *d***j** = - *dθ* **i**

Dividing by *dt* and replacing

$d\mathbf{i}/dt$ by $\dot{\mathbf{i}}$, $d\mathbf{j}/dt$ by $\dot{\mathbf{j}}$, and $d\theta/dt$ by $\dot{\theta} = \omega$

→ $\dot{\mathbf{i}} = \omega \mathbf{j}$ and $\dot{\mathbf{j}} = -\omega \mathbf{i}$

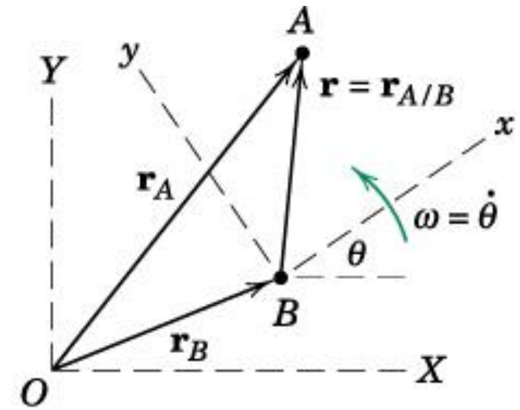


Using cross-product: ω x **i** = ω**j** and ω x **j** = - ω**i**

→ $\dot{\mathbf{i}} = \boldsymbol{\omega} \times \mathbf{i}$ and $\dot{\mathbf{j}} = \boldsymbol{\omega} \times \mathbf{j}$

Plane Kinematics of Rigid Bodies

Motion Relative to Rotating Axes



Relative Velocity Relations

Differentiating $\mathbf{r}_A = \mathbf{r}_B + (x\mathbf{i} + y\mathbf{j})$ wrt time:

$$\dot{\mathbf{r}}_A = \dot{\mathbf{r}}_B + \frac{d}{dt}(x\mathbf{i} + y\mathbf{j}) \quad = \dot{\mathbf{r}}_B + (x\dot{\mathbf{i}} + y\dot{\mathbf{j}}) + (\dot{x}\mathbf{i} + \dot{y}\mathbf{j})$$

The second term:

$$x\dot{\mathbf{i}} + y\dot{\mathbf{j}} = \boldsymbol{\omega} \times x\mathbf{i} + \boldsymbol{\omega} \times y\mathbf{j} = \boldsymbol{\omega} \times (x\mathbf{i} + y\mathbf{j}) = \boldsymbol{\omega} \times \mathbf{r}$$

Since the observer in x-y measures vel components $\dot{x}$ and $\dot{y}$

Third term: $\dot{x}\mathbf{i} + \dot{y}\mathbf{j} = \mathbf{v}_{\text{rel}}$ = vel relative to x-y frame

→ Relative Velocity Equation: $\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r} + \mathbf{v}_{\text{rel}}$

Comparing with the eqn for non-rotating reference axes: → $\mathbf{v}_{A/B} = \boldsymbol{\omega} \times \mathbf{r} + \mathbf{v}_{\text{rel}}$

→$\boldsymbol{\omega} \times \mathbf{r}$ = difference betn the relative velocities as measured from non-rotating and rotating axes.

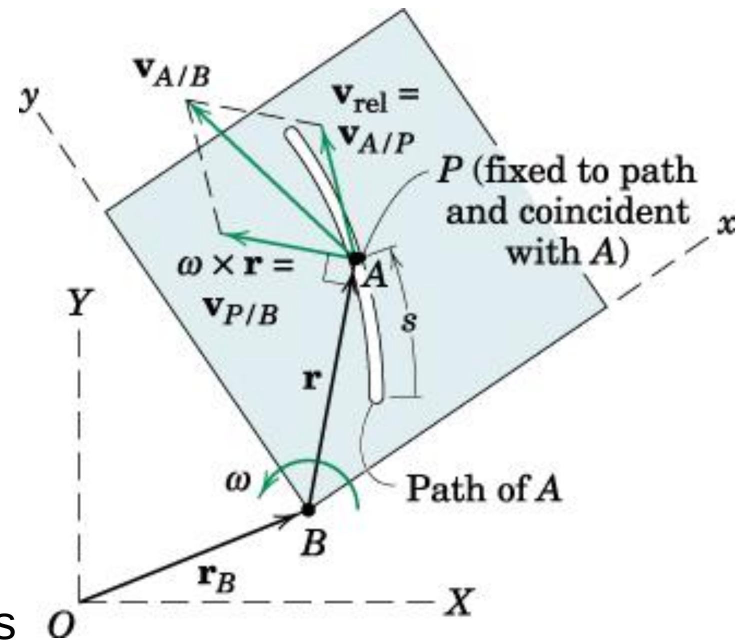
$\mathbf{v}_A$ = Absolute vel of A (motion of A observed from X-Y frame)
$\mathbf{v}_B$ = Absolute vel of origin of x-y frame (motion of x-y frame observed from X-Y frame)
$\boldsymbol{\omega} \times \mathbf{r}$ = Angular velocity effect caused by rotation of x-y frame (motion of x-y frame observed from X-Y frame)
$\mathbf{v}_{\text{rel}}$ = Relative velocity of A wrt B (motion of A observed from x-y frame)

Plane Kinematics of Rigid Bodies

Motion Relative to Rotating Axes

$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r} + \mathbf{v}_{rel}$$

Relative Velocity Relations



- The curved slot represents rotating *x-y* frame
- The x-y axes are not rotating themselves.
- Vel of *A* measured relative to the plate = $\mathbf{v}_{rel}$.
- $\mathbf{v}_{rel}$ will be tangent to the path fixed in *x-y* plate
- Magnitude of $\mathbf{v}_{rel}$ will be *ds/dt*
- $\mathbf{v}_{rel}$ may also be viewed as the vel $\mathbf{v}_{A/P}$ relative to a point *P* attached to the plate and coincident with *A* at the instant under consideration.
- $\boldsymbol{\omega}$ x **r** has dirn normal to **r**

  = $\mathbf{v}_{P/B}$ vel of *P* rel to origin *B* of non-rotating axes

Comparison betn relative vel eqns for rotating and non-rotating reference axes

$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r} + \mathbf{v}_{rel}$$

$$\mathbf{v}_A = \underbrace{\mathbf{v}_B + \mathbf{v}_{P/B}} + \mathbf{v}_{A/P}$$

$$\mathbf{v}_A = \mathbf{v}_P + \underbrace{\quad \mathbf{v}_{A/P}}$$

$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$$

- $\mathbf{v}_{P/B}$ is measured from a non-rotating posn
- $\mathbf{v}_P$ = absolute velocity of P and represent the effect of the moving coordinate system (both translational @ rotational)
- Last eqn is the same as that developed for non-rotating axes $\mathbf{v}_{A/B} = \mathbf{v}_{P/B} + \mathbf{v}_{A/P} = \boldsymbol{\omega} \times \mathbf{r} + \mathbf{v}_{rel}$

Plane Kinematics of Rigid Bodies

Motion Relative to Rotating Axes

$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r} + \mathbf{v}_{rel}$$

$$\mathbf{v}_A = \mathbf{v}_B + \boldsymbol{\omega} \times \mathbf{r} + \mathbf{v}_{rel}$$
$$\mathbf{v}_A = \underbrace{\mathbf{v}_B + \mathbf{v}_{P/B}}_{} + \mathbf{v}_{A/P}$$
$$\mathbf{v}_A = \mathbf{v}_P + \underbrace{\mathbf{v}_{A/P}}_{}$$
$$\mathbf{v}_A = \mathbf{v}_B + \mathbf{v}_{A/B}$$

Relative Velocity Relations

Transformation of a time derivative:

•These two eqns represent a transformation of the time derivative of the position vector between rotating and non-rotating axes.

•Generalized for any vector: $\mathbf{V} = V_x\mathbf{i} + V_y\mathbf{j}$

The total time derivative wrt *X*-*Y* system:

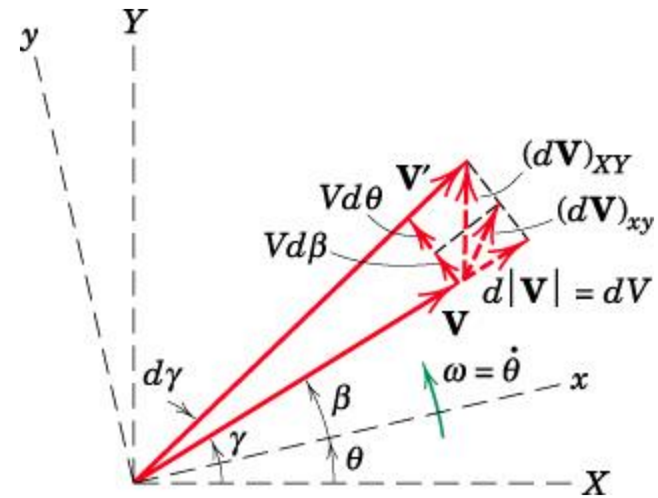
$$\left(\frac{d\mathbf{V}}{dt}\right)_{XY} = (\dot{V}_x\mathbf{i} + \dot{V}_y\mathbf{j}) + (V_x\dot{\mathbf{i}} + V_y\dot{\mathbf{j}})$$

First two terms represent that part of total derivative of **V** that is measured relative to the *x*-*y* frame.
Second two terms represent that part of derivative due to the rotation of the reference system.

Since $\dot{\mathbf{i}} = \omega\mathbf{j}$ $\quad \dot{\mathbf{j}} = -\omega\mathbf{i}$

$$\left(\frac{d\mathbf{V}}{dt}\right)_{XY} = \left(\frac{d\mathbf{V}}{dt}\right)_{xy} + \boldsymbol{\omega} \times \mathbf{V}$$

ωx**V** represents the diff betn time derivative of the vector measured in fixed and in rotating reference system



Physical Significance