EE540 Advance Electromagnetic Theory & Antennas

Prof. Rakhesh S. Kshetrimayum

Dept. of EEE, IIT Guwahati, India

भारतीय प्रौद्योगिकी संस्थान गुवाहाटी
INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI

Helmholtz Wave Equation

- *Spherical waves*
  - Like in cylindrical waves case
  - We will consider TE^r and TM^r spherical waves
- For TE^r spherical waves
  - $\vec{F} = F_r(r, \theta, \varphi)\hat{r}, \vec{A} = 0$
  - Note that $\mathrm{E_r} = 0$ for TE^r modes since $\vec{E} = -\frac{1}{\varepsilon}(\nabla \times \vec{F})$
- Solving the source free region, for this case,
  - $\nabla \times \nabla \times \vec{F} - \omega^2 \mu\varepsilon \vec{F} = -j\omega\mu\varepsilon\nabla\Phi_m$
- We get,
  - $(\nabla^2 + \beta^2)\frac{F_r}{r} = 0$
- Hence $\frac{\mathrm{F_r}}{\mathrm{r}}$ (not $\mathrm{F_r}$) satisfies Helmholtz wave equation

TE^r spherical waves

$$\vec{F} = F_r(r, \theta, \varphi)\hat{r}, \vec{A} = 0$$

For source free region, $\frac{\mathrm{F_r}}{\mathrm{r}}$ satisfies Helmholtz wave equation and find the fields

$$(\nabla^2 + \beta^2)\frac{F_r}{r} = 0$$

Helmholtz Wave Equation

भारतीय प्रौद्योगिकी संस्थान गुवाहाटी
INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI

- Solve $\frac{\mathrm{F_r}}{\mathrm{r}}$ by solving Helmholtz wave equation and find
  - $\vec{E}$
    - $\vec{E} = \vec{E}_F + \vec{E}_A = -\frac{1}{\varepsilon}(\nabla \times \vec{F}) + \frac{\nabla\times\nabla\times\vec{A}}{j\omega\mu\varepsilon}$
  - $\vec{H}$
    - $\vec{H} = \vec{H}_F + \vec{H}_A = \frac{\nabla\times\nabla\times\vec{F}}{j\omega\mu\varepsilon} + \frac{1}{\mu}(\nabla \times \vec{A})$
  - Let us solve Helmholtz wave equation for
    - Scalar function Ψ which could be equated to
    - either $\frac{\mathrm{F_r}}{\mathrm{r}}$ or $\frac{\mathrm{A_r}}{\mathrm{r}}$
    - for TE^r and TM^r modes respectively

For source free region, solve Helmholtz wave equation for Ψ which could be equated to either $\frac{\mathrm{F_r}}{\mathrm{r}}$ or $\frac{\mathrm{A_r}}{\mathrm{r}}$ for TE^r and TM^r modes respectively

↓

Find electric field ($\vec{E}_A$) and magnetic field ($\vec{H}_A$) as

$$\vec{E} = -\frac{1}{\varepsilon}(\nabla \times \vec{F}) + \frac{\nabla \times \nabla \times \vec{A}}{j\omega\mu\varepsilon}$$

$$\vec{H} = \frac{\nabla \times \nabla \times \vec{F}}{j\omega\mu\varepsilon} + \frac{1}{\mu}(\nabla \times \vec{A})$$

Helmholtz Wave Equation

- Wave equation:
  - $\nabla^2\Psi + \beta^2\Psi=0$
  - Laplacian in General curvilinear coordinates

$$\nabla^2\psi = \nabla \bullet \nabla\psi = \frac{1}{s_1 s_2 s_3}\left[\frac{\partial}{\partial a_1}\left(\frac{s_2 s_3}{s_1}\frac{\partial\psi}{\partial a_1}\right) + \frac{\partial}{\partial a_2}\left(\frac{s_1 s_3}{s_2}\frac{\partial\psi}{\partial a_2}\right) + \frac{\partial}{\partial a_3}\left(\frac{s_1 s_2}{s_3}\frac{\partial\psi}{\partial a_3}\right)\right]$$

  - For Laplacian in spherical coordinates:
  - Substitute
    - $s_1 = 1, s_2 = r, s_3 = r sin\theta$
    - $a_1 = r, a_2 = \theta, a_3 = \varphi$
  - Wave equation becomes
    - $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2\Psi}{\partial\varphi^2}+\beta^2\Psi=0$

Helmholtz Wave Equation

भारतीय प्रौद्योगिकी संस्थान गुवाहाटी
INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI

- Multiplying by $r^2 sin^2\theta$
  - $sin^2\theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial \Psi}{\partial r}\right) + sin\theta \frac{\partial}{\partial \theta}\left(sin\theta \frac{\partial \Psi}{\partial \theta}\right) + \frac{\partial^2 \Psi}{\partial \varphi^2} + \beta^2 r^2 sin^2\theta \Psi = 0$
  - Applying method of separation of variables
    - Substitute $\Psi = R(r)\Theta(\theta)\Phi(\varphi)$
    - $\Theta\Phi sin^2\theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + R\Phi sin\theta \frac{\partial}{\partial \theta}\left(sin\theta \frac{\partial \Theta}{\partial \theta}\right) + R\Theta \frac{\partial^2 \Phi}{\partial \varphi^2} + \beta^2 r^2 sin^2\theta (R\Theta\Phi) = 0$
  - Dividing by $\Psi = R(r)\Theta(\theta)\Phi(\varphi)$, we have
  - $\frac{1}{R} sin^2\theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + \frac{sin\theta}{\Theta} \frac{\partial}{\partial \theta}\left(sin\theta \frac{\partial \Theta}{\partial \theta}\right) + \frac{1}{\Phi} \frac{\partial^2 \Phi}{\partial \varphi^2} + \beta^2 r^2 sin^2\theta = 0$

Helmholtz Wave Equation

- Since the third term is dependent of φ
- but all other terms of independent of φ
  - $\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\varphi^2} = -m^2$
  - where $-m^2$ is an arbitrary constant
- Hence,
  - $\frac{1}{R}sin^2\theta\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{sin\theta}{\Theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\Theta}{\partial\theta}\right) - m^2 + \beta^2 r^2 sin^2\theta = 0$
- Divide by $sin^2\theta$, we have,
- $\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{\Theta sin}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\Theta}{\partial\theta}\right) - \frac{m^2}{sin^2\theta} + \beta^2 r^2 = 0$

Helmholtz Wave Equation

भारतीय प्रौद्योगिकी संस्थान गुवाहाटी
INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI

- First and 4th term dependent on r
- 2nd and 3rd term dependent on φ
- Can be converted into two equations as
  - $\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\beta^2 r^2=n(n+1)$
  - $\frac{1}{\Theta sin}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\Theta}{\partial\theta}\right)-\frac{m^2}{sin^2\theta}=-n(n+1)$
  - where $n(n+1)$ is an arbitrary constant
- Rewritten as
  - $\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\{\beta^2 r^2-n(n+1)\}R=0$
  - Spherical Bessel Equations

Helmholtz Wave Equation

भारतीय प्रौद्योगिकी संस्थान गुवाहाटी
INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI

- *First solution:*
  - $R(r) = a_1 j_n(\beta r) + b_1 y_n(\beta r)$
  - Note the small letter symbol of j_n and y_n
  - They are called Spherical Bessel functions of first and second kind
- Spherical Bessel functions represent
  - radial standing waves
  - Related to regular Bessel function as
  - $j_n(\beta r) = \sqrt{\frac{\pi}{2\beta r}} J_{n+\frac{1}{2}}(\beta r)$
  - $y_n(\beta r) = \sqrt{\frac{\pi}{2\beta r}} Y_{n+\frac{1}{2}}(\beta r)$

Helmholtz Wave Equation

- *Second solution:*
  - $R(r) = a_2 h_n^{(1)}(r) + b_2 h_n^{(2)}(r)$
  - They are called Spherical Hankel functions of first and second kind
  - radial propagating waves
  - Related to regular Hankel function as
  - $h_n^{(1)}(\beta r) = \sqrt{\frac{\pi}{2\beta r}} H_{n+\frac{1}{2}}^{(1)}(\beta r)$
  - $h_n^{(2)}(\beta r) = \sqrt{\frac{\pi}{2\beta r}} H_{n+\frac{1}{2}}^{(2)}(\beta r)$

Helmholtz Wave Equation

- Legendre's equations
  - $\frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\frac{\partial\Theta}{\partial\theta}\right)+\left\{n(n+1)-\frac{m^2}{sin^2\theta}\right\}\Theta=0$
    - whose solutions are associated Legendre functions
    - $\Theta=e_1P_n^m(cos\theta)+f_1Q_n^m(cos\theta)$
    - where P_n^m and Q_n^m are associated Legendre functions of first and second kind
    - n is degree and m is the order
    - e_1 and f_1 are any arbitrary constants
  - Note that
    - $P_n^m(cos\theta)$ is finite for $\theta=0,\pi$ only if *n* is an integer
    - $Q_n^m(cos\theta)$ tends to infinity for $\theta=0,\pi$

Helmholtz Wave Equation

भारतीय प्रौद्योगिकी संस्थान गुवाहाटी
INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI

- When *n* is an integer (*n=0,1,2,…*)
  - $P_n^m(cos\theta)$ also called as associated Legendre polynomial
    - can be obtained from Legendre polynomial $P_n(x)$ for $-\mathrm{n} \leq m \leq n$ as
    - $P_n^m(x) = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m}\left(P_n(x)\right) = \frac{1}{2^n n!}(1-x^2)^{\frac{m}{2}} \frac{d^{n+m}}{dx^{n+m}}\left((x^2-1)^n\right)$
    - Note that *m* is the order of derivative of $P_n(x)$ w.r.t. *x* (*m=0*, it reduce to $P_n(x)$)
  - Also Legendre polynomial can be obtained from Rodriguez formula as
    - $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left((x^2-1)^n\right)$
    - Note that *n* is the degree of derivative of $(x^2-1)^n$ w.r.t. *x*
    - and $P_n(x)$ is a polynomial of degree *n*
    - and it can be obtained from recurrence relation as well
    - $P_n(x)$ form an orthogonal set of polynomials on [−1, 1]

Helmholtz Wave Equation

- Typical solutions of Ψ
  - Fields inside a sphere
  - with finite fields for *r=0* and $\theta = 0, \pi$
  - $\Psi(r,\theta,\varphi) = C_{mn1} j_n(\beta r) P_n^m(cos\theta) \times \big(C_1 cos(m\varphi) + D_1 sin(m\varphi)\big)$
  - Fields outside a sphere
  - with finite fields for *r=0* and $\theta = 0, \pi$
  - $\Psi(r,\theta,\varphi) = C_{mn2} h_n^{(2)}(\beta r) P_n^m(cos\theta) \times \big(C_1 cos(m\varphi) + D_1 sin(m\varphi)\big)$