EE540 Advance Electromagnetic Theory & Antennas

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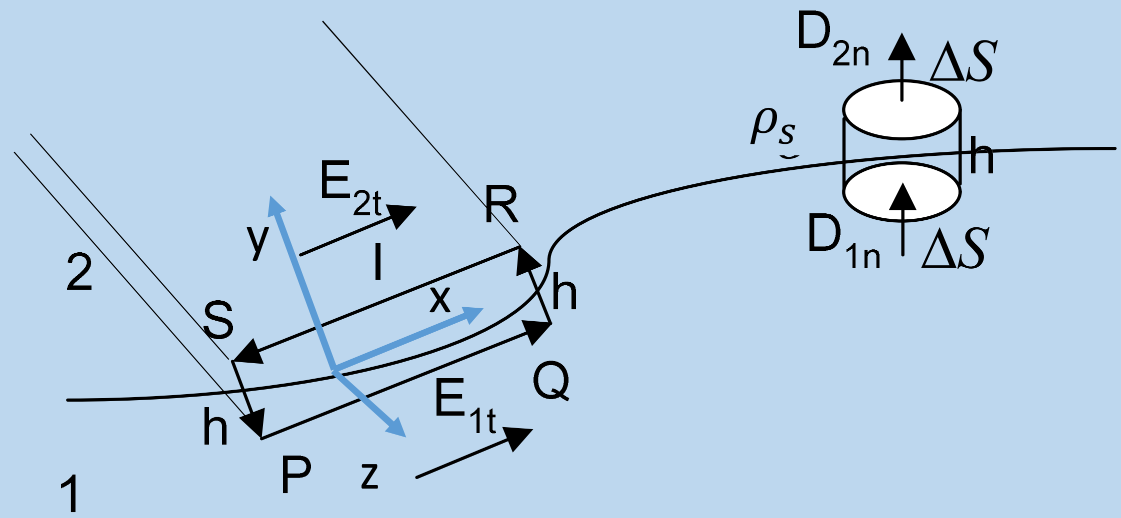
Boundary conditions for electric fields



- Fig. Boundary for electric fields at the interface of two media (Interface at z=0, z>0 is region 1 and z<0 is region 2)

Boundary conditions for electric fields

- Use Faraday's law over loop PQRSP (LHS), note that loop PQRSP is in x-y plane

$$\oint_C \vec{E} \bullet d\vec{l} = -\int_S \left(\frac{\partial \vec{B}}{\partial t} \right) \bullet d\vec{s}$$

- Note that h→0 at the boundary interface and
- therefore there is no contribution from
  - QR and SP in the above line integral
- Also note that the direction of the line integral along
  - PQ and RS are in the opposite direction

$$\because \oint_{PQRSP} \vec{E} \bullet d\vec{l} = \int_P^Q \vec{E}_1 \bullet d\vec{l}_1 + \int_R^S \vec{E}_2 \bullet d\vec{l}_2 = E_{1t} l - E_{2t} l$$

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Boundary conditions for electric fields

- Use Faraday's law over surface PQRS (RHS)

$$\oint_C \vec{E} \bullet d\vec{l} = -\int_S \left(\frac{\partial \vec{B}}{\partial t} \right) \bullet d\vec{s}$$

- If time and space dependence of $\vec{B}$ are independent
- We can take the time derivative outside the integral

$$\oint_C \vec{E} \bullet d\vec{l} = -\frac{\partial}{\partial t} \int_S \vec{B} \bullet d\vec{s}$$

  - Also area of PQRS is small enough that $\vec{B}$ is same w.r.t. space
  - We can take the $\vec{B}$ outside the integration

$$\oint_C \vec{E} \bullet d\vec{l} = -\frac{\partial}{\partial t} \left(\vec{B} \bullet \int_S d\vec{s} \right) = -\frac{\partial}{\partial t} \left(\vec{B} \bullet \vec{S} \right)$$

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Boundary conditions for electric fields

- Let us denote area of PQRS by A_{PQRS} and its direction will be normal to the surface i.e. along $\hat{z}$

$$\oint_C \vec{E} \bullet d\vec{l} = -\frac{\partial}{\partial t}\left(\vec{B} \bullet \vec{S}\right) = -\frac{\partial\left(\vec{B} \bullet \hat{z} A_{PQRS}\right)}{\partial t}$$

- We have the area of PQRS, $A_{PQRS} = lh$, hence

$$\oint_C \vec{E} \bullet d\vec{l} = -\frac{\partial\left(\vec{B} \bullet \hat{z} lh\right)}{\partial t} = -\frac{\partial\left(B_z lh\right)}{\partial t}$$

- Note that h→0, area of PQRS, A_{PQRS} →0
- So RHS surface integral is negligible, therefore

$$E_{1t} l - E_{2t} l = 0 \Rightarrow E_{1t} = E_{2t}$$

- *First boundary condition of electric field*

Boundary conditions for electric fields

- Boundary conditions for electric flux density

(*Second boundary condition of electric field*)

$$\oint_S \vec{D} \bullet d\vec{s} = \int_V \rho_v dv$$

  - Let us consider a small cylinder at the interface
  - Cross section of the cylinder must be such that
  - vector $\vec{D}$ is the same
- Note that h→0 at the boundary interface at z=0 (use cylindrical coordinate, z>0 is region 1 and z<0 is region 2)
  - therefore, there are no contribution from the curved surface of the pillbox in the above surface integral
- So only the top and bottom surfaces remains in the surface integral

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Boundary conditions for electric fields

$$\oint_{pillbox} \vec{D} \bullet d\vec{s} = \int_{top\ surface} \vec{D}_2 \bullet d\vec{s}_2 + \int_{bottom\ surface} \vec{D}_1 \bullet d\vec{s}_1 = Q_{enclosed}$$

- The normal is in the upward direction in the top surface
- and downward direction in the bottom surface
- $D_{2n}\Delta S - D_{1n}\Delta S = \rho_s \Delta S$
- $D_{2n} - D_{1n} = \rho_s$

- the normal component of electric flux density
  - can only change at the interface
- if there is charge on the interface, i.e.,
  - surface charge is present

Magnetic boundary conditions

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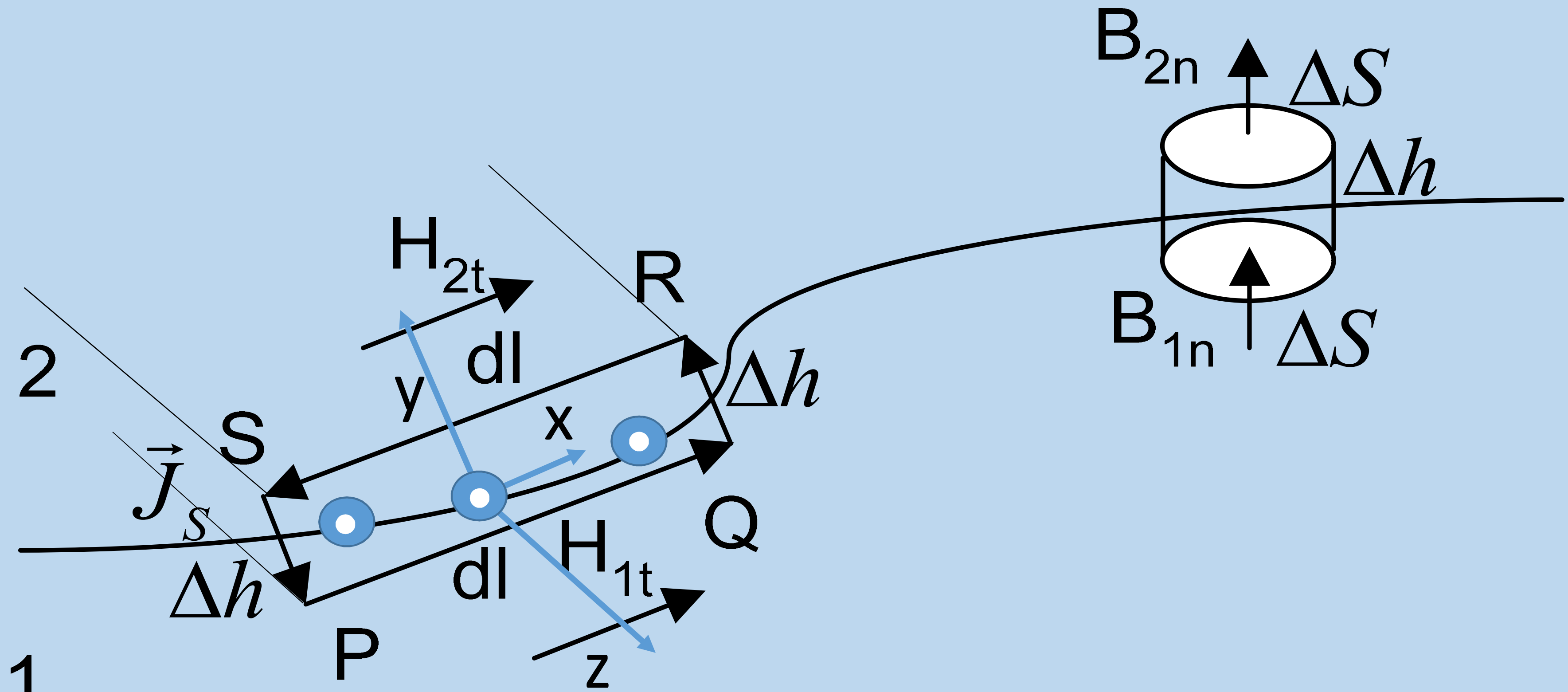

Fig. Magnetic boundary conditions

Boundary conditions for magnetic fields

- Use Modified Ampere's law over loop PQRSP (LHS)

$$\oint_C \vec{H} \bullet d\vec{l} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \bullet d\vec{s}$$

- Note that Δh→0 at the boundary interface and
- therefore there is no contribution from
  - QR and SP in the above line integral
- Also note that the direction of the line integral along
  - PQ and RS are in the opposite direction

$$\because \oint_{PQRSP} \vec{H} \bullet d\vec{l} = \int_P^Q \vec{H}_1 \bullet d\vec{l}_1 + \int_R^S \vec{H}_2 \bullet d\vec{l}_2 = H_{1t} l - H_{2t} l$$

Boundary conditions for magnetic fields

- Use Modified Ampere's law over surface PQRS (RHS)

$$\oint_C \vec{H} \bullet d\vec{l} = \int_S \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) \bullet d\vec{s}$$

- If time and space dependence of $\vec{D}$ are independent
  - Also area of PQRS is small enough that $\vec{D}$ is same w.r.t. space

$$\oint_C \vec{H} \bullet d\vec{l} = \int_S \vec{J} \bullet d\vec{s} + \frac{\partial}{\partial t} \int_S \vec{D} \bullet d\vec{s} = \int_S \vec{J} \bullet d\vec{s} + \frac{\partial \left(\vec{D} \bullet \int_S d\vec{s} \right)}{\partial t} = \int_S \vec{J} \bullet d\vec{s} + \frac{\partial \left(\vec{D} \bullet \vec{S} \right)}{\partial t}$$

$$\Rightarrow \oint_C \vec{H} \bullet d\vec{l} = \int_S \vec{J} \bullet d\vec{s} + \frac{\partial \left(\vec{D} \bullet \hat{z} (\Delta h l) \right)}{\partial t} = \int_S \vec{J} \bullet d\vec{s} + \frac{\partial \left(D_z (\Delta h l) \right)}{\partial t}$$

- Note that Δ h→0, area of PQRS, A_{PQRS} →0
- So the 2nd term of RHS surface integral is negligible, so

$$\oint_C \vec{H} \bullet d\vec{l} = \int_S \vec{J} \bullet d\vec{s} + 0$$

Boundary conditions for magnetic fields

- Note that $\vec{J}$ here is volume current density due to free charge
- Maxwell's equations we have considered is for
  - free charge and current density
  - volume current density is often referred to as current density
- What is volume current density?
- It is the amount of current passing through a unit area
  - *normal/perpendicular* to the direction of current flow
  - Its unit is A/m^2
- Volume current density at a point $\vec{r}$ is defined as
  - In other words you may consider ΔS perpendicular
  - to the current flow

$$\vec{J}(\vec{r}) = \lim_{\Delta S^{\perp} \to 0} \frac{\Delta I}{\Delta S^{\perp}} \hat{n} \; (A/m^2)$$

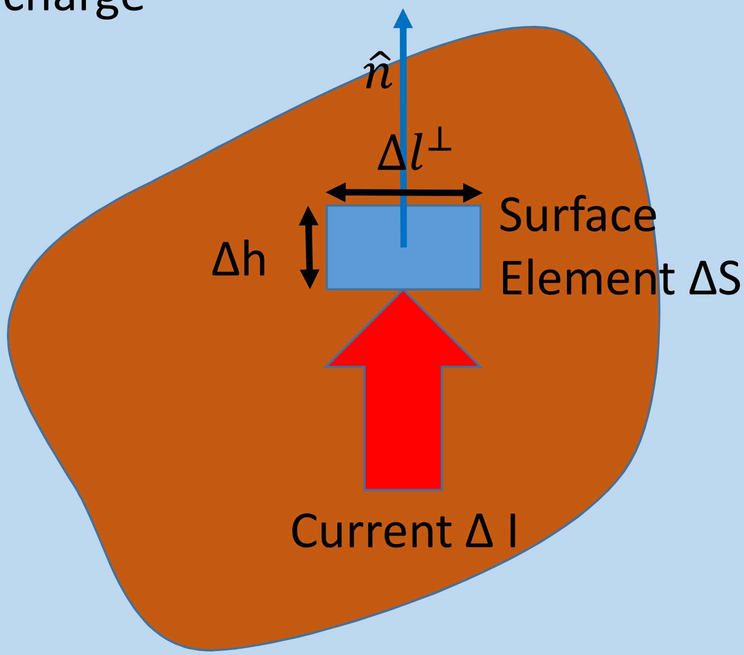

Fig. Definition of volume current density (assume $\hat{n}$ is normal to surface element ΔS and current ΔI is flowing in that direction)

Boundary conditions for magnetic fields

- What is surface current density?
- Surface current density is the amount of current passing through a unit width
  - *normal/perpendicular* to the direction of current flow
  - Its unit is A/m

Alternate way of looking at surface current density
  - Consider current flow in a thin layer
  - Imagine you squeeze the height of the surface element
  - considered perpendicular to the current flow
  - to zero (Δh→0), then it will form a line of length $\Delta l^{\perp}$
- In this case, we can consider a surface current density
- It is defined at a point $\vec{r}$ as

$$\vec{J}_S(\vec{r}) = \lim_{\Delta l^{\perp} \to 0} \frac{\Delta I}{\Delta l^{\perp}} \hat{n} \ (A/m)$$

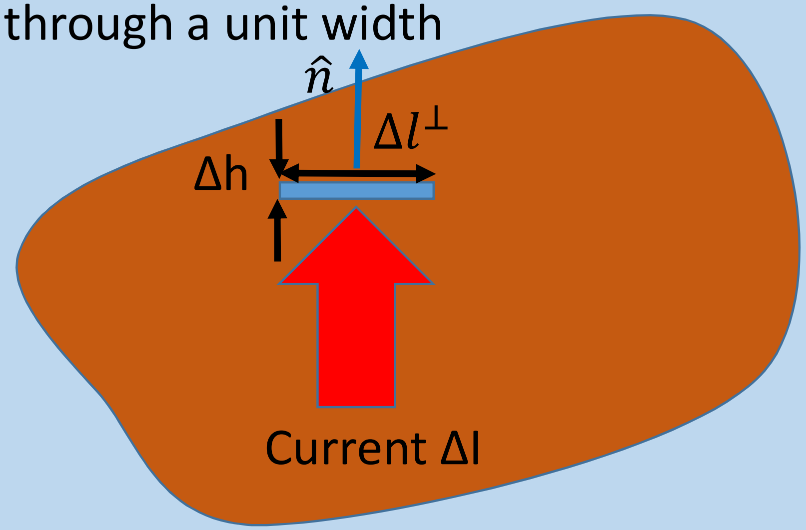

Fig. Definition of surface current density (assume $\Delta l^{\perp}$ is perpendicular to the flow of current and ΔI is the current flowing)

Boundary conditions for magnetic fields

- Note that for small area of PQRS $\vec{J}$ (volume current density) remains the same within PQRS

$$\oint_C \vec{H} \bullet d\vec{l} = \int_S \vec{J} \bullet d\vec{s} = \vec{J} \bullet \int_S d\vec{s} = \vec{J} \bullet \vec{S} = \vec{J} \bullet \Delta h l \hat{z} = \left(\vec{J}\Delta h\right) \bullet l\hat{z}$$

- When Δ h→0, we can define

$$\lim_{\Delta h \to 0} \vec{J}\Delta h = \vec{J}_S$$

- We call this impressed surface electric current density $\overrightarrow{J_S}$ at the interface
  - and its unit will be A/m
- Hence RHS after noting that $\overrightarrow{J_S}$ is along $\hat{z}$ direction, we will now have

$$\oint_C \vec{H} \bullet d\vec{l} = \left(\vec{J}_S\right) \bullet l\hat{z} = (J_S)\hat{z} \bullet l\hat{z}$$

- Therefore,

$$H_{1t} l - H_{2t} l = J_S l \Rightarrow H_{1t} - H_{2t} = J_S$$

- *First boundary condition of magnetic field*

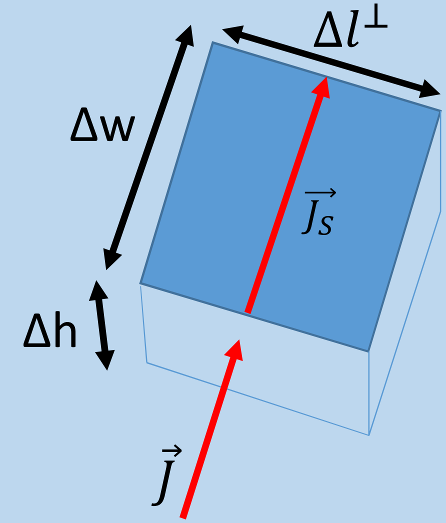

Fig. Illustration of volume and surface current density and its equivalence with current *I* $(I = (J\Delta h)\Delta l^{\perp} = J_S \Delta l^{\perp})$

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Boundary conditions for magnetic fields

- Boundary conditions for magnetic flux density

(*Second boundary condition of magnetic field*)

$$\oint_S \vec{B} \bullet d\vec{s} = 0$$

  - Let us consider a small cylinder at the interface
  - Cross section of the cylinder must be such that
  - vector $\vec{B}$ is the same
- Note that Δh→0 at the boundary interface
  - therefore, there are no contribution from the curved surface of the pillbox in the above surface integral
- So only the top and bottom surfaces remains in the surface integral

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Boundary conditions for magnetic fields

$$\oint_{pillbox} \vec{B} \bullet d\vec{s} = \int_{top\ surface} \vec{B}_2 \bullet d\vec{s}_2 + \int_{bottom\ surface} \vec{B}_1 \bullet d\vec{s}_1 = 0$$

- The normal is in the upward direction in the top surface
- and downward direction in the bottom surface
- $B_{2n}\Delta S - B_{1n}\Delta S = 0$
- $B_{2n} = B_{1n}$

- the normal component of magnetic flux density
  - are continuous at the boundary

Boundary conditions for current density

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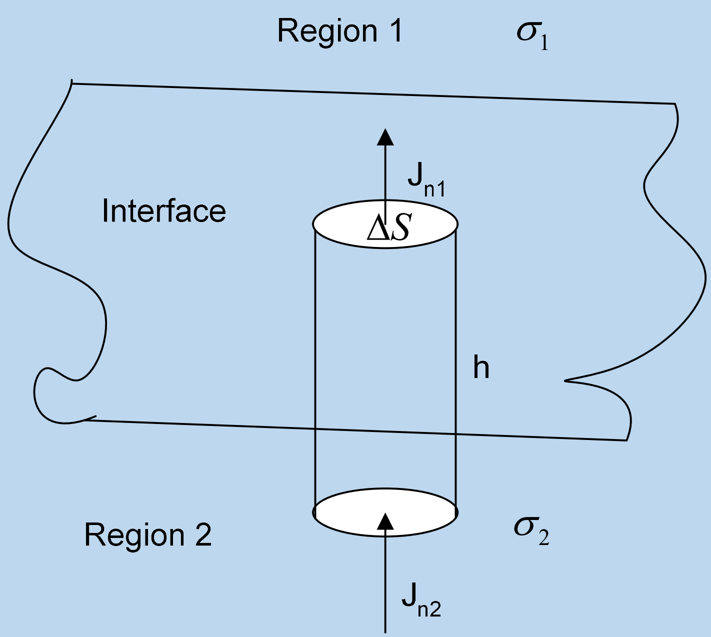

Fig. Boundary conditions for current density

Boundary conditions for current density

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- Let us construct a pillbox
  - whose height is so small that the contribution from the
  - curved surface of the cylinder to the current can be neglected
- Applying equation of continuity
  - points with changing ρ_v is the source for current density
  - for steady state (there are no points with changing ρ_v w.r.t. time)

$$i = -\frac{dq}{dt} = -\frac{d}{dt}\left(\int_V \rho_v dv\right) \Rightarrow \oint_S \vec{j} \bullet d\vec{s} = -\frac{d}{dt}\left(\int_V \rho_v dv\right)$$

  - and computing the surface integrals, we have,

$$i = \oint_S \vec{j} \bullet d\vec{s} = 0 \Rightarrow \hat{n} \bullet \vec{J}_1 \Delta s - \hat{n} \bullet \vec{J}_2 \Delta s = 0 \Rightarrow \hat{n} \bullet \left(\vec{J}_1 - \vec{J}_2\right) = 0 \Rightarrow J_{n1} = J_{n2}$$

Boundary conditions for current density

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- It states that the normal component
  - of electric current density is
  - continuous across the boundary
- Since, we have another boundary condition
  - that the tangential component of the
  - electric field is continuous across the boundary,
  - and applying Ohm's law in point form $\vec{J} = \sigma \vec{E}$
- We have,

$$\hat{n} \times \left(\vec{E}_1 - \vec{E}_2 \right) = 0 \Rightarrow \hat{n} \times \left(\frac{\vec{J}_1}{\sigma_1} - \frac{\vec{J}_2}{\sigma_2} \right) = 0 \Rightarrow \frac{J_{t1}}{\sigma_1} - \frac{J_{t2}}{\sigma_2} = 0 \Rightarrow \frac{J_{t1}}{J_{t2}} = \frac{\sigma_1}{\sigma_2}$$