EE540 Advance Electromagnetic Theory & Antennas

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Electromagnetic Theorems and Concepts

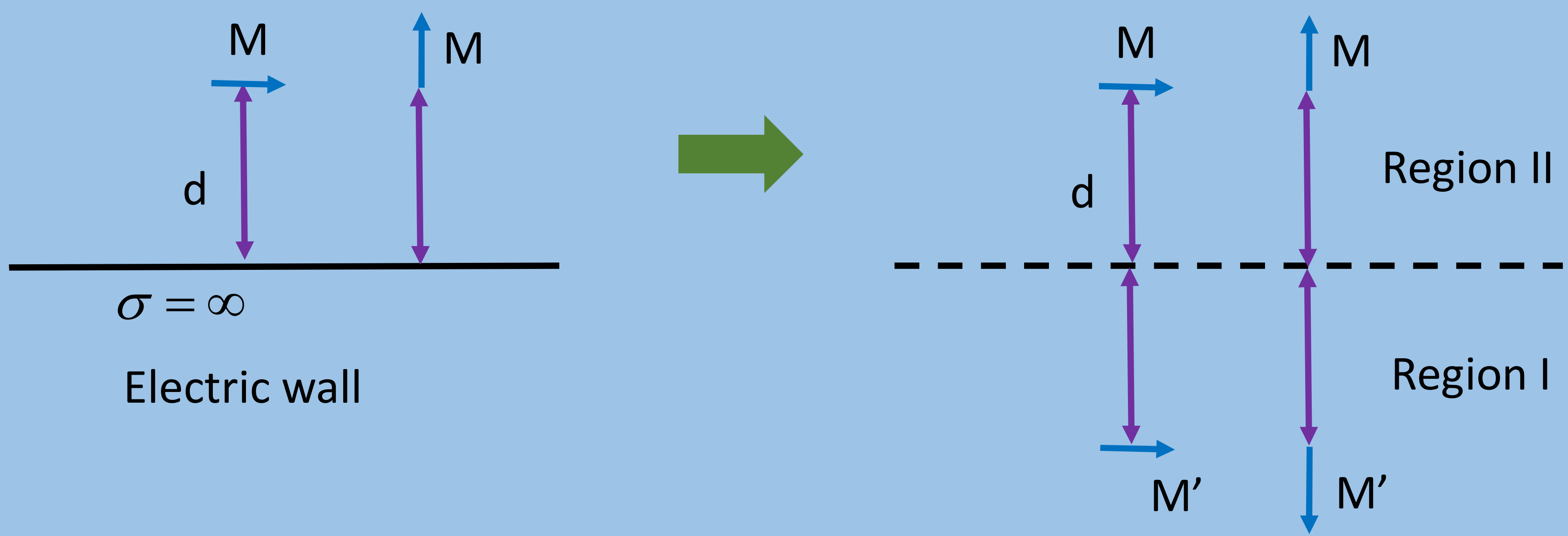

Fig. Classic image current problem (HMD and VMD) for electric wall

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Electromagnetic Theorems and Concepts

- *Magnetic field due to $\vec{J}(\vec{r}')$:*
  - $\vec{H} = -\frac{1}{4\pi}\iiint_{V'} e^{-j\beta r_d}\left(\frac{1}{r_d^3}+\frac{j\beta}{r_d^2}\right)\vec{r}_d \times \vec{J}(\vec{r}')dv'$
- Apply duality theorem (Electric field due to $\vec{M}(\vec{r}')$)
- $\vec{E} = \frac{1}{4\pi}\iiint_{V'} e^{-j\beta r_d}\left(\frac{1}{r_d^3}+\frac{j\beta}{r_d^2}\right)\vec{r}_d \times \vec{M}(\vec{r}')dv'$
- For any arbitrary oriented magnetic current I_m
- along $\hat{d}$ of length dl at the position $\vec{r}'$
- $\vec{E} = \frac{1}{4\pi}e^{-j\beta r_d}\left(\frac{1}{r_d^3}+\frac{j\beta}{r_d^2}\right)\vec{r}_d \times I_m\hat{d}dl$

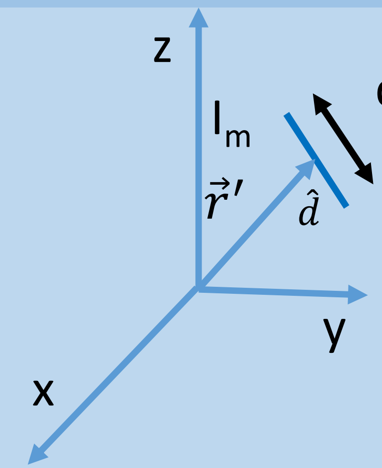

Fig. An arbitrary oriented $\hat{d}$ current carrying element carrying magnetic current I_m of length dl at the position $\vec{r}'$

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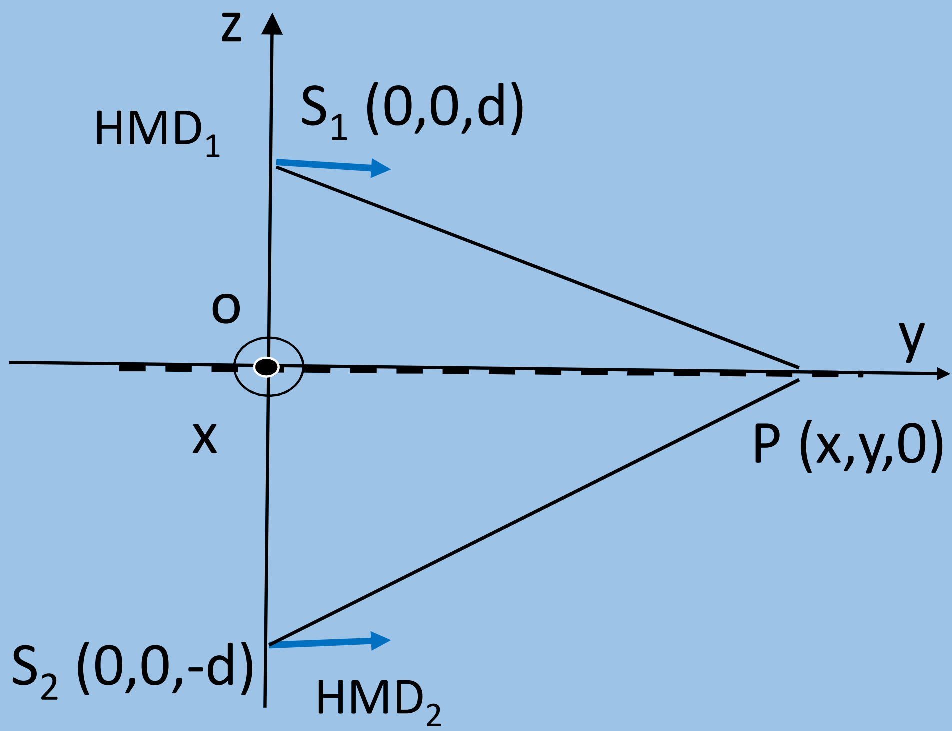

- Interface is at z=0
- For HMD$_1$, $S_1P = r_d = \sqrt{x^2 + y^2 + d^2}$
- $\vec{r}_d = x\hat{x} + y\hat{y} - d\hat{z}$
- For HMD$_2$, $S_2P = r_d = \sqrt{x^2 + y^2 + d^2}$
- $\vec{r}_d = x\hat{x} + y\hat{y} + d\hat{z}$
- Note that r_d is equal for both HMD$_s$
- but $\vec{r}_d$ is not the same for both HMD$_s$

Fig. Classic image current problem for electric wall

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- $\vec{E} = \frac{1}{4\pi} e^{-j\beta r_d} \left(\frac{1}{{r_d}^3} + \frac{j\beta}{{r_d}^2} \right) \vec{r}_d \times I_m \hat{d} dl$
- For both HMD$_s$, we have $\vec{r}_d \times I_m \hat{d}$
- For HMD$_1$,
- $\vec{r}_d \times I_m \hat{y} = \left((x\hat{x} + y\hat{y} - d\hat{z}) \times I_m \hat{y} \right) = I_m (x\hat{z} + d\hat{x})$
- $\vec{E}_{tan}^{HMD_1} = \frac{1}{4\pi} e^{-j\beta r_d} \left(\frac{1}{{r_d}^3} + \frac{j\beta}{{r_d}^2} \right) I_m (d\hat{x}) dl$
- For HMD$_2$,
- $\vec{r}_d \times (I_m \hat{y}) = \left((x\hat{x} + y\hat{y} + d\hat{z}) \times (I_m \hat{y}) \right) = I_m (x\hat{z} - d\hat{x})$
- $\vec{E}_{tan}^{HMD_2} = \frac{1}{4\pi} e^{-j\beta r_d} \left(\frac{1}{{r_d}^3} + \frac{j\beta}{{r_d}^2} \right) I_m (-d\hat{x}) dl$, therefore, $\vec{E}_{tan}^{HMD_1} + \vec{E}_{tan}^{HMD_2} = 0$

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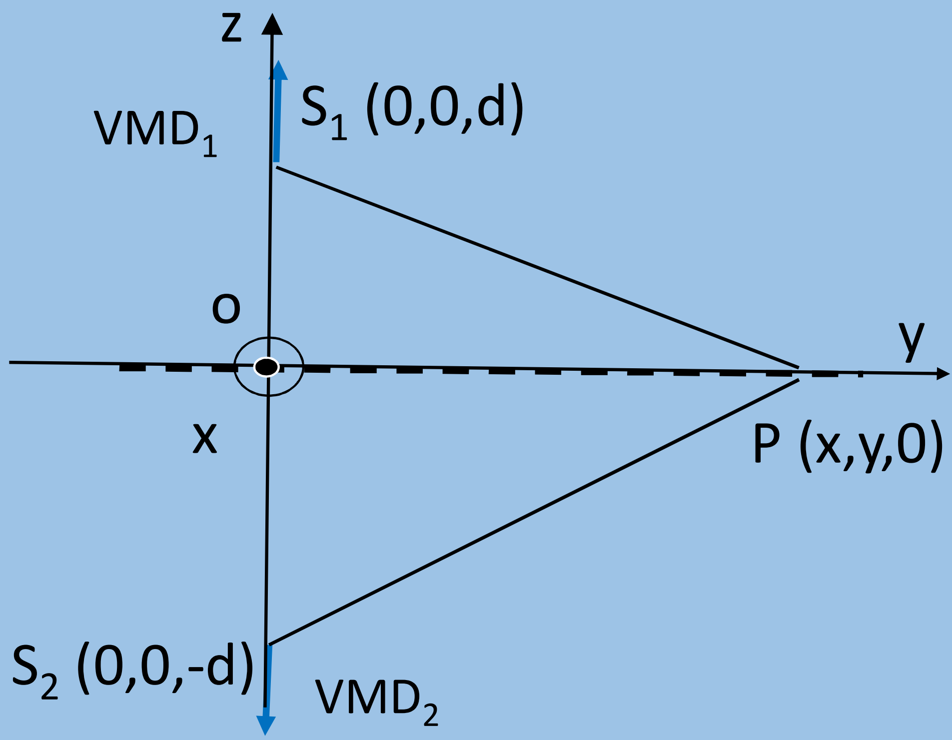

- Interface is at z=0
- For VMD$_1$, $S_1P = r_d = \sqrt{x^2 + y^2 + d^2}$
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- $\vec{r}_d = x\hat{x} + y\hat{y} + d\hat{z}$
- Note that r_d is equal for both VMD$_s$
- but $\vec{r}_d$ is not the same for both VMD$_s$

Fig. Classic image current problem for electric wall

Electromagnetic Theorems and Concepts

- $\vec{E} = \frac{1}{4\pi} e^{-j\beta r_d} \left(\frac{1}{{r_d}^3} + \frac{j\beta}{{r_d}^2} \right) \vec{r}_d \times I_m \hat{d} dl$
- For both VMD$_s$, we have $\vec{r}_d \times I_m \hat{d}$
- For VMD$_1$,
- $\vec{r}_d \times I_m \hat{z} = \left((x\hat{x} + y\hat{y} - d\hat{z}) \times I_m \hat{z} \right) = I_m(-x\hat{y} + y\hat{x})$
- $\vec{E}_{tan}^{VMD_1} = \frac{1}{4\pi} e^{-j\beta r_d} \left(\frac{1}{{r_d}^3} + \frac{j\beta}{{r_d}^2} \right) I_m \boxed{(-x\hat{y} + y\hat{x})} dl$
- For VMD$_2$,
- $\vec{r}_d \times (-I_m \hat{z}) = \left((x\hat{x} + y\hat{y} + d\hat{z}) \times (-I_m \hat{z}) \right) = I_m(x\hat{y} - y\hat{x})$
- $\vec{E}_{tan}^{VMD_2} = \frac{1}{4\pi} e^{-j\beta r_d} \left(\frac{1}{{r_d}^3} + \frac{j\beta}{{r_d}^2} \right) I_m \boxed{(x\hat{y} - y\hat{x})} dl,$
- therefore, $\vec{E}_{tan}^{VMD_1} + \vec{E}_{tan}^{VMD_2} = 0$

Electromagnetic Theorems and Concepts

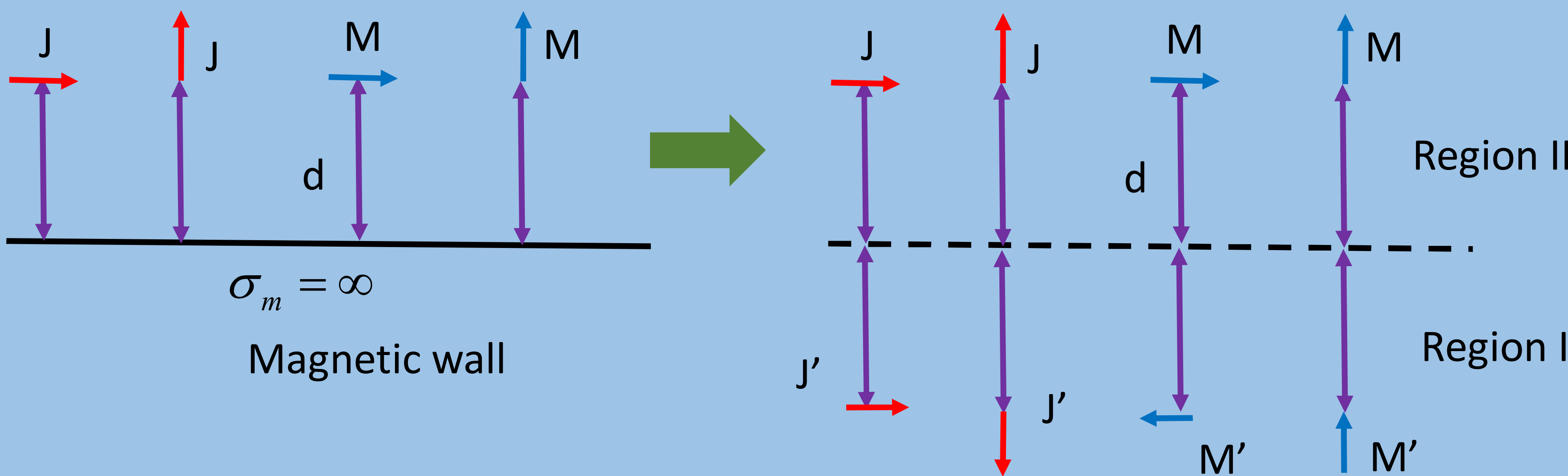

Fig. Classic image current (HED, HMD, VED, VMD) problem for magnetic wall

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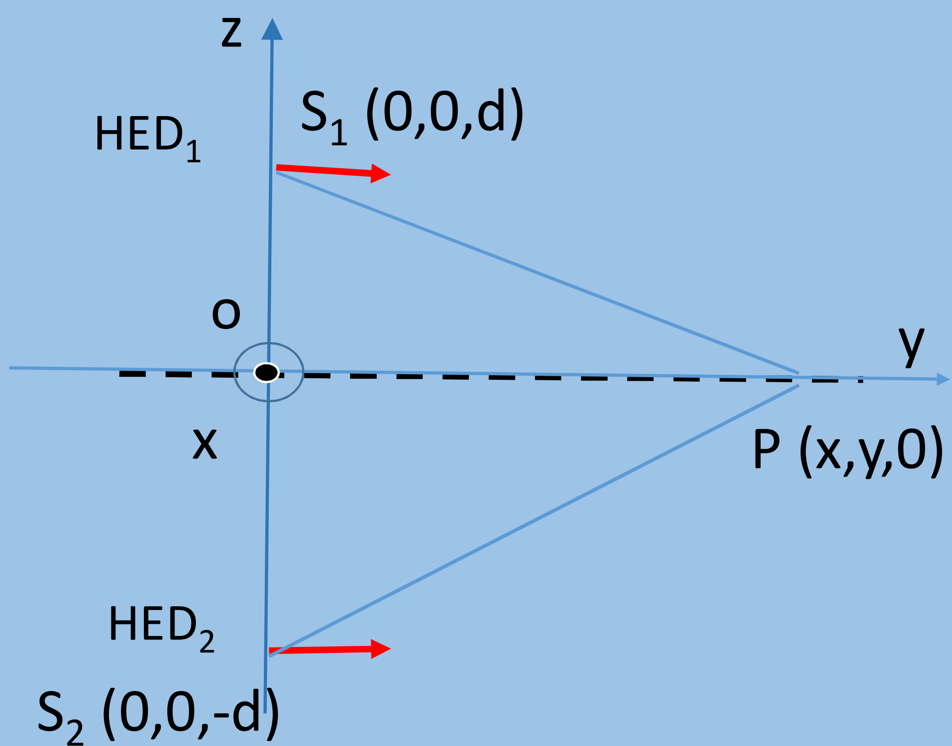

- Interface is at z=0
- For HED_1, $S_1P = r_d = \sqrt{x^2 + y^2 + d^2}$
- $\vec{r}_d = x\hat{x} + y\hat{y} - d\hat{z}$
- For HED_2, $S_2P = r_d = \sqrt{x^2 + y^2 + d^2}$
- $\vec{r}_d = x\hat{x} + y\hat{y} + d\hat{z}$
- Note that r_d is equal for both HED_s
- but $\vec{r}_d$ is not the same for both HED_s

Fig. Classic image current problem for magnetic wall

Electromagnetic Theorems and Concepts

- $\vec{H}=\frac{1}{4\pi}\iiint_{V'} e^{-j\beta r_d}\left(-\frac{1}{{r_d}^3}-\frac{j\beta}{{r_d}^2}\right)\vec{r}_d\times\vec{J}(\vec{r}')dv'=\frac{1}{4\pi}e^{-j\beta r_d}\left(-\frac{1}{{r_d}^3}-\frac{j\beta}{{r_d}^2}\right)\vec{r}_d\times I_0\hat{d}dl$
- For both HED$_s$, we have $\vec{r}_d\times I_0\hat{d}$
- For HED$_1$,
- $\vec{r}_d\times I_0\hat{y}=\left((x\hat{x}+y\hat{y}-d\hat{z})\times I_0\hat{y}\right)=I_0(x\hat{z}+d\hat{x})$
- $\vec{H}_{tan}^{HED_1}=\frac{1}{4\pi}e^{-j\beta r_d}\left(-\frac{1}{{r_d}^3}-\frac{j\beta}{{r_d}^2}\right)I_0(d\hat{x})dl$
- For HED$_2$,
- $\vec{r}_d\times(I_0\hat{y})=\left((x\hat{x}+y\hat{y}+d\hat{z})\times(I_0\hat{y})\right)=I_0(x\hat{z}-d\hat{x})$
- $\vec{H}_{tan}^{HED_2}=\frac{1}{4\pi}e^{-j\beta r_d}\left(-\frac{1}{{r_d}^3}-\frac{j\beta}{{r_d}^2}\right)I_0(-d\hat{x})dl,$
- therefore, $\vec{H}_{tan}^{HED_1}+\vec{H}_{tan}^{HED_2}=0$

Electromagnetic Theorems and Concepts

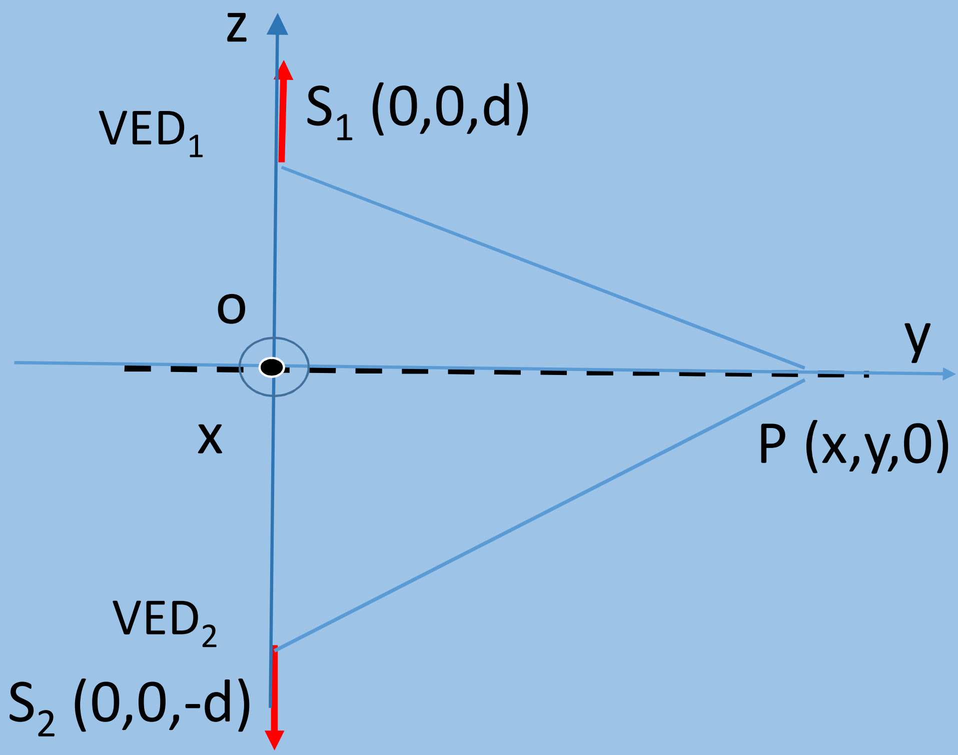

Fig. Classic image current problem for magnetic wall

- Interface is at z=0
- For VED_1, $S_1P = r_d = \sqrt{x^2 + y^2 + d^2}$
- $\vec{r}_d = x\hat{x} + y\hat{y} - d\hat{z}$
- For VED_2, $S_2P = r_d = \sqrt{x^2 + y^2 + d^2}$
- $\vec{r}_d = x\hat{x} + y\hat{y} + d\hat{z}$
- Note that r_d is equal for both VED_s
- but $\vec{r}_d$ is not the same for both VED_s

Electromagnetic Theorems and Concepts

- $\vec{H} = \frac{1}{4\pi}\iiint_{V'} e^{-j\beta r_d}\left(-\frac{1}{{r_d}^3} - \frac{j\beta}{{r_d}^2}\right)\vec{r}_d \times \vec{J}(\vec{r}')dv' = \frac{1}{4\pi}e^{-j\beta r_d}\left(-\frac{1}{{r_d}^3} - \frac{j\beta}{{r_d}^2}\right)\vec{r}_d \times I_0\hat{d}dl$
- For both VED$_s$, we have $\vec{r}_d \times I_0\hat{d}$
- For VED$_1$,
- $\vec{r}_d \times I_0\hat{z} = \left((x\hat{x} + y\hat{y} - d\hat{z}) \times I_0\hat{z}\right) = I_0(-x\hat{y} + y\hat{x})$
- $\vec{H}_{tan}^{VE\ 1} = \frac{1}{4\pi}e^{-j\beta r_d}\left(-\frac{1}{{r_d}^3} - \frac{j\beta}{{r_d}^2}\right)I_0(-x\hat{y} + y\hat{x})dl$
- For VED$_2$,
- $\vec{r}_d \times (-I_0\hat{z}) = \left((x\hat{x} + y\hat{y} + d\hat{z}) \times (-I_0\hat{z})\right) = I_0(x\hat{y} - y\hat{x})$
- $\vec{H}_{tan}^{VE\ 2} = \frac{1}{4\pi}e^{-j\beta r_d}\left(-\frac{1}{{r_d}^3} - \frac{j\beta}{{r_d}^2}\right)I_0(x\hat{y} - y\hat{x})dl,$
- therefore, $\vec{H}_{tan}^{VED_1} + \vec{H}_{tan}^{VED_2} = 0$

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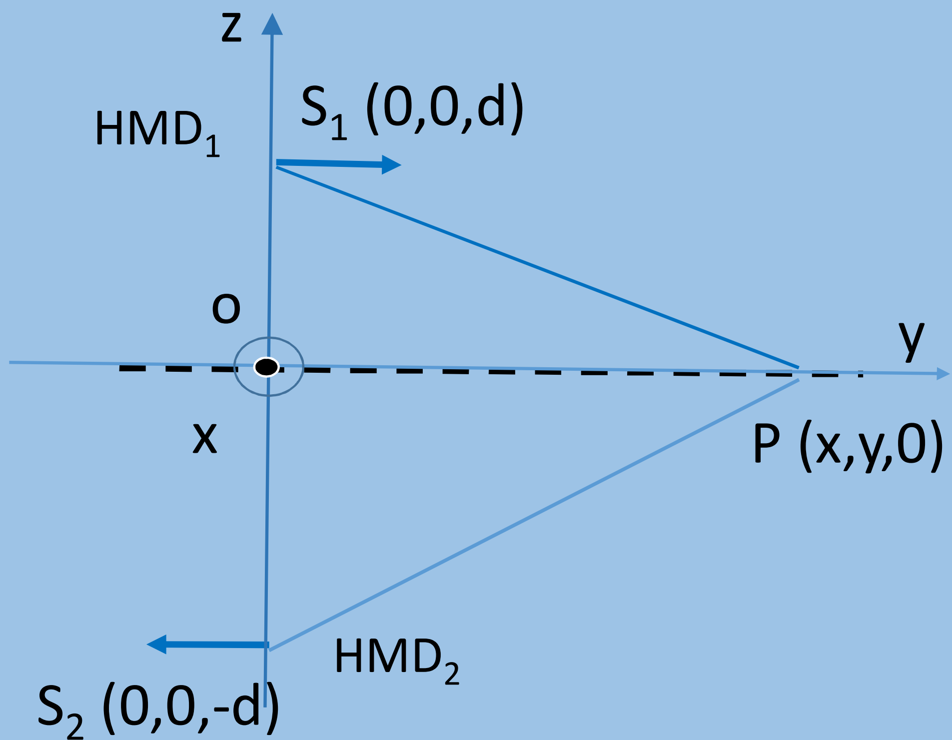

- Interface is at z=0
- For HED$_1$, $S_1P = r_d = \sqrt{x^2 + y^2 + d^2}$
- $\vec{r}_d = x\hat{x} + y\hat{y} - d\hat{z}, \hat{r}_d = \frac{\vec{r}_d}{r_d}$
- For HED$_2$, $S_2P = r_d = \sqrt{x^2 + y^2 + d^2}$
- $\vec{r}_d = x\hat{x} + y\hat{y} + d\hat{z}, \hat{r}_d = \frac{\vec{r}_d}{r_d}$
- Note that r_d is equal for both HED$_s$
- but $\vec{r}_d$ is not the same for both HED$_s$

Fig. Classic image current problem for magnetic wall

Electromagnetic Theorems and Concepts

- $\vec{E} = \frac{1}{j4\pi\omega\varepsilon}\iiint_{V'}\left\{\frac{\beta^2}{r_d}e^{-j\beta r_d}\left(-1-\frac{3}{j\beta r_d}+\frac{3}{\beta^2 {r_d}^2}\right)\hat{r}_d\times\hat{r}_d\times\vec{J}(\vec{r}') - 2e^{-j\beta r_d}\left(-\frac{1}{{r_d}^3}-\frac{j\beta}{{r_d}^2}\right)\vec{J}(\vec{r}')\right\}dv'$

Apply Duality Principles ($\vec{E}\rightarrow\vec{H}$, $\varepsilon\rightarrow\mu$, $\vec{J}(\vec{r}')\rightarrow\vec{M}(\vec{r}')$)

- $\vec{H} = \frac{1}{j4\pi\omega\mu}\iiint_{V'}\left\{\frac{\beta^2}{r_d}e^{-j\beta r_d}\left(-1-\frac{3}{j\beta r_d}+\frac{3}{\beta^2 {r_d}^2}\right)\hat{r}_d\times\hat{r}_d\times\vec{M}(\vec{r}') - 2e^{-j\beta r_d}\left(-\frac{1}{{r_d}^3}-\frac{j\beta}{{r_d}^2}\right)\vec{M}(\vec{r}')\right\}dv'$

- Replace $\vec{M}(\vec{r}')$ by $I_m\hat{d}$ and dv' by dl
- $\vec{H} = \frac{1}{j4\pi\omega\mu}\left\{\frac{\beta^2}{r_d}e^{-j\beta r_d}\left(-1-\frac{3}{j\beta r_d}+\frac{3}{\beta^2 {r_d}^2}\right)\hat{r}_d\times\hat{r}_d\times I_m\hat{d} - 2e^{-j\beta r_d}\left(-\frac{1}{{r_d}^3}-\frac{j\beta}{{r_d}^2}\right)I_m\hat{d}\right\}dl$

Electromagnetic Theorems and Concepts

- For HMD$_1$, $\hat{d} = \hat{y}$ and for HMD$_2$, $\hat{d} = -\hat{y}$,
- So the second term cancels out
- For both HMD$_s$, we have $\hat{r}_d \times \hat{r}_d \times \hat{d}$ in first term, let us calculate
- For HMD$_1$,
- $\hat{r}_d \times \hat{r}_d \times \hat{d} = \hat{r}_d \times \hat{r}_d \times \hat{y} = \frac{1}{r_d^2}(x\hat{x} + y\hat{y} - d\hat{z}) \times (x\hat{x} + y\hat{y} - d\hat{z}) \times \hat{y} = \frac{1}{r_d^2}(x\hat{x} + y\hat{y} - d\hat{z}) \times (d\hat{x} + x\hat{z}) = \frac{1}{r_d^2}(-yd\hat{z} + yx\hat{x} - x^2\hat{y} - d^2\hat{y})$
- $\vec{H}_{tan}^{HMD_1} = \frac{dl\ \ m}{j4\pi\omega} \frac{\beta^2}{r_d} e^{-j\beta r_d}\left(-1 - \frac{3}{j\beta r_d} + \frac{3}{\beta^2 {r_d}^2}\right)\frac{1}{r_d^2}(yxd\hat{x} - x^2\hat{y} - d^2\hat{y})$
$$-2\frac{dlI_m}{j4\pi\omega\mu} e^{-j\beta r_d}\left(-\frac{1}{{r_d}^3} - \frac{j\beta}{{r_d}^2}\right)\hat{y}$$

Electromagnetic Theorems and Concepts

- For HMD_2,
- $\hat{r}_d \times \hat{r}_d \times \hat{d} = \hat{r}_d \times \hat{r}_d \times (-\hat{y}) = \frac{1}{r_d^2}(x\hat{x} + y\hat{y} + d\hat{z}) \times (x\hat{x} + y\hat{y} + d\hat{z}) \times (-\hat{y}) = \frac{1}{r_d^2}(x\hat{x} + y\hat{y} + d\hat{z}) \times (d\hat{x} - x\hat{z}) = \frac{1}{r_d^2}(-yd\hat{z} - yx\hat{x} + x^2\hat{y} + d^2\hat{y})$
- $\vec{H}_{tan}^{HMD_2} = \frac{dlI_m}{j4\pi\omega\mu}\frac{\beta^2}{r_d}e^{-j\beta r_d}\left(-1 - \frac{3}{j\beta r_d} + \frac{3}{\beta^2 {r_d}^2}\right)\frac{1}{r_d^2}(-yxd\hat{x} + x^2\hat{y} + d^2\hat{y}) - 2\frac{dlI_m}{j4\pi\omega\mu}e^{-j\beta r_d}\left(-\frac{1}{{r_d}^3} - \frac{j\beta}{{r_d}^2}\right)(-\hat{y})$
- therefore, $\vec{H}_{tan}^{HMD_1} + \vec{H}_{tan}^{HMD_2} = 0$

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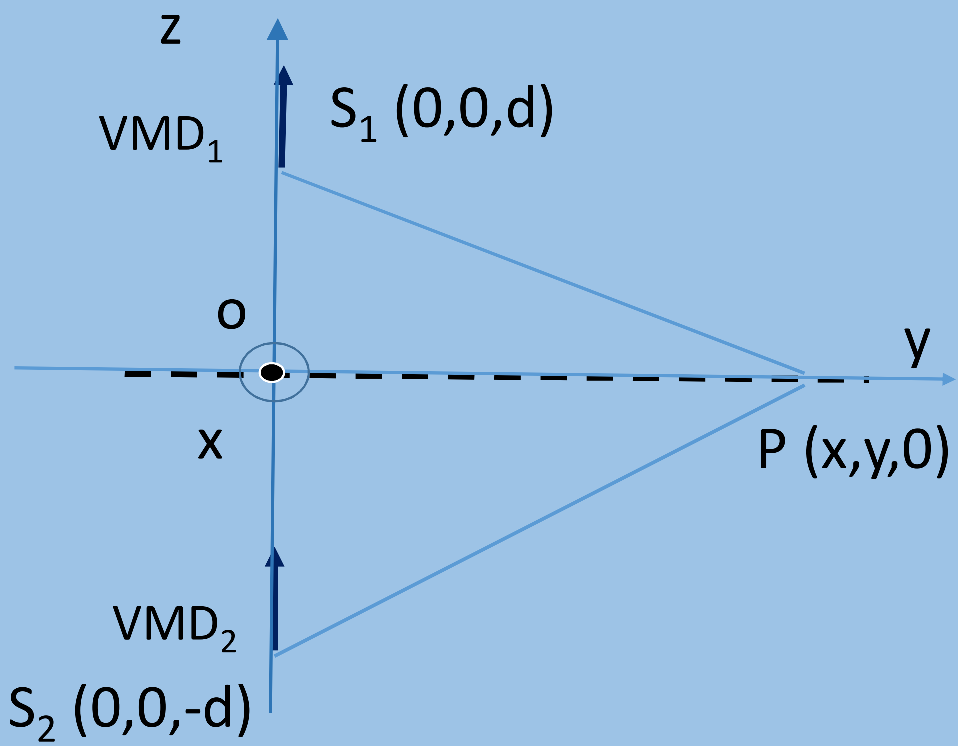

Fig. Classic image current problem for magnetic wall

- Interface is at z=0
- For VED$_1$, $S_1P = r_d = \sqrt{x^2 + y^2 + d^2}$
- $\vec{r}_d = x\hat{x} + y\hat{y} - d\hat{z}, \hat{r}_d = \frac{\vec{r}_d}{r_d}$
- For VED$_2$, $S_2P = r_d = \sqrt{x^2 + y^2 + d^2}$
- $\vec{r}_d = x\hat{x} + y\hat{y} + d\hat{z}, \hat{r}_d = \frac{\vec{r}_d}{r_d}$
- Note that r_d is equal for both VED$_s$
- but $\vec{r}_d$ is not the same for both VED$_s$

Electromagnetic Theorems and Concepts

- $\vec{H} = \frac{1}{j4\pi\omega}\left\{\frac{\beta^2}{r_d} e^{-j\beta r_d}\left(-1 - \frac{3}{j\beta r_d} + \frac{3}{\beta^2 {r_d}^2}\right)\hat{r}_d \times \hat{r}_d \times I_m \hat{d} - 2e^{-j\beta r_d}\left(-\frac{1}{{r_d}^3} - \frac{j\beta}{{r_d}^2}\right) I_m \hat{d}\right\} dl$
- For VMD_1, $\hat{d} = \hat{z}$ and for VMD_2, $\hat{d} = \hat{z}$,
- So the second term does not contribute to tangential components of magnetic field
- For both HMD_s, we have $\hat{r}_d \times \hat{r}_d \times \hat{d}$ in first term, let us calculate
- For HMD_1,
- $\hat{r}_d \times \hat{r}_d \times \hat{d} = \hat{r}_d \times \hat{r}_d \times \hat{z} = \frac{1}{r_d^2}(x\hat{x} + y\hat{y} - d\hat{z}) \times (x\hat{x} + y\hat{y} - d\hat{z}) \times \hat{z} = \frac{1}{r_d^2}(x\hat{x} + y\hat{y} - d\hat{z}) \times (y\hat{x} - x\hat{y}) = \frac{1}{r_d^2}(-x^2\hat{z} - y^2\hat{z} - xd\hat{x} - yd\hat{y})$
- $\vec{H}_{tan}^{HMD_1} = \frac{dlI_m}{j4\pi\omega\mu}\frac{\beta^2}{r_d} e^{-j\beta r_d}\left(-1 - \frac{3}{j\beta r_d} + \frac{3}{\beta^2 {r_d}^2}\right)\frac{1}{r_d^2}(-xd\hat{x} - yd\hat{y})$

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- For HMD_2,
- $\hat{r}_d \times \hat{r}_d \times \hat{d} = \hat{r}_d \times \hat{r}_d \times \hat{z} = \frac{1}{r_d^2}(x\hat{x} + y\hat{y} + d\hat{z}) \times (x\hat{x} + y\hat{y} + d\hat{z}) \times \hat{z} = \frac{1}{r_d^2}(x\hat{x} + y\hat{y} + d\hat{z}) \times (y\hat{x} - x\hat{y}) = \frac{1}{r_d^2}(-x^2\hat{z} - y^2\hat{z} + xd\hat{x} + yd\hat{y})$
- $\vec{H}_{tan}^{HMD_1} = \frac{dlI_m}{j4\pi\omega\mu}\frac{\beta^2}{r_d}e^{-j\beta r_d}\left(-1 - \frac{3}{j\beta r_d} + \frac{3}{\beta^2 {r_d}^2}\right)\frac{1}{r_d^2}(xd\hat{x} + yd\hat{y})$
- therefore, $\vec{H}_{tan}^{HMD_1} + \vec{H}_{tan}^{HMD_2} = 0$