EE540 Advance Electromagnetic Theory & Antennas

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Aperture antennas

- **Far Fields of aperture antenna**
- Step 1:
  - Establish the EM fields specified on the aperture surface
- Step 2:
  - Calculate the 2-D FT of the aperture fields
- Step 3:
  - Find the far field electric fields from equations (24.1) and (24.2),
  - Far field magnetic fields can be found from the electric field (if required) from $\vec{H}_{ff} = \frac{\hat{r} \times \vec{E}_{ff}}{\eta}$

Aperture antennas

- **Radiation from Uniform rectangular aperture**
- $\vec{E}_a = E_{ay}\hat{y} = \begin{cases} E_0\hat{y}, & |x| \le \frac{L_x}{2}, |y| \le \frac{L_y}{2}, z = 0 \\ 0, & otherwise \end{cases}$
- Hence, from equations (24.1) and (24.2), we have,
- $E_{ff\theta} \cong 2\frac{j\beta e^{(-j\beta r)}}{4\pi r}\left[\left(\tilde{E}_{ay} sin\phi\right)\right]$

and $E_{ff\phi} \cong 2\frac{j\beta e^{(-j\beta r)}}{4\pi r}\left[\left(\tilde{E}_{ay} cos\theta cos\phi\right)\right]$
- Why this factor of 2?

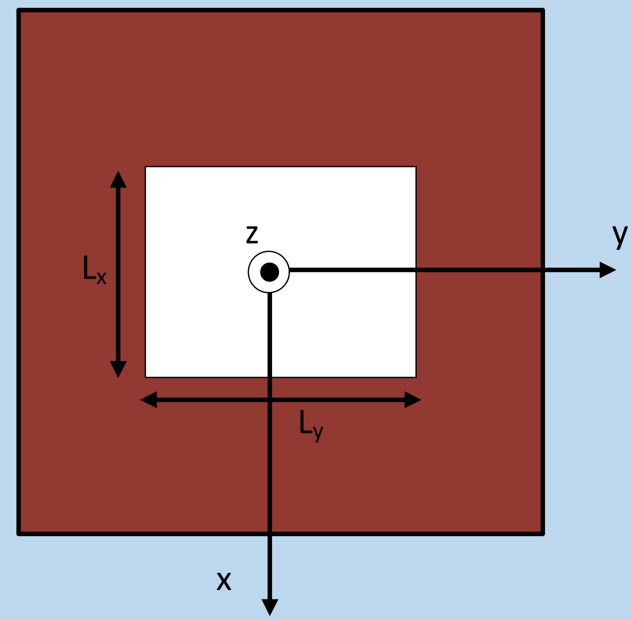

Fig. Uniform rectangular aperture (uniform amplitude and phase of the aperture field)

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Electromagnetic Theorems and Concepts

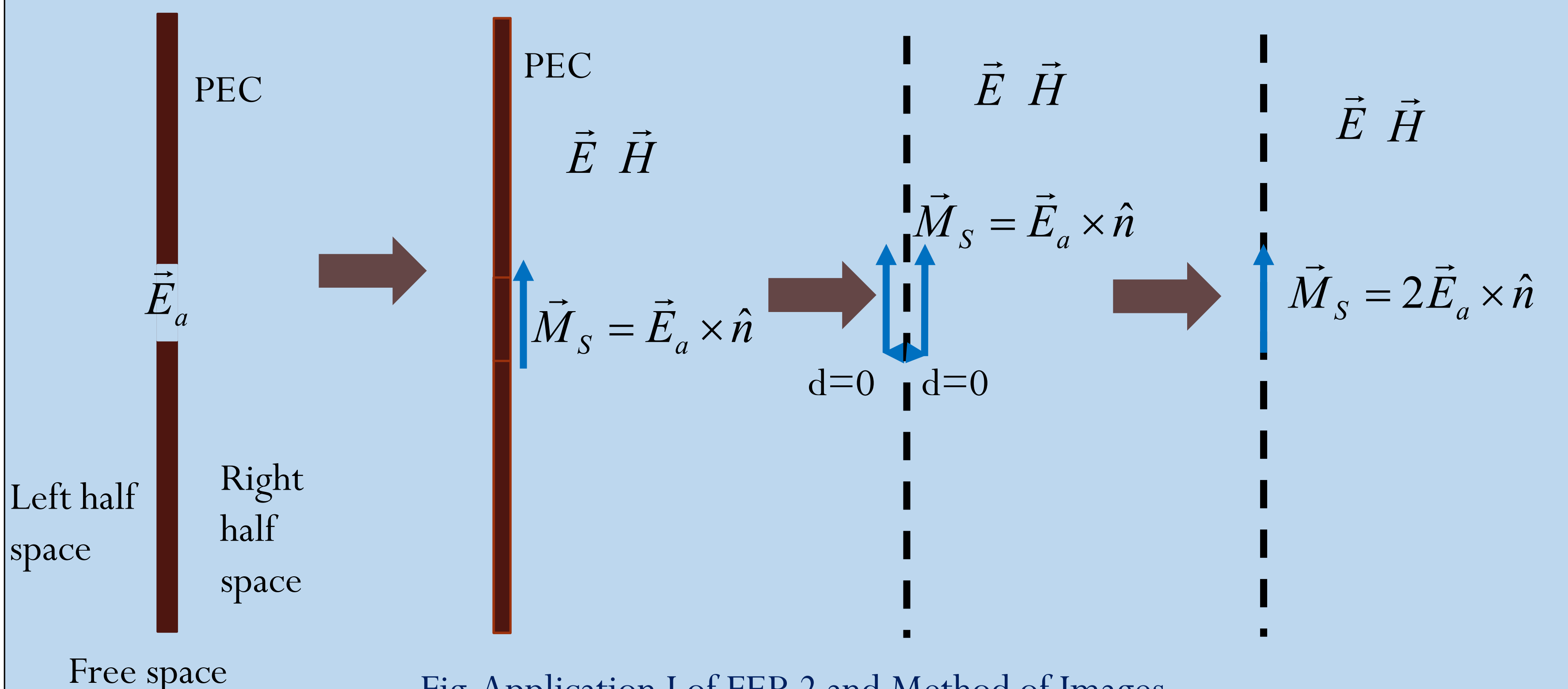

Fig. Application I of FEP 2 and Method of Images

Aperture antennas

- Let us find the 2 − D FT of the aperture field,
- $\tilde{E}_{ay} = \int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} \int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} E_0 \, e^{(j\beta_x x' + j\beta_y y')} dx' dy'$

$$= E_0 \int_{-\frac{L_x}{2}}^{\frac{L_x}{2}} e^{(j\beta_x x')} dx' \int_{-\frac{L_y}{2}}^{\frac{L_y}{2}} e^{(j\beta_y y')} dy'$$

$$= E_0 \left\{ \frac{e^{\left(j\beta_x \frac{L_x}{2}\right)} - e^{\left(-j\beta_x \frac{L_x}{2}\right)}}{j\beta_x} \right\} \left\{ \frac{e^{\left(j\beta_y \frac{L_y}{2}\right)} - e^{\left(-j\beta_y \frac{L_y}{2}\right)}}{j\beta_y} \right\}$$

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$$\tilde{E}_{ay} = 4E_0 \frac{1}{\beta_x}\left\{\frac{e^{\left(j\beta_x \frac{L_x}{2}\right)} - e^{\left(-j\beta_x \frac{L_x}{2}\right)}}{2j}\right\}\frac{1}{\beta_y}\left\{\frac{e^{\left(j\beta_y \frac{L_y}{2}\right)} - e^{\left(-j\beta_y \frac{L_y}{2}\right)}}{2j}\right\}$$

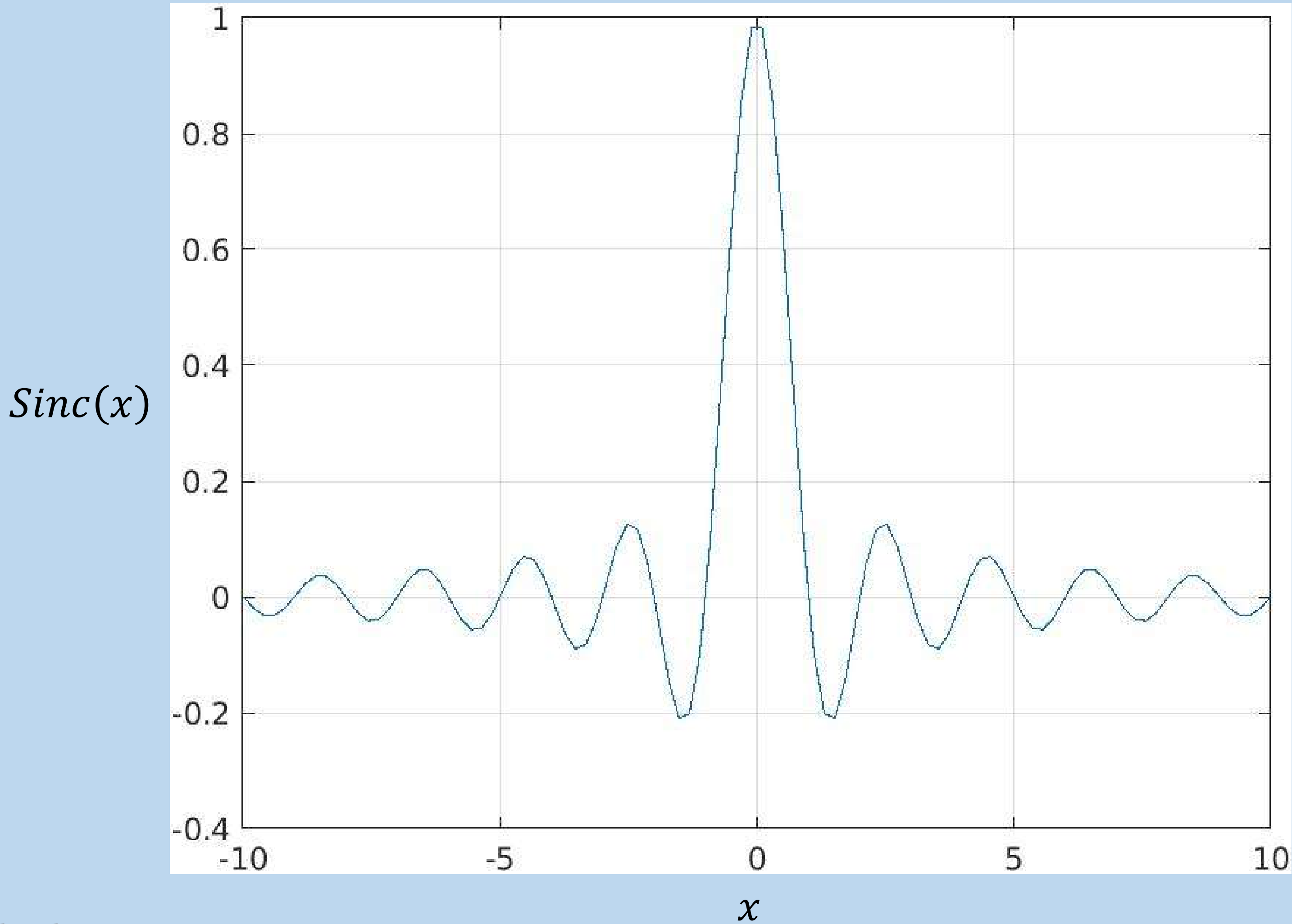
$$= 4E_0\left(\frac{sin\beta_x \frac{L_x}{2}}{\beta_x}\right)\left(\frac{sin\beta_y \frac{L_y}{2}}{\beta_y}\right) = 4E_0 \frac{L_x}{2}\left(\frac{sin\beta_x \frac{L_x}{2}}{\beta_x \frac{L_x}{2}}\right)\frac{L_y}{2}\left(\frac{sin\beta_y \frac{L_y}{2}}{\beta_y \frac{L_y}{2}}\right)$$

$$= E_0 L_x L_y sinc\left(\beta_x \frac{L_x}{2}\right) sinc\left(\beta_y \frac{L_y}{2}\right)$$

- where $\vec{\beta} = \beta_x\hat{x} + \beta_y\hat{y} + \beta_z\hat{z}$ *and*
- $\beta_x = \beta sin\theta cos\phi, \beta_y = \beta sin\theta sin\phi, \beta_z = \beta cos\theta$

Aperture antennas

- Fig. Sinc function



Aperture antennas

- ***E-plane (electric is maximum along y-axis, hence y-z plane, $\boldsymbol{\phi = \frac{\pi}{2}}$)***
- $\tilde{E}_{ay} = E_0 L_x L_y sinc\left(\beta_x \frac{L_x}{2}\right) sinc\left(\beta_y \frac{L_y}{2}\right)$
- where $\beta_x = 0$ and $\beta_y = \beta sin\theta$
- Since $sinc\left(\beta_x \frac{L_x}{2}\right) = 1, sinc\left(\beta_y \frac{L_y}{2}\right) = sinc\left(\beta sin\theta \frac{L_y}{2}\right)$, then
- $E_{ff\theta} \cong 2 \frac{j\beta e^{(-j\beta r)}}{4\pi r}\left[\left(\tilde{E}_{ay} sin\phi\right)\right] = \frac{j\beta e^{(-j\beta r)}}{2\pi r}\left[E_0 L_x L_y sinc\left(\beta sin\theta \frac{L_y}{2}\right)\right]$
- and $E_{ff\phi} \cong 2 \frac{j\beta e^{(-j\beta r)}}{4\pi r}\left[\left(\tilde{E}_{ay} cos\theta cos\phi\right)\right] = 0$

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Aperture antennas

- ***H-plane (x-z plane, $\boldsymbol{\phi = 0}$)***
- $\tilde{E}_{ay} = E_0 L_x L_y sinc\left(\beta_x \frac{L_x}{2}\right) sinc\left(\beta_y \frac{L_y}{2}\right)$
- where $\beta_x = \beta sin\theta$ and $\beta_y = 0$
- $\tilde{E}_{ay} = E_0 L_x L_y sinc\left(\beta sin\theta \frac{L_x}{2}\right)$
- $E_{ff\theta} \cong 2 \frac{j\beta e^{(-j\beta r)}}{4\pi r}\left[\left(\tilde{E}_{ay} sin\phi\right)\right] = 0$

  and $E_{ff\phi} \cong 2 \frac{j\beta e^{(-j\beta r)}}{4\pi r}\left[\left(\tilde{E}_{ay} cos\theta cos\phi\right)\right]$

$$= \frac{j\beta e^{(-j\beta r)}}{2\pi r}\left[\left(E_0 L_x L_y cos\theta sinc\left(\beta sin\theta \frac{L_x}{2}\right)\right)\right]$$

Aperture antennas

- Considering E-plane and maximum for $\theta = 0$
- $E_{ff\theta} \cong \frac{j\beta e^{(-j\beta r)}}{2\pi r}\left[E_0 L_x L_y sinc\left(\beta sin\theta \frac{L_y}{2}\right)\right]$
- Nulls occur at $\beta \frac{L_y}{2} sin\theta_n = n\pi, n = 1,2,\cdots$
- Thus $sin\theta_n = \frac{n\pi}{\beta \frac{L_y}{2}} = \frac{n\lambda}{L_y}$
- $BWFN_{E-plane} = 2sin^{-1}\left(\frac{\lambda}{L_y}\right) rad =$

  $114.59 sin^{-1}\left(\frac{\lambda}{L_y}\right) degrees$
- Similarly, $BWFN_{H-plane} = 114.59 sin^{-1}\left(\frac{\lambda}{L_x}\right) degrees$

Aperture antennas

- For HPBW, $sinc^2\left(\beta sin\theta\frac{L_y}{2}\right)=\frac{1}{2}$
- $\beta sin\theta\frac{L_y}{2}=1.391 \Rightarrow sin\theta=1.391\frac{\lambda}{\pi L_y}=\frac{0.443\lambda}{L_y}$
- $HPBW_{E-plane}=2sin^{-1}\left(\frac{0.443\lambda}{L_y}\right)rad$

$$=114.59sin^{-1}\left(\frac{0.443\lambda}{L_y}\right)degrees$$

- Similarly,
- $HPBW_{H-plane}=114.59sin^{-1}\left(\frac{0.443\lambda}{L_x}\right)degrees$
- It is inversely proportional to length in the corresponding cutting plane

Aperture antennas

- How do one calculate the directivity (for main beam in broadside direction of the aperture)?
- $D = \frac{4\pi}{\lambda^2} \frac{\left|\iint_{S_a} \vec{E}_a ds'\right|^2}{\iint_{S_a} \left|\vec{E}_a\right|^2 ds'}$
- For uniform rectangular aperture

$$\vec{E}_a = E_{ay}\hat{y} = \begin{cases} E_0\hat{y}, & |x| \leq \frac{L_x}{2}, |y| \leq \frac{L_y}{2}, z = 0 \\ 0, & otherwise \end{cases}$$

- Hence $D = \frac{4\pi L_x L_y}{\lambda^2}$
- SLL is greater than -13.26 dB (Proof is given in next slide)

Aperture antennas

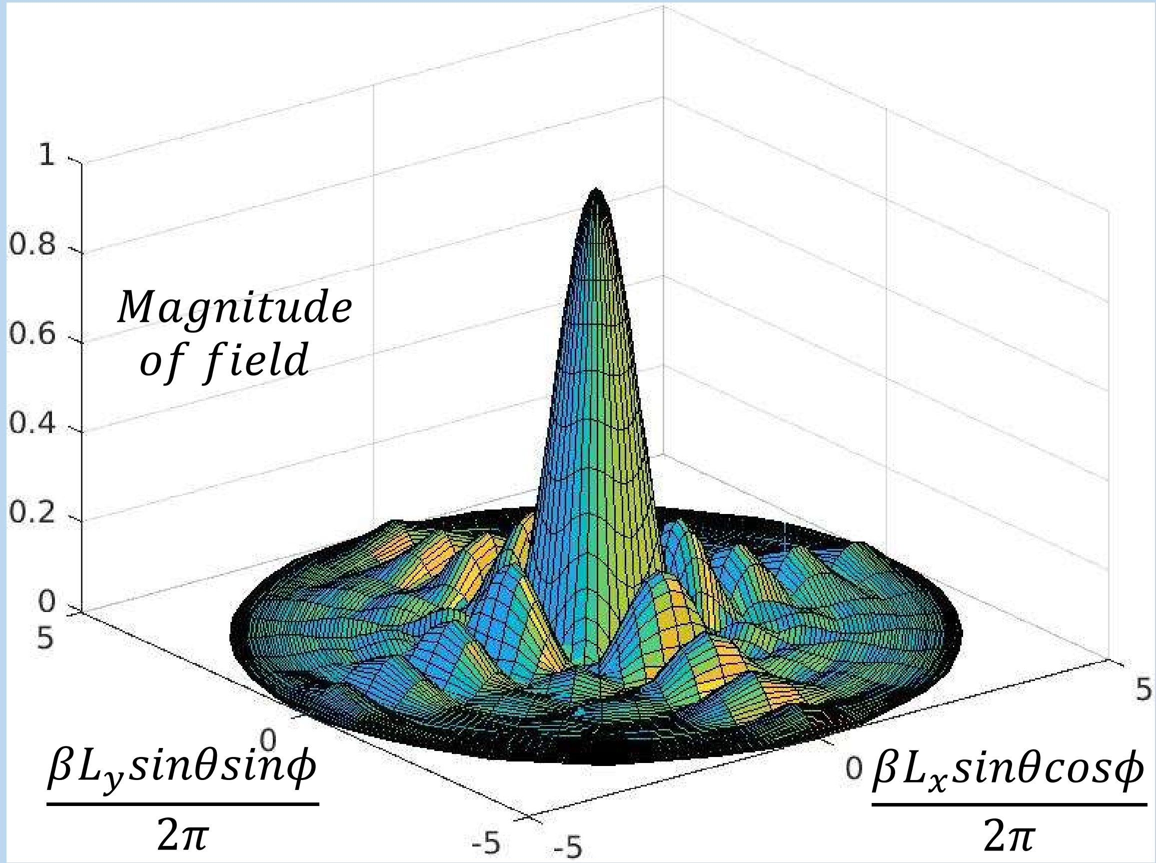
- It is a sinc function, to find the maxima
- $\frac{d}{d}\left(\frac{sinx}{x}\right) = \frac{xcosx-sinx}{x^2} = 0$
- First maxima solution: $\mathrm{x} = 0$
- is for the main beam
- Second maxima solution: $\mathrm{x} \cong 4.5$
- The value of the second maxima
- $\left|\frac{sin4.5}{4.5}\right| = 0.217$
- In dB,
- $20log(0.217) = -13.26dB$

Aperture antennas

Fig. Radiation pattern of a uniform rectangular aperture of dimension $5\lambda \times 5\lambda$

$$\left(-\frac{L_x}{\lambda} \leq \frac{\beta L_x sin\theta cos\phi}{2\pi} \leq \frac{L_x}{\lambda}, -\frac{L_y}{\lambda} \leq \frac{\beta L_y sin\theta sin\phi}{2\pi} \leq \frac{L_y}{\lambda}\right)$$

$$\left(0 \leq \theta \leq \frac{\pi}{2}, 0 \leq \phi \leq 2\pi\right)$$



$$\frac{1}{L_x^2}\left(\frac{\beta L_x sin\theta cos\phi}{2\pi}\right)^2 + \frac{1}{L_y^2}\left(\frac{\beta L_y sin\theta sin\phi}{2\pi}\right)^2 \leq \frac{1}{\lambda^2}$$

Aperture antennas

- **Uniform circular aperture**
- $\vec{E}_a = \begin{cases} E_0\hat{y}, & \rho' \leq a \\ 0, & otherwise \end{cases}$
- Let us find the $2-\mathrm{D}$ FT
  - of the aperture field,
- $\tilde{E}_a(\beta_x, \beta_y)$
- $= \iint E_a(x', y') e^{(j\beta_x x' + j\beta_y y')} dx' dy'$

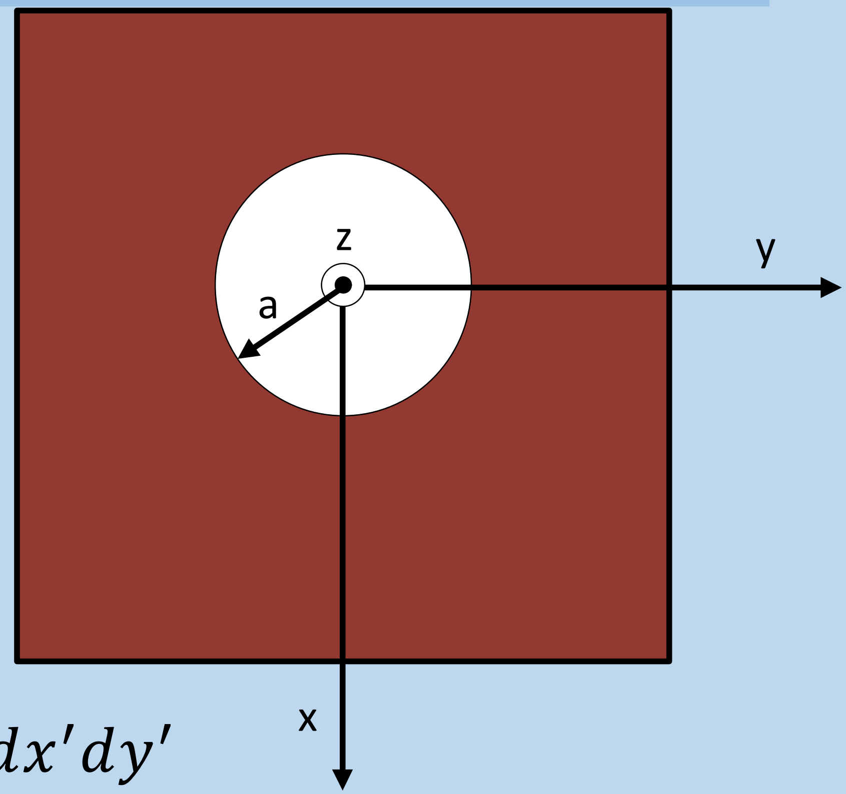


Fig. Circular aperture of radius a in x-y plane

Aperture antennas

- Better to change to
  - spherical coordinates
- $x' = r' sin\theta' cos\phi'$ and $y' = r' sin\theta' sin\phi'$
- Since aperture is in x-y plane, we have, $\theta' = \frac{\pi}{2}$, hence
- $x' = r' cos\phi'$ and $y' = r' sin\phi'$
- We know that,
- $\beta_x = \beta sin\theta cos\phi$ and $\beta_y = \beta sin\theta sin\phi$
- Hence,
- $\beta_x x' + \beta_y y' = \beta r' sin\theta cos\phi cos\phi' + \beta r' sin\theta sin\phi sin\phi'$
  $= \beta r' sin\theta cos(\phi - \phi')$

Aperture antennas

- We also know that area element in x-y plane $dx'dy' = d\phi' r' dr'$
- Hence
- $\tilde{E}_a(\beta sin\theta cos\phi, \beta sin\theta sin\phi) = \int_0^a \int_0^{2\pi} E_a(r', \phi') e^{j\beta r' sin\theta cos(\phi-\phi')} d\phi' r' dr'$
- For aperture field distribution with circular symmetry (no field variation w.r.t. ϕ'),
- $E_a(r', \phi') = E_a(r')$
- Then
- $\tilde{E}_a(\theta, \phi) = \int_0^a \left(\int_0^{2\pi} e^{j\beta r' sin\theta cos(\phi-\phi')} d\phi' \right) E_a(r') r' dr'$

Aperture antennas

- Note that the inner integral looks like Bessel's integral
- and gives 2π times the Bessel's function of first kind
- and of zero order
- Hence,
- $\tilde{E}_a(\theta,\phi) = \int_0^a 2\pi J_0(\beta r' sin\theta) E_a(r') r' dr'$
- Recall that J_0 is like a cosine function
- but with irregular roots and amplitude taper

Aperture antennas

- For uniform distribution,
- $E_a(r') = E_0$ along y direction
- $\tilde{E}_{ay}(\theta, \phi) = 2\pi E_0 \int_0^a r' J_0(\beta r' sin\theta) dr'$
- Using the Bessel's identity
- $\int x J_0(x)\, dx = x J_1(x)$
- Recall that J_1 is like a sine function
  - but with irregular roots and amplitude taper

Aperture antennas

- Substitute $\beta r' sin\theta = \xi, d\xi = \beta sin\theta dr'$,
- We have,
- $\tilde{E}_{ay}(\theta, \phi) = 2\pi E_0 \int_0^{\beta a sin\theta} \frac{\xi}{\beta sin\theta} J_0(\xi) \frac{d\xi}{\beta sin\theta}$

$$= 2\pi E_0 \frac{\beta a sin\theta J_1(\beta a sin\theta)}{(\beta sin\theta)^2} = 2\pi E_0 \frac{a J_1(\beta a sin\theta)}{\beta sin\theta}$$

- From (24.1) and (24.2), we have,
- $E_{ff\theta} \cong 2 \frac{j\beta e^{(-j\beta r)}}{4\pi r} \left[\left(\tilde{E}_{ay} sin\phi \right) \right]$
- Similarly, $E_{ff\phi} \cong 2 \frac{j\beta e^{(-j\beta r)}}{4\pi r} \left[\left(\tilde{E}_{ay} cos\theta cos\phi \right) \right]$

Aperture antennas

- Hence,
- $E_{ff\theta} \cong \frac{j\beta e^{(-j\beta r)}}{r}\left[\left(E_0 \frac{aJ_1(\beta a sin\theta)}{\beta sin\theta} sin\phi\right)\right]$
- Similarly, $E_{ff\phi} \cong \frac{j\beta e^{(-j\beta r)}}{r}\left[\left(E_0 \frac{aJ_1(\beta a sin\theta)}{\beta sin\theta} cos\theta cos\phi\right)\right]$
- $D = \frac{4\pi}{\lambda^2} \frac{\left|\iint_{S_a} \vec{E}_a ds'\right|^2}{\iint_{S_a} \left|\vec{E}_a\right|^2 ds'}$
- $\vec{E}_a = \begin{cases} E_0 \hat{y}, & \rho' \leq a \\ 0, & otherwise \end{cases}$
- Hence $D = \frac{4\pi^2 a^2}{\lambda^2}$

Aperture antennas

- The 3-dB pattern point in E-plane ($\phi = \frac{\pi}{2}$ or y-z plane) is
- $E_{ff\theta} \cong \frac{j\beta e^{(-j\beta r)}}{r}\left[\left(E_0 \frac{aJ_1(\beta a sin\theta)}{\beta sin\theta}\right)\right]$
- Similarly, $E_{ff\phi} \cong 0$
- $\beta a sin\theta_1 = 1.6162$
- Or, $sin\theta_1 = \frac{1.6162}{\beta a} = \frac{1.6162\lambda}{\pi 2a} = \frac{1.6162\lambda}{\pi D} = 0.5145\frac{\lambda}{D}$
- Hence, $HPBW_{E-plane} = 2sin^{-1}\left(0.5145\frac{\lambda}{D}\right) rad = 114.59 sin^{-1}\left(0.5145\frac{\lambda}{D}\right) degrees$
- where D is diameter

Aperture antennas

- For BWFN, the first zero or null point is
- $\beta a sin\theta_{null} = 3.8317$
- $BWFN_{E-plane} = 2sin^{-1}\left(\frac{3.8317\lambda}{\pi D}\right) = 2sin^{-1}\left(\frac{1.2197\lambda}{D}\right) rad$

$$= 114.592 sin^{-1}\left(\frac{1.2197\lambda}{D}\right)\text{degrees}$$

Aperture antennas

- Fig. Radiation pattern of a uniform circular aperture of $a = 5\lambda$

(please note SLL is 0.1323 or -17.56 dB)