# EE540 Advance Electromagnetic Theory & Antennas

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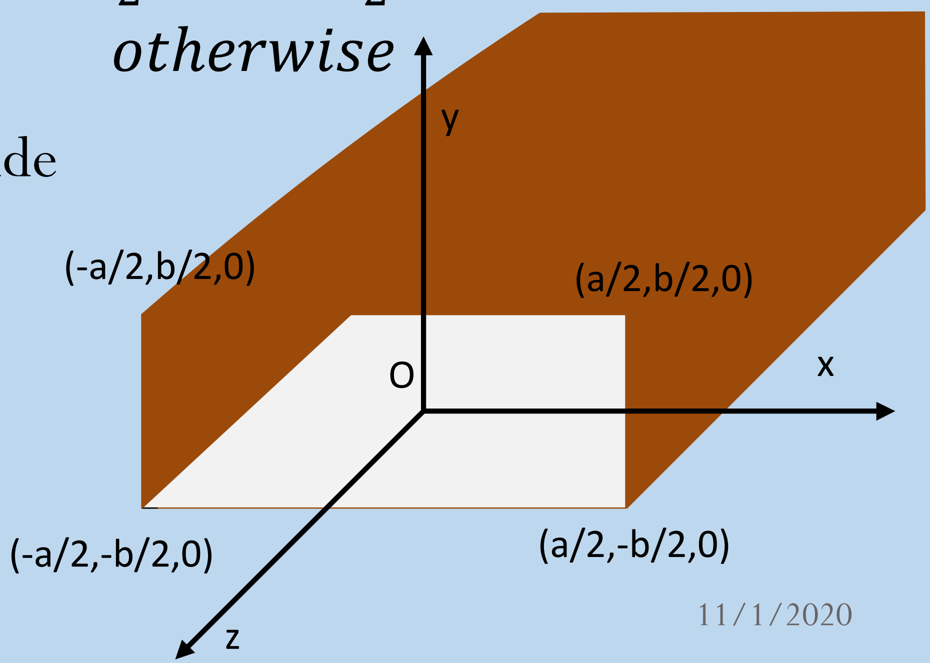
# Aperture antennas

- **Radiation from open ended waveguide**
- For dominant $TE_{10}$ mode of rectangular waveguide, aperture fields are
- $$\vec{E}_a = E_{ay}\hat{y} = \begin{cases} E_0 cos\frac{\pi x'}{a}\hat{y}, & |x'| \le \frac{a}{2}, |y'| \le \frac{b}{2}, z = 0 \\ 0, & otherwise \end{cases}$$
- Fig. Open ended rectangular waveguide



# Aperture antennas

- Let us find the 2 – D FT
  - of the aperture field,
- $\tilde{E}(\beta_x, \beta_y)$
- $= \iint E(x', y') e^{(j\beta_x x' + j\beta_y y')} dx' dy'$
- $\tilde{E}_{ay}(\beta_x, \beta_y) = E_0 \int_{-\frac{a}{2}}^{\frac{a}{2}} cos \frac{\pi x'}{a} e^{(j\beta_x x')} dx' \int_{-\frac{b}{2}}^{\frac{b}{2}} e^{(j\beta_y y')} dy'$
- Note that $\int_{-\frac{b}{2}}^{\frac{b}{2}} e^{(j\beta_y y')} dy' = \frac{e^{\left(j\beta_y \frac{b}{2}\right)} - e^{\left(-j\beta_y \frac{b}{2}\right)}}{j\beta_y} =$

$\frac{2}{\beta_y} \frac{e^{\left(j\beta_y \frac{b}{2}\right)} - e^{\left(-j\beta_y \frac{b}{2}\right)}}{2j\beta_y} = \frac{2}{\beta_y} \sin\left(\beta_y \frac{b}{2}\right) = b \frac{\sin\left(\beta_y \frac{b}{2}\right)}{\beta_y \frac{b}{2}} = bsinc\left(\beta_y \frac{b}{2}\right)$

# Aperture antennas

- Also
- $\int_{-\frac{a}{2}}^{\frac{a}{2}} cos\frac{\pi x'}{a} e^{(j\beta_x x')} dx' = 2\pi a \frac{cos\left(\frac{\beta_x a}{2}\right)}{\pi^2 - (\beta_x a)^2}$
- Hence,
- $\tilde{E}_{ay}(\beta_x, \beta_y) = 2\pi ab E_0 sinc\left(\beta_y \frac{b}{2}\right) \frac{cos\left(\frac{\beta_x a}{2}\right)}{\pi^2 - (\beta_x a)^2}$
- Hence, from equations (24.1) and (24.2), we have,
- $E_{ff\theta} \cong 2 \frac{j\beta e^{(-j\beta r)}}{4\pi r} \left[\left(\tilde{E}_{ay} sin\phi\right)\right]$

  and $E_{ff\phi} \cong 2 \frac{j\beta e^{(-j\beta r)}}{4\pi r} \left[\left(\tilde{E}_{ay} cos\theta cos\phi\right)\right]$

# Aperture antennas

- $\tilde{E}_{ay}(\beta_x, \beta_y) = 2\pi ab E_0 sinc\left(\beta_y \frac{b}{2}\right) \frac{cos\left(\frac{\beta_x a}{2}\right)}{\pi^2 - (\beta_x a)^2}$
- Hence, from equations (24.1) and (24.2), we have,
- $E_{ff\theta} \cong \frac{j\beta e^{(-j\beta r)}}{r}\left[\left(abE_0 sinc\left(\beta_y \frac{b}{2}\right) \frac{cos\left(\frac{\beta_x a}{2}\right)}{\pi^2 - (\beta_x a)^2} sin\phi\right)\right]$

  and $E_{ff\phi} \cong \frac{j\beta e^{(-j\beta r)}}{r}\left[\left(abE_0 sinc\left(\beta_y \frac{b}{2}\right) \frac{cos\left(\frac{\beta_x a}{2}\right)}{\pi^2 - (\beta_x a)^2} cos\theta cos\phi\right)\right]$
- where $\beta_x = \beta sin\theta cos\phi$ and $\beta_y = \beta sin\theta sin\phi$

# Aperture antennas

- E-plane ($\phi = \frac{\pi}{2}$ or y-z plane)
- $\beta_x = 0$ and $\beta_y = \beta sin\theta$
- $E_{ff\theta} \cong \frac{j\beta e^{(-j\beta r)}}{r}\left[\left(abE_0 \frac{1}{\pi^2} sinc\left(\beta \frac{b}{2} sin\theta\right)\right)\right]$ and $E_{ff\phi} \cong 0$
- SLL is -13.2 dB
- H-plane ($\phi = 0$ or x-z plane)
- $\beta_x = \beta sin\theta$ and $\beta_y = 0$
- $E_{ff\theta} \cong 0$ and $E_{ff\phi} \cong \frac{j\beta e^{(-j\beta r)}}{r}\left[\left(abE_0 \frac{cos\left(\frac{\beta a sin\theta}{2}\right)}{\pi^2 - (\beta sin\theta a)^2} cos\theta\right)\right]$
- SLL is -23 dB and H-plane pattern is close to $cos\theta$

# Aperture antennas

- Fig. E-plane and H-plane

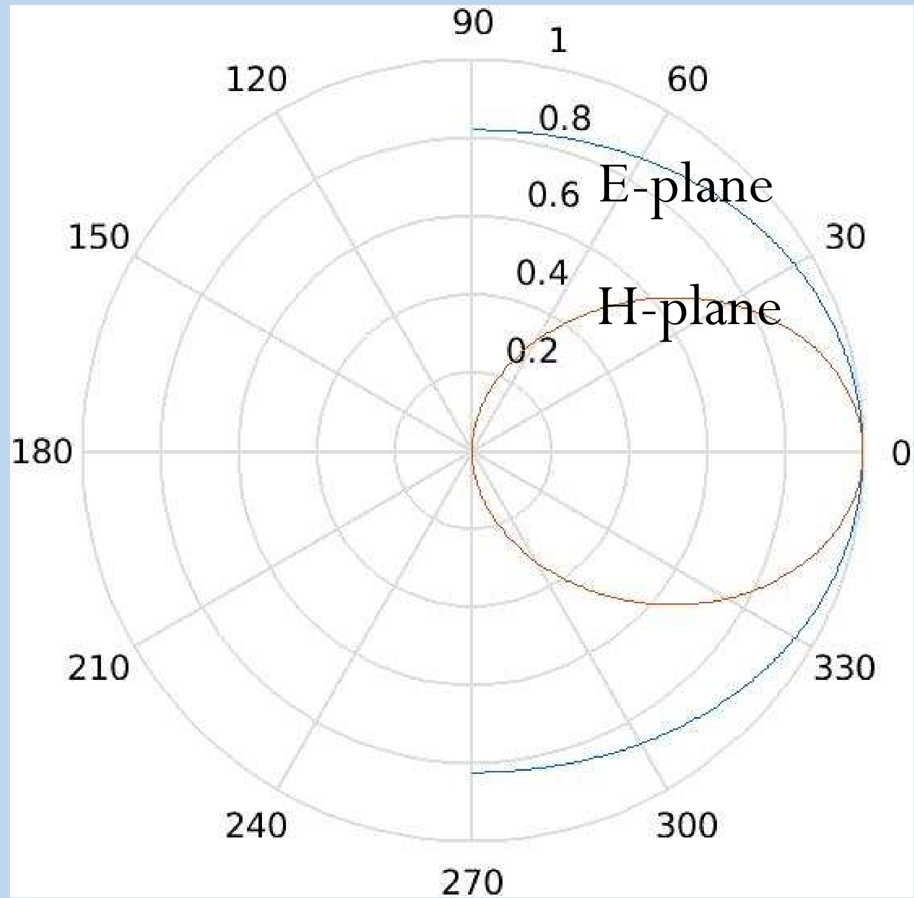
of open ended X-band waveguide at 10 GHz

E-plane

$$E_{ff\theta} \cong \frac{j\beta e^{(-j\beta r)}}{r}\left[\left(abE_0\frac{1}{\pi^2} sinc\left(\beta\frac{b}{2}sin\theta\right)\right)\right]$$

H-plane

$$E_{ff\phi} \cong \frac{j\beta e^{(-j\beta r)}}{r}\left[\left(abE_0\frac{cos\left(\frac{\beta a sin\theta}{2}\right)}{\pi^2-(\beta sin\theta a)^2}cos\theta\right)\right]$$



# Aperture antennas

- $D = \frac{4\pi}{\lambda^2} \frac{\left| \iint_{S_a} \vec{E}_a ds' \right|^2}{\iint_{S_a} \left| \vec{E}_a \right|^2 ds'} = \frac{4\pi}{\lambda^2} \frac{\left| \int_{-\frac{a}{2}}^{\frac{a}{2}} E_0 cos\frac{\pi x'}{a} dx' \int_{-\frac{b}{2}}^{\frac{b}{2}} dy' \right|^2}{\int_{-\frac{a}{2}}^{\frac{a}{2}} \left| E_0 cos\frac{\pi x'}{a} \right|^2 dx' \int_{-\frac{b}{2}}^{\frac{b}{2}} dy'}$
- $= \frac{4\pi}{\lambda^2} \frac{\left\{ E_0 b \frac{\left( sin\frac{\pi}{2} + sin\frac{\pi}{2} \right)}{\frac{\pi}{a}} \right\}^2}{(E_0)^2 \left( \frac{1}{2} a + \frac{1}{2} \frac{(sin\pi + sin\pi)}{\frac{2\pi}{a}} \right) b} = \frac{4\pi}{\lambda^2} \frac{8ab}{\{\pi\}^2} = \frac{4\pi}{\lambda^2} \frac{8ab}{\{\pi\}^2} \cong \frac{4\pi}{\lambda^2} (0.81ab)$
- It can be observed that aperture efficiency is reduced to 81% w.r.t. uniform rectangular aperture case
- Directivity is so small, needs to flare out the waveguide to create horn antennas

# Aperture antennas

- **Open ended circular waveguide**
- Aperture field for dominant TE11 mode
- $\vec{E}_a = \begin{cases} E_{a\rho}\hat{\rho} + E_{a\phi}\hat{\phi}, & \rho' \le a \\ 0, & otherwise \end{cases}$
- where $E_{a\rho} = E_0 J_1\left(\frac{1.841\rho'}{a}\right)\frac{sin\phi'}{\rho'}, E_{a\phi} = E_0 \frac{\partial J_1\left(\frac{1.841\rho'}{a}\right)}{\partial \rho'} cos\phi'$
- $p'_{nm} = \chi'_{11} = 1.841$ {(m$^{th}$ root of the derivative of the Bessel's function) and n (order of the Bessel's function)}
- Denote $\frac{\partial J_1\left(\frac{1.841\rho'}{a}\right)}{\partial \rho'} = J'_1\left(\frac{1.841\rho'}{a}\right)$

# Aperture antennas

- We need FT of aperture fields ($E_{ax}$ and $E_{ay}$) in equations {(24.1) and (24.2)} for finding far fields of aperture antenna
- Coordinate transformation from Cylindrical to Cartesian
- $$\begin{bmatrix} E_{ax} \\ E_{ay} \\ E_{az} \end{bmatrix} = \begin{bmatrix} cos\phi' & -sin\phi' & 0 \\ sin\phi' & cos\phi' & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} E_{a\rho} = E_0 J_1\left(\frac{1.841\rho'}{a}\right)\frac{sin\phi'}{\rho'} \\ E_{a\phi} = E_0 J_1'\left(\frac{1.841\rho'}{a}\right) cos\phi' \\ E_{az} = 0 \end{bmatrix}$$
- $$E_{ax} = \frac{E_0 J_1\left(\frac{1.841\ '}{a}\right)}{2\rho'} \boxed{sin2\phi'} - \frac{E_0}{2} J_1'\left(\frac{1.841\rho'}{a}\right)\boxed{sin2\phi'}$$
- $$E_{ay} = \frac{E_0 J_1\left(\frac{1.841\ '}{a}\right)}{\rho'} \boxed{sin^2\phi'} + E_0 J_1'\left(\frac{1.841\rho'}{a}\right)\boxed{cos^2\phi'}$$

# Aperture antennas

- Let us find the 2 – D FT
  - of the aperture field,
- $\tilde{E}_{ax}(\theta,\phi) =$

$$\int_0^a \int_0^{2\pi} \left\{ \frac{E_0 J_1\left(\frac{1.841\rho'}{a}\right)}{2\rho'} - \frac{E_0}{2} J_1'\left(\frac{1.841\rho'}{a}\right) \right\} sin2\phi' e^{j\beta\rho' sin\theta cos(\phi-\phi')} d\phi' \rho' d\rho'$$

- Note that
- $I_1 = \int_0^{2\pi} sin2\phi' e^{j\beta\rho' sin\theta cos(\phi-\phi')} d\phi' =$

$$\int_0^{2\pi} \frac{e^{2j\ '} - e^{-2j\phi'}}{2j} e^{j\beta\rho' sin\theta cos(\phi-\phi')} d\phi'$$

# Aperture antennas

- Put $\phi' - \phi = \zeta, d\phi' = d\zeta$
- $I_1 = \int_0^{2\pi} sin2\phi' e^{j\beta\rho' sin\theta cos(\phi-\phi')} d\phi' =$
  $\frac{e^{2j\phi}}{2j} \int_{-\phi}^{2\pi-\phi} e^{j\beta\rho' sin\theta cos\zeta + j2\zeta} d\zeta - \frac{e^{-2j\phi}}{2j} \int_{-\phi}^{2\pi-\phi} e^{j\beta\rho' sin\theta cos\zeta - j2\zeta} d\zeta$
- Since, $2\pi j^n J_n(x) = \int_0^{2\pi} e^{jxcos\psi}\, e^{jn\psi} d\psi$, we have, $n = \pm 2, x = \beta\rho' sin\theta, \psi = \zeta$
- $I_1 = -\frac{e^{2j\phi}}{2j} 2\pi J_2(\beta\rho' sin\theta) + \frac{e^{-2j\phi}}{2j} 2\pi J_{-2}(\beta\rho' sin\theta)$
- Also note that $J_{-n}(z) = (-1)^n J_n(z)$
- $I_1 = -2\pi sin(2\phi) J_2(\beta\rho' sin\theta)$

# Aperture antennas

- $\tilde{E}_{ax}(\theta,\phi) =$

$$-E_0\pi sin(2\phi)\int_0^a\left\{\frac{J_1\left(\frac{1.841\rho'}{a}\right)}{\rho'}-J_1'\left(\frac{1.841\rho'}{a}\right)\right\}\rho' J_2(\beta\rho' sin\theta)d\rho'$$

- Use Bessel identity I: $\alpha J_{p+1}(\alpha x)=\frac{p}{x}J_p(\alpha x)-\frac{dJ_p(\alpha x)}{dx}$
- Take $\alpha=\frac{1.841}{a}, x=\rho', p=1$ and we have, $\tilde{E}_{ax}(\theta,\phi)=$

$$-E_0\pi sin(2\phi)\frac{1.841}{a}\int_0^a \rho' J_2(\beta\rho' sin\theta)\left\{J_2\left(\frac{1.841\rho'}{a}\right)\right\}d\rho'$$

- Note that $\int xJ_n(\alpha x)J_n(\gamma x)dx=\frac{\gamma xJ_n(\alpha x)J_{n-1}(\gamma x)-\alpha xJ_{n-1}(\alpha x)J_n(\gamma x)}{\alpha^2-\gamma^2}$
- In this case, take $\alpha=\beta sin\theta, x=\rho', \gamma=\frac{1.841}{a}, n=2$, hence

$$\int_0^a \rho' J_2(\beta sin\theta\rho')\left\{J_2\left(\frac{1.841\rho'}{a}\right)\right\}d\rho'=$$

$$\frac{1.841J_2(\beta asin\theta)J_1(1.841)-\beta asin\theta J_1(\beta asin\theta)J_2(1.841)}{(\beta sin\theta)^2-\left(\frac{1.841}{a}\right)^2}$$

# Aperture antennas

- Use the Bessel identity I for $\alpha = 1$: $J_{p+1}(x) = \frac{p}{x}J_p(x) - \frac{dJ_p(x)}{dx}$
- Take $x = 1.841 = \chi'_{11}, p = 1$
- Since, $J_2(1.841) = \frac{J_1(1.841)}{1.841} - J'_1(1.841) = \frac{J_1(1.841)}{1.841}$,
- Hence, $J_1(1.841) = 1.841 J_2(1.841)$
- Therefore,
- $\int_0^a \rho' J_2(\beta sin\theta\rho')\left\{J_2\left(\frac{1.841\rho'}{a}\right)\right\}d\rho' =$

$$-\frac{a^2 J_2(1.841)}{1.841^2}\left\{\frac{1.841^2 J_2(\beta asin\theta) - \beta asin\theta J_1(\beta asin\theta)}{1 - \left(\frac{\beta asin\theta}{1.841}\right)^2}\right\}$$

# Aperture antennas

- $\tilde{E}_{ax}(\theta,\phi) =$
$E_0 2\pi a sin(\phi) cos(\phi) \frac{J_1(1.841)}{1.841^2} \left\{ \frac{1.841^2 J_2(\beta a sin\theta) - \beta a sin\theta J_1(\beta a sin\theta)}{1-\left(\frac{\beta a sin\theta}{1.841}\right)^2} \right\}$
- Similarly,
- $\tilde{E}_{ay}(\theta,\phi) =$
$\int_0^a \int_0^{2\pi} \left\{ E_0 J_1\left(\frac{1.841\rho'}{a}\right) \frac{sin^2\phi'}{\rho'} + E_0 J_1'\left(\frac{1.841\rho'}{a}\right) cos^2\phi' \right\} sin2\phi' e^{j\beta\rho' sin\theta cos(\phi-\phi')} d\phi' \rho' d\rho'$

$$= E_0 2\pi a \frac{J_1(1.841)}{1.841^2} \left\{ \frac{sin^2\phi J_1(\beta a sin\theta)\left(\beta a sin\theta - \frac{1.841^2}{\beta a sin\theta}\right) - cos^2\phi 1.841^2 J_1'(\beta a sin\theta)}{1-\left(\frac{\beta a sin\theta}{1.841}\right)^2} \right\}$$

# Aperture antennas

- Using equation (24.1) and (24.2), we have far fields of open ended circular waveguide
- $E_{ff\theta} \cong \frac{2j\beta e^{(-j\beta r)}}{4\pi r}\left[\left(\tilde{E}_{ay} sin\phi + \tilde{E}_{ax} cos\phi\right)\right]$
- and $E_{ff\phi} \cong \frac{2j\beta e^{(-j\beta r)}}{4\pi r}\left[\left(\tilde{E}_{ay} cos\theta cos\phi - \tilde{E}_{ax} cos\theta sin\phi\right)\right]$
- It can be shown that
- $\tilde{E}_{ay} sin\phi + \tilde{E}_{ax} cos\phi = E_0 2\pi a J_1(1.841) sin(\phi) \frac{J_1(\beta a sin\theta)}{\beta a sin\theta}$
- $\tilde{E}_{ay} cos\theta cos\phi - \tilde{E}_{ax} cos\theta sin\phi =$

  $E_0 2\pi a J_1(1.841) cos(\theta) cos(\phi) \frac{J_1'(\beta a sin\theta)}{1-\left(\frac{\beta a sin\theta}{1.841}\right)^2}$

# Aperture antennas

- Using equation (24.1) and (24.2), we have far fields of open ended circular waveguide
- $E_{ff\theta} \cong \frac{j\beta E_0 a J_1(1.841)e^{(-j\beta r)}}{r}\left[\left(\frac{J_1(\beta a sin\theta)}{\beta a sin\theta}\right) sin(\phi)\right]$
- and $E_{ff\phi} \cong \frac{j\beta E_0 a J_1(1.841)e^{(-j\beta r)}}{r}\left[\left(cos(\theta)\frac{J_1'(\beta a sin\theta)}{1-\left(\frac{\beta a sin\theta}{1.841}\right)^2}\right) cos(\phi)\right]$
- For E-plane (y-z plane and $\phi = \frac{\pi}{2}$)

$E_{ff\theta} \cong \frac{j\beta E_0 a J_1(1.841)e^{(-j\beta r)}}{r}\left[\left(\frac{J_1(\beta a sin\theta)}{\beta a sin\theta}\right)\right]$ and $E_{ff\phi} \cong 0$

# Aperture antennas

- The 3-dB pattern point in E-plane ($\phi = \frac{\pi}{2}$ or y-z plane) is
- $\beta a sin\theta_1 = 1.6162$
- Or, $sin\theta_1 = \frac{1.6162}{\beta a} = \frac{1.6162\lambda}{\pi 2a} = \frac{1.6162\lambda}{\pi D} = 0.5145\frac{\lambda}{D}$
- Hence, $HPBW_{E-plane} = 2sin^{-1}\left(0.5145\frac{\lambda}{D}\right) rad = 114.59sin^{-1}\left(0.5145\frac{\lambda}{D}\right) degrees$
- where D is diameter

# Aperture antennas

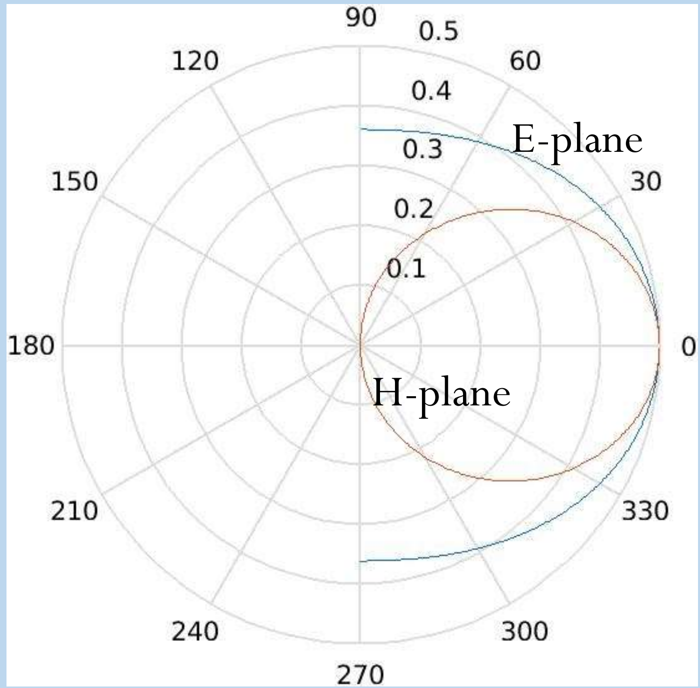
- SLL is -17.6 dB (please note that it is first order Bessel's function of first kind not sinc function)
- For BWFN, the first zero or null point is
- $\beta a sin\theta_{null} = 3.8317$
- $BWFN_{E-plane} = 2sin^{-1}\left(\frac{3.8317\lambda}{\pi D}\right) = 2sin^{-1}\left(\frac{1.2197\lambda}{D}\right) rad$

  $= 114.592 sin^{-1}\left(\frac{1.2197\lambda}{D}\right)$degrees
- Directivity
- $D = 0.836\frac{4\pi}{\lambda^2}(\pi a^2) = 0.836\left(\frac{2\pi a}{\lambda}\right)^2$

# Aperture antennas

- Fig. E-plane and H-plane

of open ended circular waveguide at 15 GHz




E-plane

$$E_{ff\theta} \cong \frac{j\beta E_0 a J_1(1.841)e^{(-j\beta r)}}{r}\left[\left(\frac{J_1(\beta a sin\theta)}{\beta a sin\theta}\right)\right]$$

H-plane

$$E_{ff\phi} \cong \frac{j\beta E_0 a J_1(1.841)e^{(-j\beta r)}}{r}\left[\left(cos(\theta)\frac{J_1'(\beta a sin\theta)}{1-\left(\frac{\beta a sin\theta}{1.841}\right)^2}\right)\right]$$