EE540 Advance Electromagnetic Theory & Antennas

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Aperture antennas

- **Motivation for Horn antennas:**
- *First motivation: Impedance matching*
- Consider a hollow rectangular waveguide operating in dominant TE_{10} mode
- What is the wave impedance?
- $Z_{TE} = \frac{\eta_0}{\sqrt{1-\left(\frac{f_c}{f}\right)^2}}$
- What is cut-off frequency for the dominant TE_{10} mode inside rectangular waveguide?

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- $f_c = \frac{1}{2a\sqrt{\mu\varepsilon}} = \frac{c}{2a}$
- For X-band waveguide with a=22.86 mm and b=10.16 mm
- $f_c = 6.56$ GHz
- Therefore around 10 GHz,
- $Z_{TE} = 499\ \Omega$
- But the wave impedance for free space is around 377 Ω
- So reflection coefficient is
- $\Gamma = \frac{377-499}{377+499} = -.139$

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- If we increase the a five times, then
- a=114.3mm
- What is new cut-off frequency for the dominant TE_{10} modes inside this new rectangular waveguide?
- $f_c = \frac{1}{2a\sqrt{\mu\varepsilon}} = \frac{c}{2a}$
- $f_c = 1.311 \text{ GHz}$
- Therefore around 10 GHz, $Z_{TE} = 380\ \Omega$
- Hence, reflection coefficient for this case is
- $\Gamma = \frac{377-380}{377+380} = -.00396$
- Impedance matching is much better in this case

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- *Second motivation:*
- Directivity enhancements:
- Directivity of open ended waveguide is given by
- $D \cong \frac{4\pi}{\lambda^2}(0.81ab)$
- What if we increase a and b (dimensions of the rectangular waveguide)?
- Say, if we double both a and b,
  - directivity will increase fourfold

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- *Third motivation:*
- Narrower beam:

E-plane

$$E_{ff\theta} \cong \frac{j\beta e^{(-j\beta r)}}{r}\left[\left(abE_0\frac{1}{\pi^2}sinc\left(\beta\frac{b}{2}sin\theta\right)\right)\right]$$

- For example, E-plane has sinc
- function and its argument
- $\beta\frac{b}{2}sin\theta$
- What will happen b becomes five times longer?

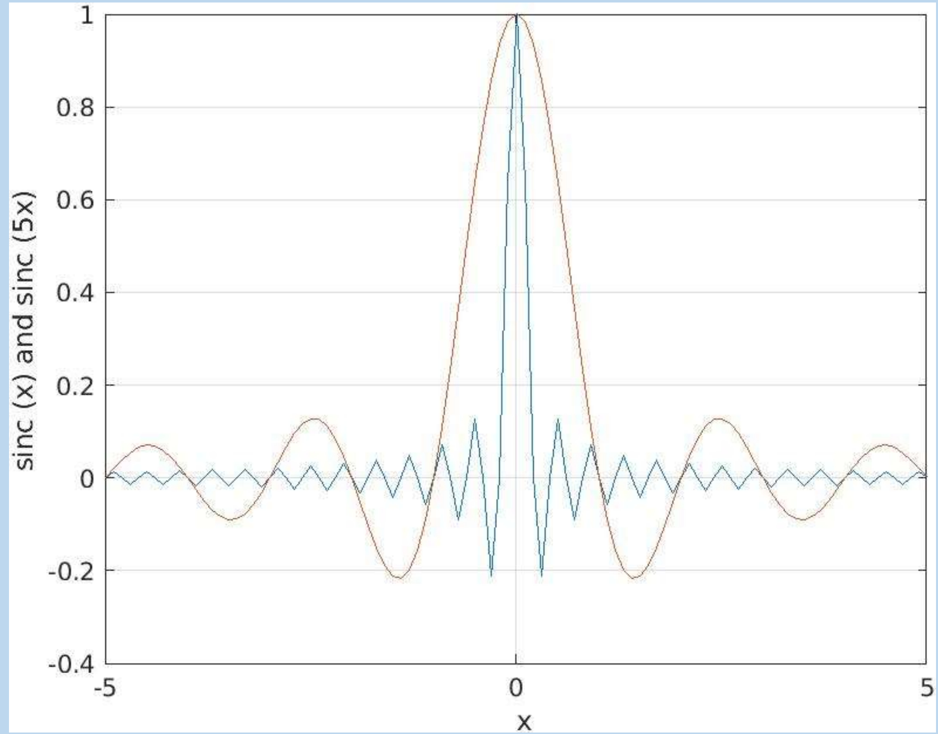

Fig. Sinc (5x) and Sinc (x) vs x

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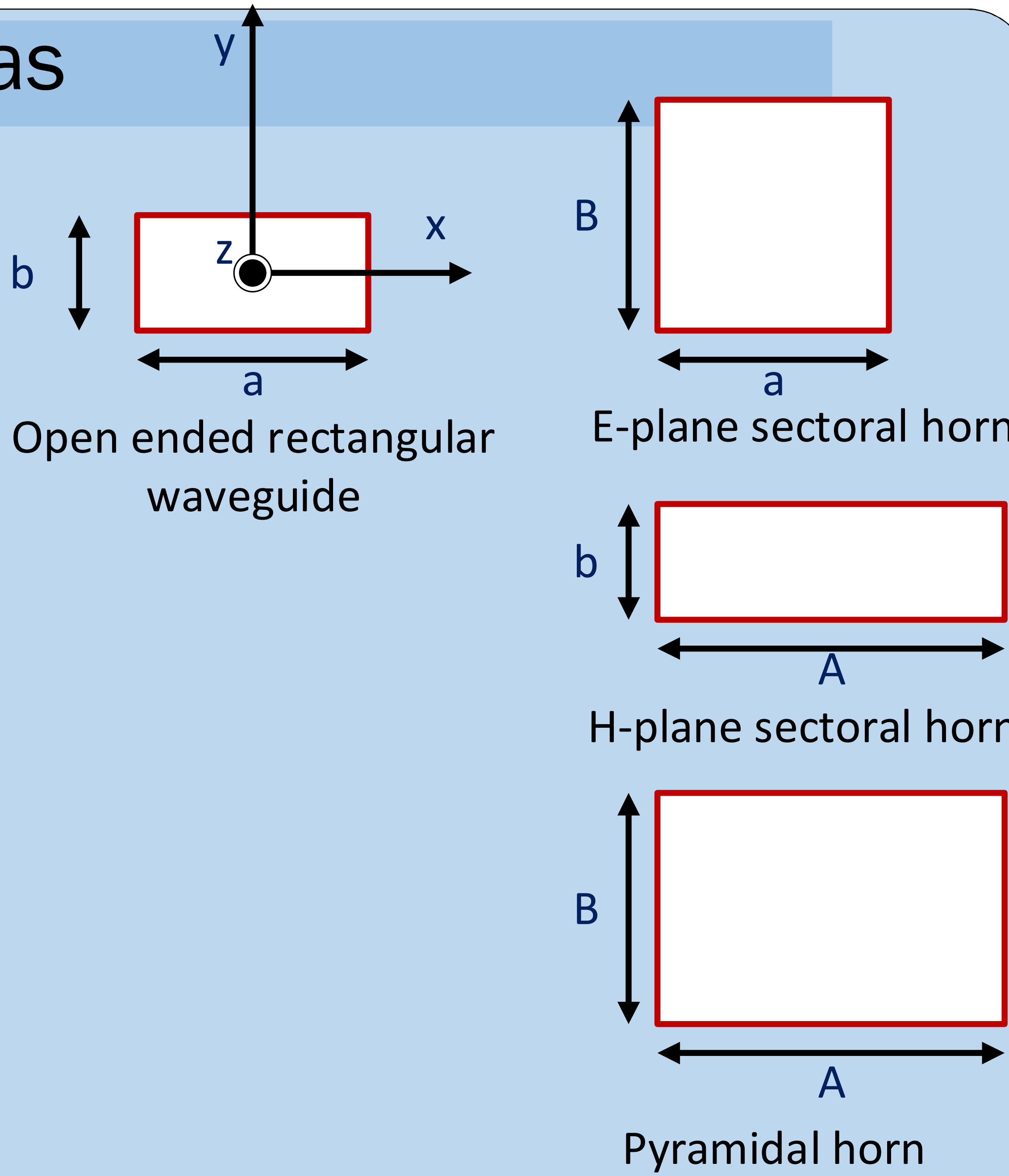
- Fig. Open ended waveguide to horn antenna (front view)



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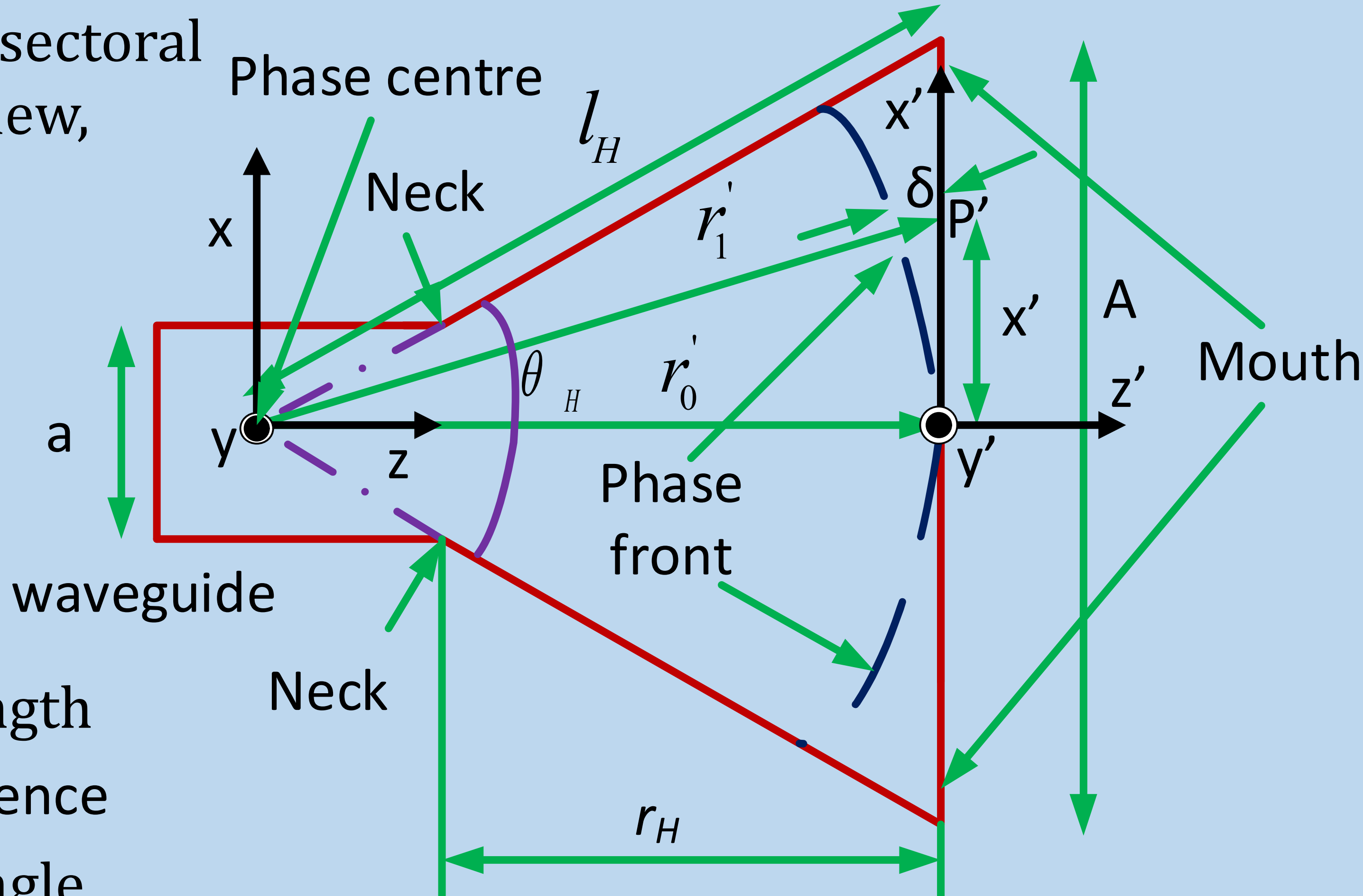
- Fig. H-plane sectoral horn (side view,
- x-z cut)



H-plane sectoral horn antenna

- l_H flaring length
- δ path difference
- θ_H flaring angle
- r_0' distance of phase centre and centre of mouth of horn

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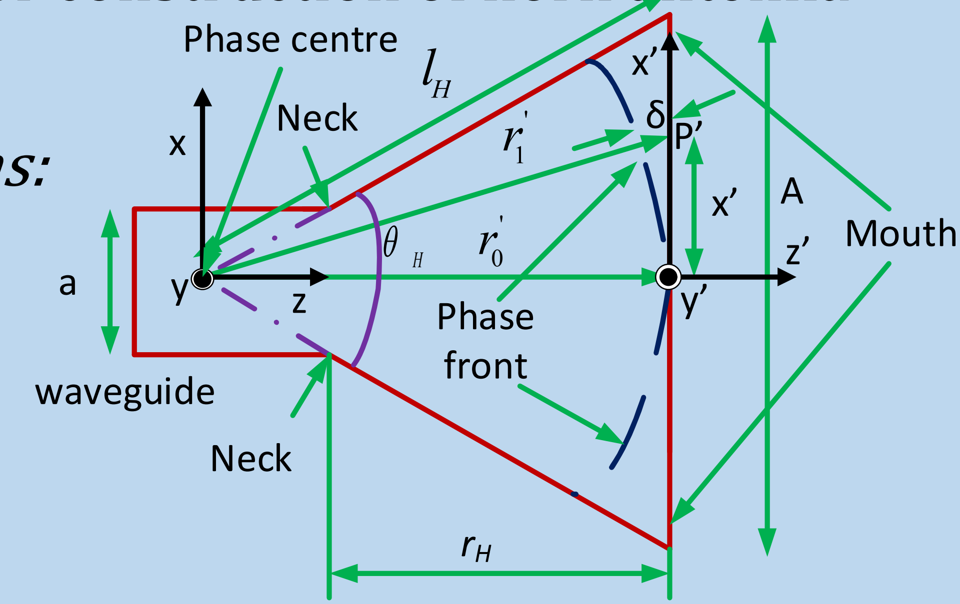
- Two dimensions required for construction of horn antenna
- A and r_H
- *Simple geometrical relations:*
- Flare angle
- $\theta_H = 2tan^{-1}\left(\frac{A}{2r_0'}\right)$
- Flaring length
- $l_H^2 = \frac{A^2}{4} + (r_0')^2$
- Distance between neck and mouth r_H
- $r_H = (A - a)\sqrt{\left(\frac{l_H}{A}\right)^2 - \frac{1}{4}}$



H-plane sectoral horn antenna

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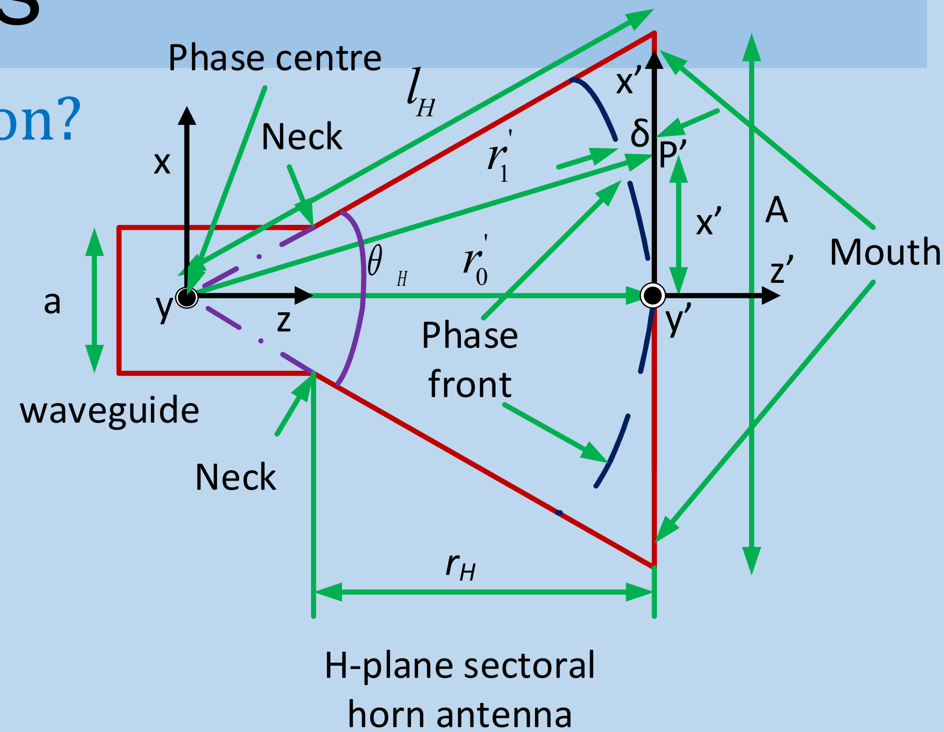
- How do you get last equation?
- Note that
- $tan\left(\frac{\theta_H}{2}\right) = \frac{A}{2r_0'} = \frac{a}{2(r_0'-r_H)}$
- Inverting and cancelling 2
- $\frac{r_0'}{A} = \frac{(r_0'-r_H)}{a}$
- Cross multiplying



- $ar_0' = Ar_0' - Ar_H \Rightarrow Ar_H = (A-a)r_0' \Rightarrow r_H = (A-a)\frac{r_0'}{A}$
- $\Rightarrow r_H = (A-a)\frac{\sqrt{l_H^2 - \frac{A^2}{4}}}{A} = (A-a)\sqrt{\left(\frac{l_H}{A}\right)^2 - \frac{1}{4}}$

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- Note that at any point P',
  - since the distance r_0' and r_1' are not same
- Hence there will be some phase change
- *How much is the phase change?*
- Let us find the difference in the distance $r_1' - r_0'$ first
- Applying the Pythagoras theorem,
- $r_1' = \sqrt{(r_0')^2 + (x')^2} = r_0'\left(1 + \left(\frac{x'}{r_0'}\right)^2\right)^{1/2} \cong r_0'\left(1 + \frac{1}{2}\left(\frac{x'}{r_0'}\right)^2\right)$
- Hence, $r_1' - r_0' \cong \frac{1}{2r_0'}(x')^2$

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- $\delta = r_1' - r_0' \cong \frac{1}{2r_0'}(x')^2$
- We can observe that $r_1' - r_0'$ has quadratic variation with x'
- Hence phase has quadratic variation
- *First step:* so we can write the aperture field variation as
- $E_{ay} = E_0 cos\left(\frac{\pi x'}{A}\right) e^{-j\beta\frac{1}{2r_0'}(x')^2}$ inside the aperture and zero elsewhere
- $E_0 cos\left(\frac{\pi x'}{A}\right)$ is the amplitude variation and
- $e^{-j\beta\frac{1}{2r_0'}(x')^2}$ is the phase variation

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Table: Amplitude and phase variation of H-sectoral horn antenna

Aperture field parameter	Expression	$x' = 0$ (centre of mouth)	$x' = -A/2$ (extreme edge of mouth)	$x' = A/2$ (extreme edge of mouth)
Amplitude	$E_0 cos\left(\frac{\pi x'}{A}\right)$	E_0	0	0
Phase	$e^{-j\beta\frac{1}{2r_0'}(x')^2}$	1	$e^{-j\beta\frac{1}{8r_0'}(A)^2}$ (maximum)	$e^{-j\beta\frac{1}{8r_0'}(A)^2}$ (maximum)

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- *Second step:*
- Take the 2-D FT of aperture field
- $\tilde{E}_{ay} = \int_{-A/2}^{A/2} E_0 cos\left(\frac{\pi x'}{A}\right) e^{-j\beta \frac{1}{2r_0'}(x')^2} e^{j\beta_x x'} dx' \int_{-b/2}^{b/2} e^{j\beta_y y'} dy'$
- $= E_0 \left\{\frac{1}{2}\sqrt{\frac{\pi r_0'}{\beta}} I(\theta, \phi)\right\} \left\{bsinc\left(\beta_y \frac{b}{2}\right)\right\}$
- where $I(\theta, \phi) = A_1(\beta_{x1})\left(f(s_2') - f(s_1')\right) + A_2(\beta_{x2})\left(f(t_2') - f(t_1')\right)$
- $A_1(\beta_{x1}) = e^{j\frac{r_0'(\beta_{x1})^2}{2\beta}}, A_2(\beta_{x2}) = e^{j\frac{r_0'(\beta_{x2})^2}{2\beta}},$
- $\beta_{x1} = \beta_x + \frac{\pi}{A}, \beta_{x2} = \beta_x - \frac{\pi}{A}$

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- $f(\xi') = C(\xi') - jS(\xi'),$
- $C(\xi') = \int_0^{\xi'} cos\left(\frac{\pi}{2}\tau^2\right)d\tau, S(\xi') = \int_0^{\xi'} sin\left(\frac{\pi}{2}\tau^2\right)d\tau$
- $s_1' = \frac{1}{\sqrt{\pi\beta r_0'}}\left(-\frac{\beta A}{2} - \beta_{x1}r_0'\right)$
- $s_2' = \frac{1}{\sqrt{\pi\beta r_0'}}\left(+\frac{\beta A}{2} - \beta_{x1}r_0'\right)$
- $t_1' = \frac{1}{\sqrt{\pi\beta r_0'}}\left(-\frac{\beta A}{2} - \beta_{x2}r_0'\right)$
- $t_2' = \frac{1}{\sqrt{\pi\beta r_0'}}\left(+\frac{\beta A}{2} - \beta_{x2}r_0'\right)$

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- *Third step:*
- Find the fields using equations (24.1) and (24.2)
- Recall for TE_{10} mode waveguide,
- $E_{ay} = E_0 cos\left(\frac{\pi x'}{A}\right) e^{-j\beta \frac{1}{2r'_0}(x')^2}$ inside the aperture and zero elsewhere
- Note that there is no infinite ground plane
- Besides, inherently, waveguide has both aperture electric and magnetic fields
- And their relation is $H_{ax} = -\frac{E_{ay}}{Z_g} \Rightarrow E_{ay} = -H_{ax} Z_g$

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- Revisiting equations (24.1) and (24.2) for this aperture antenna,
- $E_{ff\theta} \cong \frac{j\beta e^{(-j\beta r)}}{4\pi r}\left[\eta\left(\tilde{H}_{ay}cos\theta cos\phi - \tilde{H}_{ax}cos\theta sin\phi\right) + \left(\tilde{E}_{ay}sin\phi + \tilde{E}_{ax}cos\phi\right)\right]$ ---(24.1)
- $E_{ff\phi} \cong \frac{j\beta e^{(-j\beta r)}}{4\pi r}\left[-\eta\left(\tilde{H}_{ay}sin\phi + \tilde{H}_{ax}cos\phi\right) + \left(\tilde{E}_{ay}cos\theta cos\phi - \tilde{E}_{ax}cos\theta sin\phi\right)\right]$ ---(24.2)
- Since there is contribution from $\tilde{E}_{ay}$ and $\tilde{H}_{ax}$, it is simplified as
- $E_{ff\theta} \cong \frac{j\beta e^{(-j\beta r)}}{4\pi r}\left[\eta\left(-\tilde{H}_{ax}cos\theta sin\phi\right) + \left(\tilde{E}_{ay}sin\phi\right)\right]$ ---(24.1)
- $E_{ff\phi} \cong \frac{j\beta e^{(-j\beta r)}}{4\pi r}\left[-\eta\left(\tilde{H}_{ax}cos\phi\right) + \left(\tilde{E}_{ay}cos\theta cos\phi\right)\right]$ ---(24.2)

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- Using the relation $E_{ay} = -H_{ax}Z_g$ (note that η for the waveguide case should be the wave impedance for the mode we are considering), it can be simplified as
- $E_{ff\theta} \cong \frac{j\beta e^{(-j\beta r)}}{4\pi r}\left[\left(\tilde{E}_{ay}cos\theta sin\phi\right) + \left(\tilde{E}_{ay}sin\phi\right)\right]$ ---(24.1)
- $E_{ff\phi} \cong \frac{j\beta e^{(-j\beta r)}}{4\pi r}\left[\left(\tilde{E}_{ay}cos\phi\right) + \left(\tilde{E}_{ay}cos\theta cos\phi\right)\right]$ ---(24.2)
- Further simplification after combining common terms,
- $E_{ff\theta} \cong \frac{j\beta e^{(-j\beta r)}}{4\pi r}(1 + cos\theta)\left[\tilde{E}_{ay}sin\phi\right]$ ---(27.1)
- $E_{ff\phi} \cong \frac{j\beta e^{(-j\beta r)}}{4\pi r}(1 + cos\theta)\left[\tilde{E}_{ay}cos\phi\right]$ ---(27.2)

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- where $\tilde{E}_{ay} = E_0 \left\{ \frac{1}{2}\sqrt{\frac{\pi r_0'}{\beta}} I(\theta,\phi) \right\} \left\{ bsinc\left(\beta_y \frac{b}{2}\right) \right\}$
- $I(\theta,\phi) = A_1(\beta_{x1})\big(f(s_2') - f(s_1')\big) + A_2(\beta_{x2})\big(f(t_2') - f(t_1')\big)$
- E-plane patterns ($\phi = \frac{\pi}{2}$):
- $E_{ff\theta} \cong \frac{j\beta e^{(-j\beta r)}}{4\pi r}(1+cos\theta)\left[\tilde{E}_{ay}\right]$ and $E_{ff\phi} \cong 0$
- $\beta_x = \beta sin\theta cos\phi = 0$ and $\beta_y = \beta sin\theta sin\phi = \beta sin\theta$
- $\beta_{x1} = \frac{\pi}{A}, \beta_{x2} = -\frac{\pi}{A}, A_1\left(\frac{\pi}{A}\right) = e^{j\frac{r_0'\left(\frac{\pi}{A}\right)^2}{2\beta}}, A_2\left(-\frac{\pi}{A}\right) = e^{j\frac{r_0'\left(\frac{\pi}{A}\right)^2}{2\beta}}$
- Hence $A_1\left(\frac{\pi}{A}\right) = A_2\left(-\frac{\pi}{A}\right)$

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- $\therefore I(\theta,\phi) = A_1\left(\frac{\pi}{A}\right)\left(f(s_2') - f(s_1')\right) + \left(f(t_2') - f(t_1')\right)$
- $s_1' = \frac{1}{\sqrt{\pi\beta r_0'}}\left(-\frac{\beta A}{2} - \frac{\pi}{A}r_0'\right)$ and $s_2' = \frac{1}{\sqrt{\pi\beta r_0'}}\left(+\frac{\beta A}{2} - \frac{\pi}{A}r_0'\right)$
- $t_1' = \frac{1}{\sqrt{\pi\beta r_0'}}\left(-\frac{\beta A}{2} + \frac{\pi}{A}r_0'\right)$ and $t_2' = \frac{1}{\sqrt{\pi\beta r_0'}}\left(+\frac{\beta A}{2} + \frac{\pi}{A}r_0'\right)$
- Hence $s_2' = -t_1'$ and $s_1' = -t_2'$
- $f(\xi') = C(\xi') - jS(\xi')$ where $C(-\xi') = -C(\xi')$ and $S(-\xi') = -S(\xi')$ and $C(\xi')$ and $S(\xi')$ are sine and cosine Fresnel integrals
- Therefore, $f(-\xi') = -f(\xi')$
- $\therefore I(\theta,\phi) = A_1\left(\frac{\pi}{A}\right)2\left(f(s_2') - f(s_1')\right)$

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- Hence
- $E_{ff\theta} \cong \frac{j\beta b E_0 e^{(-j\beta r)}}{4\pi r} \sqrt{\frac{\pi r_0'}{\beta}} \left[A_1\left(\frac{\pi}{A}\right) 2\left(f(s_2') - f(s_1')\right)\right] \left(\frac{1+cos\theta}{2}\right) \left\{sinc\left(\beta sin\theta \frac{b}{2}\right)\right\}$
- H-plane patterns ($\phi = 0$):
- $E_{ff\theta} \cong 0$ and $E_{ff\phi} \cong \frac{j\beta e^{(-j\beta r)}}{4\pi r}(1 + cos\theta)\left[\tilde{E}_{ay}\right]$
- $\beta_x = \beta sin\theta cos\phi = \beta sin\theta$ and $\beta_y = \beta sin\theta sin\phi = 0$
- $\beta_{x1} = \beta_x + \frac{\pi}{A}, \beta_{x2} = \beta_x - \frac{\pi}{A}, A_1(\beta_{x1}) = e^{j\frac{r_0'(\beta_{x1})^2}{2\beta}}, A_2(\beta_{x2}) = e^{j\frac{r_0'(\beta_{x2})^2}{2\beta}}$

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- $\tilde{E}_{ay} = E_0 \left\{ \frac{1}{2} \sqrt{\frac{\pi r_0'}{\beta}} I(\theta, \phi) \right\} \left\{ bsinc\left(\beta_y \frac{b}{2}\right) \right\}$
- $I(\theta, \phi) = A_1(\beta_{x1})\left(f(s_2') - f(s_1')\right) + A_2(\beta_{x2})\left(f(t_2') - f(t_1')\right)$
- Hence
- $E_{ff\phi} \cong \frac{j\beta b E_0 e^{(-j\beta r)}}{4\pi r} \sqrt{\frac{\pi r_0'}{\beta}} \left[A_1(\beta_{x1})\left(f(s_2') - f(s_1')\right) + A_2(\beta_{x2})\left(f(t_2') - f(t_1')\right)\right] \left(\frac{1 + \cos\theta}{2}\right)$
- Directivity
- $D_H = \frac{4\pi}{\lambda^2} \epsilon_t \epsilon_{ph}^H A b$

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- $\epsilon_t = \frac{8}{\pi^2} \cong 0.81$
- $\epsilon_{ph}^H = \frac{\pi^2}{64t}\left[\left(C(p_1) - C(p_2)\right)^2 + \left(S(p_1) - S(p_2)\right)^2\right]$

How do we find t for maximum phase error?

- Maximum phase error is for $x' = \frac{A}{2}$,
- hence $\beta \frac{1}{2r_0'}(x')^2 = \beta \frac{1}{8r_0'}(A)^2 = 2\pi \frac{A^2}{8\lambda r_0'}$
- Let us equate the maximum phase error to $2\pi t$
- So, $\frac{A^2}{8\lambda r_0'} = t$
-

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- *What are parameters p_1 and p_2?*
- $s_1' = \frac{1}{\sqrt{\pi\beta r_0'}}\left(-\frac{\beta A}{2} - \beta_{x1} r_0'\right) = \frac{1}{\sqrt{\pi\beta r_0'}}\left(-\frac{\beta A}{2} - \left(\beta sin\theta cos\phi + \frac{\pi}{A}\right) r_0'\right) = \frac{1}{\sqrt{\pi\beta r_0'}}\left(-\frac{\beta A}{2} - \left(\beta u + \frac{\pi}{A}\right) r_0'\right)$
- For $u = sin\theta cos\phi = 0,$
- $p_1 = -s_1' = \frac{1}{\sqrt{\pi\beta r_0'}}\left(\frac{\beta A}{2} + \left(\frac{\pi}{A}\right) r_0'\right) = t_2'$
- $p_2 = s_2' = \frac{1}{\sqrt{\pi\beta r_0'}}\left(+\frac{\beta A}{2} - \left(\frac{\pi}{A}\right) r_0'\right) = -t_1'$

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- *Can we express* p_1 *and* p_2 *in terms of t* ($\sqrt{8\lambda r_0' t} = A$)*?*
- $p_1 = \frac{1}{\sqrt{\pi\beta r_0'}}\left(\frac{\beta\sqrt{8\lambda r_0' t}}{2} + \left(\frac{\pi}{\sqrt{8\lambda r_0' t}}\right) r_0'\right) = 2\sqrt{t}\left(1 + \frac{1}{8t}\right)$
- $p_2 = s_2' = \frac{1}{\sqrt{\pi\beta r_0'}}\left(+\frac{\beta A}{2} - \left(\frac{\pi}{A}\right) r_0'\right) = 2\sqrt{t}\left(-1 + \frac{1}{8t}\right)$
- *Is directivity dependent on A?*
- Yes and optimal value is for $A = \sqrt{3\lambda r_0'}$
- *What is the value of t for this?* $t = \frac{A^2}{8\lambda r_0'} = \frac{3\lambda r_0'}{8\lambda r_0'} = \frac{3}{8}$
- $D_H = \frac{4\pi}{\lambda^2}\epsilon_t \epsilon_{ph}^H A b$