# EE540 Advance Electromagnetic Theory & Antennas

Prof. Rakhesh Singh Kshetrimayum

Dept. of EEE, IIT Guwahati, India

# Aperture antennas

- Pyramidal horn antenna:
- *First step:*
- E-plane and H-plane sectoral horn antennas together make pyramidal horn antenna
- so we can write the aperture field variation as
- $E_{ay} = E_0 cos\left(\frac{\pi x'}{A}\right) e^{-j\beta\frac{1}{2r'_{0e}}(y')^2} e^{-j\beta\frac{1}{2r'_{0h}}(x')^2}$ inside the aperture and zero elsewhere
- $E_0 cos\left(\frac{\pi x'}{A}\right)$ is the amplitude variation and
- $e^{-j\beta\frac{1}{2r'_{0e}}(y')^2} e^{-j\beta\frac{1}{2r'_{0h}}(x')^2}$ is the phase variation

# Aperture antennas

Table: Amplitude and phase variation of Pyramidal horn antenna

| Aperture field parameter | Expression | $x' = 0$ (amplitude) $x', y' = 0$ (phase) | $x' = -A/2$ (amplitude) $y' = -B/2$ & $x' = -A/2$ (phase) | $x' = A/2$ (amplitude) $y' = B/2$ & $x' = A/2$ (phase) |
|---|---|---|---|---|
| Amplitude | $E_0 cos\left(\frac{\pi x'}{A}\right)$ | $E_0$ | 0 | 0 |
| Phase | $e^{-j\beta\frac{1}{2r'_{0e}}(y')^2} e^{-j\beta\frac{1}{2r'_{0h}}(x')^2}$ | 1 | $e^{-j\beta\frac{1}{8r'_{0e}}(B)^2} e^{-j\beta\frac{1}{8r'_{0h}}(A)^2}$ (maximum) | |

# Aperture antennas

- *Second step:*
- Take the 2-D FT of aperture field
- $\tilde{E}_{ay} = \int_{-A/2}^{A/2} E_0 cos\left(\frac{\pi x'}{A}\right) e^{-j\beta \frac{1}{2r'_{0h}}(x')^2} e^{j\beta_x x'} dx' \int_{-B/2}^{B/2} e^{-j\beta \frac{1}{2r'_{0e}}(y')^2} e^{j\beta_y y'} dy'$
- $= E_0 I_2 I_3$
- where
- $I_2 = \frac{1}{2}\sqrt{\frac{\pi r'_{0h}}{\beta}}\left\{e^{j\frac{\beta_{x1}^2 r'_{0h}}{2\beta}}\left(f(s'_2) - f(s'_1)\right) + e^{j\frac{\beta_{x2}^2 r'_{0h}}{2\beta}}\left(f(t'_2) - f(t'_1)\right)\right\}$

# Aperture antennas

- $I_3 = \sqrt{\frac{\pi r'_{0e}}{\beta}} \left\{ e^{j\frac{\beta_y^2 r'_{0e}}{2\beta}} \left( f(s'_4) - f(s'_3) \right) \right\}$
- *Third step:*
- Use equations (27.1) and (27.2) to find the FF electric fields
- $E_{ff\theta} \cong \frac{j\beta e^{(-j\beta r)}}{4\pi r} (1 + cos\theta) \left[ \tilde{E}_{ay} sin\phi \right]$ ---(27.1)
- $E_{ff\phi} \cong \frac{j\beta e^{(-j\beta r)}}{4\pi r} (1 + cos\theta) \left[ \tilde{E}_{ay} cos\phi \right]$ ---(27.2)
- where $\tilde{E}_{ay} = E_0 I_2 I_3$

# Aperture antennas

- E-plane patterns ($\phi = \frac{\pi}{2}$):
- $E_{ff\theta} \cong \frac{j\beta e^{(-j\beta r)}}{4\pi r}(1 + cos\theta)\left[\tilde{E}_{ay}\right]$ and $E_{ff\phi} \cong 0$
- where $v = sin\theta sin\phi = sin\theta$ and $\beta_x = \beta sin\theta cos\phi = 0$
- $I_2 = \frac{1}{2}\sqrt{\frac{\pi r'_{0h}}{\beta}}\left\{e^{j\frac{\beta_{x1}^2 r'_{0h}}{2\beta}}\left(f(s'_2) - f(s'_1)\right) + e^{j\frac{\beta_{x2}^2 r'_{0h}}{2\beta}}\left(f(t'_2) - f(t'_1)\right)\right\}$
- $\beta_{x1} = \beta_x + \frac{\pi}{A} = \frac{\pi}{A}, \beta_{x2} = \beta_x - \frac{\pi}{A} = -\frac{\pi}{A}$
- $e^{j\frac{\beta_{x1}^2 r'_{0h}}{2\beta}}$ and $e^{j\frac{\beta_{x2}^2 r'_{0h}}{2\beta}}$ are independent of $\theta$

# Aperture antennas

- $s_1' = \frac{1}{\sqrt{\pi\beta r_0'}}\left(-\frac{\beta A}{2} - \beta_{x1} r_0'\right) = \frac{1}{\sqrt{\pi\beta r_0'}}\left(-\frac{\beta A}{2} - \frac{\pi}{A} r_0'\right)$
- $s_2' = \frac{1}{\sqrt{\pi\beta r_0'}}\left(+\frac{\beta A}{2} - \beta_{x1} r_0'\right) = \frac{1}{\sqrt{\pi\beta r_0'}}\left(+\frac{\beta A}{2} - \frac{\pi}{A} r_0'\right)$
- $t_1' = \frac{1}{\sqrt{\pi\beta r_0'}}\left(-\frac{\beta A}{2} - \beta_{x2} r_0'\right) = \frac{1}{\sqrt{\pi\beta r_0'}}\left(-\frac{\beta A}{2} + \frac{\pi}{A} r_0'\right)$
- $t_2' = \frac{1}{\sqrt{\pi\beta r_0'}}\left(+\frac{\beta A}{2} - \beta_{x2} r_0'\right) = \frac{1}{\sqrt{\pi\beta r_0'}}\left(+\frac{\beta A}{2} + \frac{\pi}{A} r_0'\right)$
- All the above parameters are independent of θ
- $I_2$ is independent of θ

# Aperture antennas

- $I_3 = \sqrt{\frac{\pi r'_{0e}}{\beta}} \left\{ e^{j\frac{\beta_y^2 r'_{0e}}{2\beta}} \left( f(s'_4) - f(s'_3) \right) \right\}$
- where $\beta_y = \beta sin\theta sin\phi = \beta sin\theta$
- $s'_3 = \sqrt{\frac{\beta}{\pi r'_0}} \left( -\frac{B}{2} - r'_0 sin\theta \right)$
- $s'_4 = \sqrt{\frac{\beta}{\pi r'_0}} \left( +\frac{B}{2} - r'_0 sin\theta \right)$
- Hence $e^{j\frac{\beta_y^2 r'_{0e}}{2\beta}}$ as well as $\left( f(s'_4) - f(s'_3) \right)$ are dependent on θ and
- It will determine the radiation pattern which is similar to E-plane pattern of E-plane sectoral horn antenna

# Aperture antennas

- $E_{ff\theta} \cong$
- $\frac{j\beta e^{(-j\beta r)}}{4\pi r} E_0 I_2 (1 + cos\theta)[I_3]$
- H-plane patterns ($\phi = 0$):
- $E_{ff\theta} \cong 0$ and $E_{ff\phi} \cong \frac{j\beta e^{(-j\beta r)}}{4\pi r}(1 + cos\theta)\left[\tilde{E}_{ay}\right]$
- where $\beta_y = \beta sin\theta sin\phi = 0$ and $\beta_x = \beta sin\theta cos\phi = \beta sin\theta$
- $I_2 = \frac{1}{2}\sqrt{\frac{\pi r'_{0h}}{\beta}}\left\{e^{j\frac{\beta_{x1}^2 r'_{0h}}{2\beta}}\left(f(s'_2) - f(s'_1)\right) + e^{j\frac{\beta_{x2}^2 r'_{0h}}{2\beta}}\left(f(t'_2) - f(t'_1)\right)\right\}$
- $\beta_{x1} = \beta sin\theta + \frac{\pi}{A}, \beta_{x2} = \beta sin\theta - \frac{\pi}{A}$

# Aperture antennas

- $s_1' = \frac{1}{\sqrt{\pi \beta r_0'}}\left(-\frac{\beta A}{2} - \beta_{x1} r_0'\right)$
- $s_2' = \frac{1}{\sqrt{\pi \beta r_0'}}\left(+\frac{\beta A}{2} - \beta_{x1} r_0'\right)$
- $t_1' = \frac{1}{\sqrt{\pi \beta r_0'}}\left(-\frac{\beta A}{2} - \beta_{x2} r_0'\right)$
- $t_2' = \frac{1}{\sqrt{\pi \beta r_0'}}\left(+\frac{\beta A}{2} - \beta_{x2} r_0'\right)$
- All the above parameters are dependent on θ and hence $I_2$ is dependent on θ
- It will determine the radiation pattern which is similar to H-plane pattern of H-plane sectoral horn antenna

# Aperture antennas

- $I_3 = \sqrt{\frac{\pi r'_{0e}}{\beta}} \left\{ e^{j\frac{\beta_y^2 r'_{0e}}{2\beta}} \left( f(s'_4) - f(s'_3) \right) \right\}$
- where $\beta_y = \beta sin\theta sin\phi = 0, e^{j\frac{\beta_y^2 r'_{0e}}{2\beta}} = 0$
- $s'_3 = \sqrt{\frac{\beta}{\pi r'_0}} \left( -\frac{B}{2} - r'_0 \nu \right) = \sqrt{\frac{\beta}{\pi r'_0}} \left( -\frac{B}{2} \right)$
- $s'_4 = \sqrt{\frac{\beta}{\pi r'_0}} \left( +\frac{B}{2} - r'_0 \nu \right) = \sqrt{\frac{\beta}{\pi r'_0}} \left( +\frac{B}{2} \right) = -s'_3$
- $\therefore I_3 = 2f(s'_4)$
- $I_3$ is independent of θ

# Aperture antennas

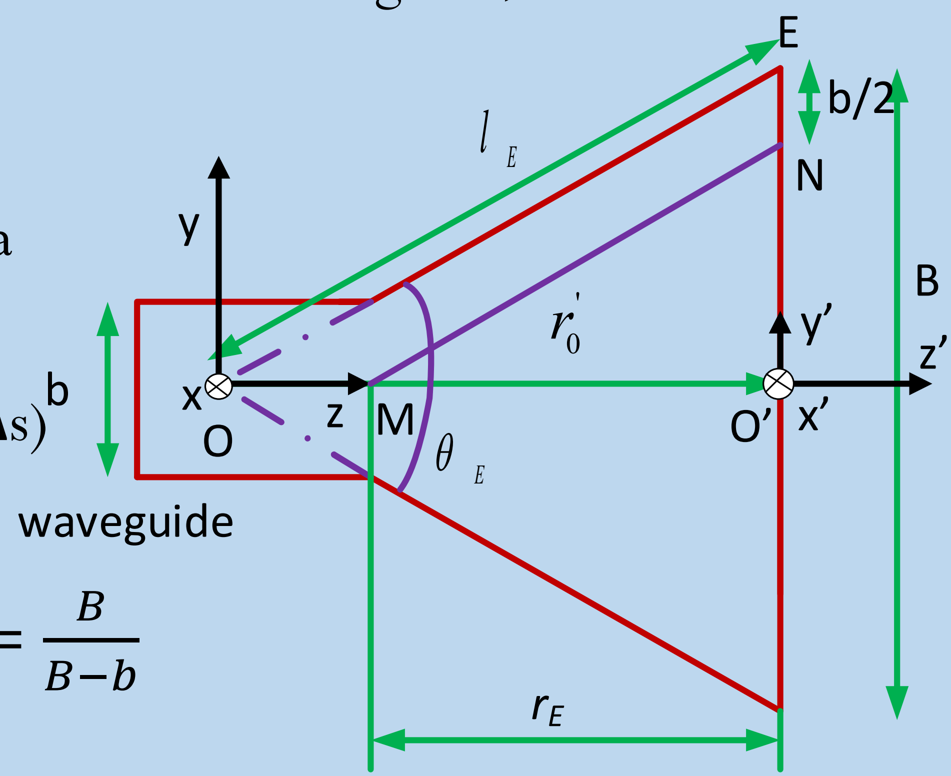
- $E_{ff\phi} \cong \frac{j\beta e^{(-j\beta r)}}{4\pi r}(1 + cos\theta)[E_0 I_2 I_3]$
- Hence
- $E_{ff\phi} \cong \frac{j\beta e^{(-j\beta r)}}{4\pi r} E_0 I_3 \boxed{(1 + cos\theta)[I_2]}$
- E-plane radiation pattern is similar to E-plane radiation pattern of E-plane sectoral horn
- Whereas H-plane radiation pattern is similar to H-plane radiation pattern of H-plane sectoral horn

# Aperture antennas

- *Gain of Pyramidal Horn*
- $D = \frac{4\pi}{\lambda^2}\epsilon_{ap}AB = \frac{4\pi}{\lambda^2}\epsilon_t\epsilon_{ph}^E\epsilon_{ph}^H AB$
- $\epsilon_t = \frac{8}{\pi^2} \cong 0.81$
- The phase efficiencies for optimum gain sectoral horns with s=1/4=0.25 and t=3/8=0.375 are
- $\epsilon_{ph}^E = 0.8$
- $\epsilon_{ph}^H = 0.79$
- Hence, $\epsilon_{ap} \cong 0.81 \times 0.8 \times 0.79 = 0.51$
- Therefore, for optimum gain pyramidal horn antenna
- $D \cong 0.51\frac{4\pi}{\lambda^2}AB$

# Aperture antennas

- E-plane and H-plane sectoral horn antennas together make pyramidal horn antenna
- In order to properly connect to the feed waveguide, to make it physically realizable
- $R_E = R_H$
- E-plane sectoral horn antenna
- Since all angles are equal
- $\Delta OO'E \sim \Delta MO'N$ (similar Δs)
- Hence
- $\dfrac{OO'}{MO'} = \dfrac{O'E}{O'N} \Rightarrow \dfrac{r_0'}{r_E} = \dfrac{B/2}{\frac{B}{2} - b/2} = \dfrac{B}{B-b}$




E-plane sectoral horn antenna

# Aperture antennas

- Similarly
- H-plane sectoral horn antenna
- $\frac{r_0'}{r_H} = \frac{A/2}{\frac{A}{2} - a/2} = \frac{A}{A-a}$
- Note that for E-plane sectoral horn antenna for optimum gain
- $\because \frac{r_0'}{r_E} = \frac{B}{B-b}$
- $B = \sqrt{2\lambda r_0'} = \sqrt{2\lambda \frac{B r_E}{B-b}} \Rightarrow \sqrt{B} = \sqrt{2\lambda \frac{r_E}{B-b}}$
- $\Rightarrow B = \frac{2\lambda r_E}{B-b}$
- $\Rightarrow B^2 - bB - 2\lambda r_E = 0$
- We know that $r_H$ and $r_E$ are equal

# Aperture antennas

- Using the relation for H-plane sectoral horn antenna
- $\frac{r_0'}{r_H} = \frac{A/2}{\frac{A}{2} - a/2} = \frac{A}{A-a} \Rightarrow \frac{r_H}{r_0'} = \frac{A-a}{A} \Rightarrow r_H = \frac{A-a}{A} r_0'$
- Consider the case for optimum gain
- $A = \sqrt{3\lambda r_0'} \Rightarrow r_0' = \frac{A^2}{3\lambda}$
- $\therefore r_H = \frac{A-a}{A}\left(\frac{A^2}{3\lambda}\right) = \frac{A-a}{3\lambda}(A)$
- Hence $B^2 - bB - 2\lambda r_H = 0$
- $\Rightarrow B^2 - bB - \frac{2}{3}(A-a)(A) = 0$
- $\Rightarrow B^2 - bB - \frac{2}{3}(A^2 - aA) = 0$
- $\Rightarrow A^2 - aA - \frac{3}{2}B^2 + \frac{3}{2}bB = 0$

# Aperture antennas

- From gain relation, $G \cong \epsilon_{ap} \frac{4\pi}{\lambda^2} AB$
- We can simplify the above equation in terms of $A$ or $B$
- Let us simplify it for $A$
- Replace $B$ as $B \cong \frac{G\lambda^2}{4\pi\epsilon_{ap} A}$
- $A^2 - aA - \frac{3}{2}\left(\frac{G\lambda^2}{4\pi\epsilon_{ap} A}\right)^2 + \frac{3}{2} b \left(\frac{G\lambda^2}{4\pi\epsilon_{ap} A}\right) = 0$
- Finally, fourth order equation in $A$ results which is the design equation for optimum pyramidal horn
- $A^4 - aA^3 + \left(\frac{3bG\lambda^2}{8\pi\epsilon_{ap}}\right) A - \frac{3G^2\lambda^4}{32\pi^2\epsilon_{ap}^2} = 0$

# Aperture antennas

- You can easily find the roots of this equation (use *roots* command in MATLAB)
- It gives four roots for a=22.86mm, b=10.16 mm, G=10dBi, $\epsilon_{ap} = 0.51$ and f=10GHz
- 44.9088 + 0.0000i
  8.6655 +39.9698i
  8.6655 -39.9698i
  -39.3798 + 0.0000i
- Only first one is possible solution hence *A*= *44.9088 mm*
- $B \cong \frac{G\lambda^2}{4\pi\epsilon_{ap}A} = 31.27\ mm$
- For E-plane sectoral case
- $B = \sqrt{2\lambda r_0'} \Rightarrow r_0' = \frac{B^2}{2\lambda} = 16.29\ mm$

# Aperture antennas

E-plane sectoral horn

- $l_E^2 = \frac{B^2}{4} + (r_0')^2 \Rightarrow l_E = 22.579\, mm$
- Distance between neck and mouth $r_E$
- $r_E = (B - b)\sqrt{\left(\frac{l_E}{B}\right)^2 - \frac{1}{4}} \Rightarrow r_E = 10.997\, mm$

H-plane sectoral horn ($A = \sqrt{3\lambda r_0'} \Rightarrow r_0' = \frac{A^2}{3\lambda} = 22.409\, mm$)

- $l_H^2 = \frac{A^2}{4} + (r_0')^2 \Rightarrow l_H = 31.723\, mm$
- Distance between neck and mouth $r_H$
- $r_H = (A - a)\sqrt{\left(\frac{l_H}{A}\right)^2 - \frac{1}{4}} \Rightarrow r_H = 11.00\, mm$
- Since $r_H \cong r_E$, this pyramidal horn antenna is practically realizable