EE540 Advance Electromagnetic Theory & Antennas

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Helmholtz Wave Equation

- **Wave equations from Maxwell's equations**
- In linear isotropic medium,
- the two Maxwell curl equations for time harmonic fields are

$$\nabla \times \vec{E} = -j\omega\mu\vec{H} \quad \text{(Equation 1)}$$

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E} + \vec{J} = j\omega\varepsilon\vec{E} + \sigma\vec{E} = \left(j\omega\varepsilon + \sigma\right)\vec{E} \quad \text{(Equation 2)}$$

- To solve for electric fields,
  - we can take curl on the first equation, for LHS

$$\nabla \times \nabla \times \vec{E} = \nabla \times \nabla \times \vec{E} = \nabla(\nabla \bullet \vec{E}) - \nabla^2\vec{E}$$

- For a charge free region, wave propagation in free space, $\nabla \bullet \vec{E} = 0$
  - therefore $\nabla \times \nabla \times \vec{E} = -\nabla^2\vec{E}$

Helmholtz Wave Equation

- eliminate the curl of magnetic field vector
- in the RHS, after taking curl of the first equation
  - by using the second equation given above

$$\therefore \nabla\times\nabla\times\vec{E} = -\nabla^2\vec{E} = -j\omega\mu(\nabla\times\vec{H}) = -j\omega\mu\left(j\omega\varepsilon+\sigma\right)\vec{E} = -\gamma^2\vec{E}$$

- Hence,

$$\nabla^2\vec{E} - \gamma^2\vec{E} = 0$$

- Similarly,

$$\nabla^2\vec{H} - \gamma^2\vec{H} = 0$$

- Note that Helmholtz wave equation is a vector wave equation
  - So Laplacian operator ∇^2 will operate on each components of vector $\vec{E}/\vec{H}$

Helmholtz Wave Equation

- *Solution of wave equation in Cartesian coordinates*

$$\nabla^2 \vec{E} - \gamma^2 \vec{E} = 0$$

  - Electric field is a vector and will have 3 components

$$\vec{E} = E_x \hat{x} + E_y \hat{y} + E_z \hat{z}$$

  - Laplacian will operate on all three components
  - Consider Laplacian operating on E_x

$$\nabla^2 E_x - \gamma^2 E_x = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) E_x - \gamma^2 E_x = 0$$

  - Applying method of separation of variables, assume $E_x = X(x)Y(y)Z(z)$
  - Rewrite wave equation as

$$\left(YZ \frac{\partial^2 X}{\partial x^2} + XZ \frac{\partial^2 Y}{\partial y^2} + XY \frac{\partial^2 Z}{\partial z^2} \right) = \gamma^2 XYZ$$

  - Divide by XYZ on both sides, we have

$$\left(\frac{1}{X} \frac{\partial^2 X}{\partial x^2} + \frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} \right) = \gamma^2$$

Helmholtz Wave Equation

- RHS is a constant, LHS first, second and third term
  - are totally dependent on x, y and z respectively
- Hence assume

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2}=\gamma_x^2,\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}=\gamma_y^2,\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2}=\gamma_z^2$$

- whose solutions are

$$X=C_x e^{\pm\gamma_x x},Y=C_y e^{\pm\gamma_y y},Z=C_z e^{\pm\gamma_z z}$$

- Therefore,

$$E_x=XYZ=C_x C_y C_z e^{\pm\gamma_x x}e^{\pm\gamma_y y}e^{\pm\gamma_z z}=C_1 e^{\pm\gamma_x x\pm\gamma_y y\pm\gamma_z z}$$

- Similarly

$$E_y=C_2 e^{\pm\gamma_x x\pm\gamma_y y\pm\gamma_z z},E_z=C_3 e^{\pm\gamma_x x\pm\gamma_y y\pm\gamma_z z}$$

- Therefore,

$$\vec{E}=E_x\hat{x}+E_y\hat{y}+E_z\hat{z}=\left(C_1\hat{x}+C_2\hat{y}+C_3\hat{z}\right)e^{\pm\left(\gamma_x x+\gamma_y y+\gamma_z z\right)}=\vec{C}e^{\pm\left(\gamma_x x+\gamma_y y+\gamma_z z\right)}$$

Helmholtz Wave Equation

- For $\vec{\gamma} = \gamma_x \hat{x} + \gamma_y \hat{y} + \gamma_z \hat{z}, \vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$
- Reduce $\vec{E} = \vec{C} e^{\pm(\vec{\gamma} \bullet \vec{r})}$
- For time harmonic fields

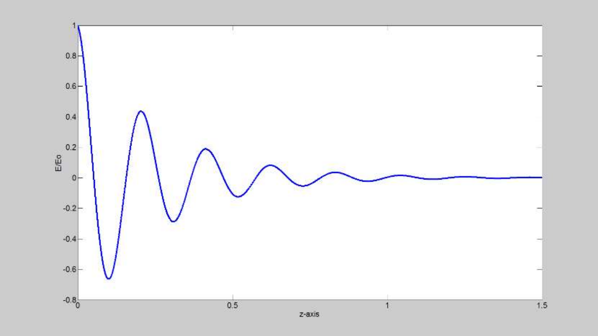
$$\vec{E} = \vec{C} e^{j\omega t \pm (\vec{\gamma} \bullet \vec{r})} = \vec{C} e^{j\omega t \pm \left((\vec{\alpha} + j\vec{\beta}) \bullet \vec{r} \right)} = \vec{C} e^{\pm \vec{\alpha} \bullet \vec{r}} e^{j\left(\omega t \pm \vec{\beta} \bullet \vec{r}\right)}$$

- Taking real parts (plane wave)

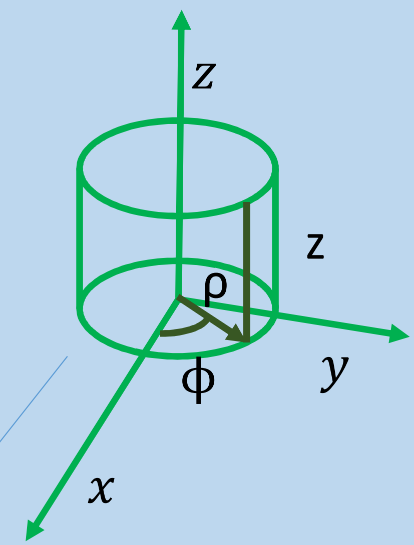
$$\mathrm{Re}\left\{\vec{E}\right\} = \mathrm{Re}\left\{\vec{C} e^{\pm \vec{\alpha} \bullet \vec{r}} e^{j\left(\omega t \pm \vec{\beta} \bullet \vec{r}\right)}\right\} = \vec{C} e^{\pm \vec{\alpha} \bullet \vec{r}} \cos\left(\omega t \pm \vec{\beta} \bullet \vec{r}\right)$$

- Simplifications:
- Assume x-polarized (electric field has only x-component, $\vec{E} = E_x \hat{x}$)
- wave propagating in positive z-direction, $\vec{\gamma} = \gamma_z \hat{z}$=$(\alpha_z + j\beta_z)\hat{z}$

$$\mathrm{Re}\left\{\vec{E}\right\} = \mathrm{Re}\left\{C\hat{x} e^{-\alpha_z z} e^{j(\omega t - \beta_z z)}\right\} = C e^{-\alpha_z z} \cos(\omega t - \beta_z z)\hat{x}$$

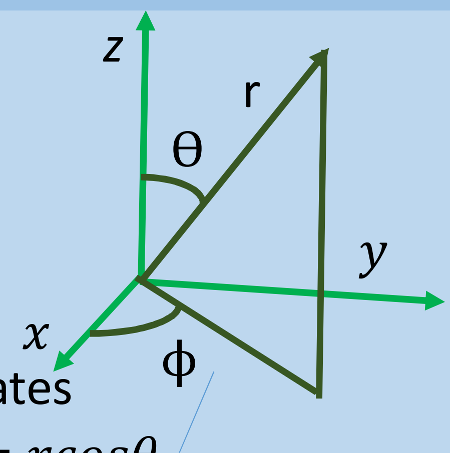


- *General Orthogonal Coordinate System*
  - Introduce a new set of coordinates
  - (extension of Cartesian coordinates (x,y,z))
    - $a_1 = a_1(x, y, z)$, $a_2 = a_2(x, y, z)$ and $a_3 = a_3(x, y, z)$
  - where the directions at any point indicated by $\widehat{a_1}, \widehat{a_2}$ and $\widehat{a_3}$
    - are orthogonal (perpendicular) to each other
  - is referred to as a set of orthogonal curvilinear coordinates
  - For example,
  - *Cylindrical coordinate system*
  - $a_1 = \rho, a_2 = \varphi, a_3 = z$
  - $\hat{a}_1 = \hat{\rho}, \hat{a}_2 = \hat{\varphi}, \hat{a}_3 = \hat{z}$
  - Relation between Cartesian and Cylindrical coordinates
  - $x = \rho cos\varphi, y = \rho sin\varphi, z = z$



Helmholtz Wave Equation

- With each coordinate is associated a scale factor
  - s_1, s_2 and s_3 respectively
  - The scale factor gives a measure of
  - how a change in the coordinate changes the position of a point
  - *Calculation of scale factors*
  - Relation between Cartesian and Cylindrical coordinates
  - $x = \rho cos\varphi, y = \rho sin\varphi, z = z$
  - $s_1 = \sqrt{\left(\frac{\partial x}{\partial \rho}\right)^2 + \left(\frac{\partial y}{\partial \rho}\right)^2 + \left(\frac{\partial z}{\partial \rho}\right)^2} = \sqrt{(cos\varphi)^2 + (sin\varphi)^2 + (0)^2} = 1$
  - $s_2 = \sqrt{\left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2 + \left(\frac{\partial z}{\partial \varphi}\right)^2} = \sqrt{(-\rho sin\varphi)^2 + (\rho cos\varphi)^2 + (0)^2} = \rho$
  - $s_3 = \sqrt{\left(\frac{\partial x}{\partial z}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + \left(\frac{\partial z}{\partial z}\right)^2} = \sqrt{(0)^2 + (0)^2 + (1)^2} = 1$



- *Spherical coordinate system*
- $a_1 = r, a_2 = \theta, a_3 = \varphi$
- $\hat{a}_1 = \hat{r}, \hat{a}_2 = \hat{\theta}, \hat{a}_3 = \hat{\varphi}$
- Relation between
    - Cartesian and Spherical coordinates
- $x = r sin\theta cos\varphi, y = r sin\theta sin\varphi, z = r cos\theta$
- *Calculation of scale factors*
- $s_1 = \sqrt{\left(\frac{\partial x}{\partial r}\right)^2 + \left(\frac{\partial y}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial r}\right)^2} = \sqrt{(sin\theta cos\varphi)^2 + (sin\theta sin\varphi)^2 + (cos\theta)^2} = 1$
- $s_2 = \sqrt{\left(\frac{\partial x}{\partial \theta}\right)^2 + \left(\frac{\partial y}{\partial \theta}\right)^2 + \left(\frac{\partial z}{\partial \theta}\right)^2} = \sqrt{(r cos\theta cos\varphi)^2 + (r cos\theta sin\varphi)^2 + (-r sin\theta)^2} = r$
- $s_3 = \sqrt{\left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2 + \left(\frac{\partial z}{\partial \varphi}\right)^2} = \sqrt{(-r sin\theta sin\varphi)^2 + (r sin\theta cos\varphi)^2 + (0)^2} = r sin\theta$

Helmholtz Wave Equation

- Laplacian of a scalar function:

$$\nabla^2\psi = \nabla \bullet \nabla \psi = \frac{1}{s_1 s_2 s_3}\left[\frac{\partial}{\partial a_1}\left(\frac{s_2 s_3}{s_1}\frac{\partial \psi}{\partial a_1}\right) + \frac{\partial}{\partial a_2}\left(\frac{s_1 s_3}{s_2}\frac{\partial \psi}{\partial a_2}\right) + \frac{\partial}{\partial a_3}\left(\frac{s_1 s_2}{s_3}\frac{\partial \psi}{\partial a_3}\right)\right]$$

Table: General Curvilinear System: Particular cases

General curvilinear system	a_1	a_2	a_3	s_1	s_2	s_3
Cartesian coordinate system	x	y	z	1	1	1
Cylindrical coordinate system	ρ	φ	z	1	ρ	1
Spherical coordinate system	r	ϴ	φ	1	r	rsinϴ

Helmholtz Wave Equation

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$$\nabla^2\psi = \nabla \bullet \nabla \psi$$

$$= \frac{1}{s_1 s_2 s_3}\left[\frac{\partial}{\partial a_1}\left(\frac{s_2 s_3}{s_1}\frac{\partial \psi}{\partial a_1}\right) + \frac{\partial}{\partial a_2}\left(\frac{s_1 s_3}{s_2}\frac{\partial \psi}{\partial a_2}\right) + \frac{\partial}{\partial a_3}\left(\frac{s_1 s_2}{s_3}\frac{\partial \psi}{\partial a_3}\right)\right]$$

Tips to memorize this formula:

- Note that in the expression of Laplacian of a scalar function above,
    - outside the third bracket, we have division by product of all scale factors, and
    - inside the third bracket there are three terms
- Each term is a partial differential with respect to a variable of the expression in a first bracket
- Inside first bracket,
    - you have multiplication of scale factors of the remaining two axes divide by the scale factor of the same variable multiplied
        - partial differential of the scalar function with the same variable

Helmholtz Wave Equation

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- Laplacian of a scalar function:

$$\nabla^2\psi = \nabla \bullet \nabla\psi = \frac{1}{s_1 s_2 s_3}\left[\frac{\partial}{\partial a_1}\left(\frac{s_2 s_3}{s_1}\frac{\partial\psi}{\partial a_1}\right) + \frac{\partial}{\partial a_2}\left(\frac{s_1 s_3}{s_2}\frac{\partial\psi}{\partial a_2}\right) + \frac{\partial}{\partial a_3}\left(\frac{s_1 s_2}{s_3}\frac{\partial\psi}{\partial a_3}\right)\right]$$

- Hence,
- Cartesian coordinates: $\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}$
- Cylindrical coordinates: $\nabla^2\psi = \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\right) + \frac{\partial}{\partial z}\left(\rho\frac{\partial\psi}{\partial z}\right)\right] = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$
- Spherical coordinates: $\nabla^2\psi = \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\psi}{\partial r}\right) + \frac{\partial}{\partial\theta}\left(\frac{r\sin\theta}{r}\frac{\partial\psi}{\partial\theta}\right) + \frac{\partial}{\partial\phi}\left(\frac{r}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\right)\right]$

$$= \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)\right] + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial\psi}{\partial\theta}\right] + \frac{1}{r^2\sin^2\theta}\left[\frac{\partial^2\psi}{\partial\phi^2}\right]$$