MoM Advances

- **EM Absorption in the Human body**
- EM absorption is specified in terms of specific absorption rate (SAR) which is the mass normalized rate of energy absorbed by the body
- At a specific location, SAR may be defined as

$$SAR = \frac{\sigma}{\rho}\left|E\right|^2$$

- where σ is tissue conductivity, ρ is tissue mass density, E=rms value of internal field strength

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- Usual MoM steps are required:
  - Deriving the appropriate IE
  - Converting IE to matrix equation & matrix elements calculation
  - Solving the set of simultaneous equations
- We will use tensor integral-equation here
- What is this?
- When some electric field is incident on human body, the induced current in the body gives scattered electric field
  - Correspondingly the body may be replaced by an equivalent current density

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- Consider Maxwell curl equation

$$\nabla \times \vec{E} = -j\omega\mu_0 \vec{H}$$

- We can derive wave equation from Maxwell curl equation as

$$\begin{aligned}
&\nabla \times \nabla \times \vec{E} = -j\omega\mu_0\left(\nabla \times \vec{H}\right) = -j\omega\mu_0\left(\sigma\vec{E} + j\omega\varepsilon\vec{E}\right) \\
&= -j\omega\mu_0\left(\sigma\vec{E} + j\omega\left(\varepsilon - \varepsilon_0\right)\vec{E} + j\omega\varepsilon_0\vec{E}\right) \\
&= -j\omega\mu_0\left(\vec{J}_{conduction} + \vec{J}_{polarization} + j\omega\varepsilon_0\vec{E}\right) \\
&= -j\omega\mu_0\left(\vec{J}_{eq} + j\omega\varepsilon_0\vec{E}\right) \\
&\Rightarrow \nabla \times \nabla \times \vec{E} - k_0^2\vec{E} = -j\omega\mu_0\vec{J}_{eq}
\end{aligned}$$

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- It is more like wave is propagating in free space
- And there is an equivalent current source which is effectively produced as an effect of the human body
- The equivalent current density can be expressed in terms of a tensor as follows

$$\vec{J}_{eq}(\vec{r}) = \sigma(\vec{r})\vec{E}(\vec{r}) + j\omega\left(\varepsilon(\vec{r}) - \varepsilon_0\right)\vec{E}(\vec{r}) = \tau(\vec{r})\vec{E}(\vec{r})$$

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- The tensor $\tau(\vec{r})$ takes into account all the effect of a human body in terms of a 3-D matrix
- Consider a biological body of arbitratry shape with constitutive parameters $\varepsilon, \mu_0, \sigma$ illuminated by an incident (or impressed) plane EM wave
- The induced current in the body gives rise to a scattered field $\vec{E}^s$

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- For time varying electric fields $\vec{E}^s = -j\omega\vec{A} - \nabla\phi$
- where

$$\vec{A} = \mu_0 \int_{V'} G_0\left(\vec{r},\vec{r}'\right)\vec{J}_{eq}\left(\vec{r}'\right)dv'$$

- and the free space scalar Green's function is given by

$$G_0\left(\vec{r},\vec{r}'\right) = \frac{e^{-jk\left|\vec{r}-\vec{r}'\right|}}{4\pi\left|\vec{r}-\vec{r}'\right|}$$

$$\nabla \bullet \vec{A} = -j\omega\mu\varepsilon\phi$$

- From Lorentz Gaug condition

$$\Rightarrow \phi = \frac{\nabla \bullet \vec{A}}{-j\omega\mu\varepsilon}$$

- Hence, the scattered fields are

$$\vec{E}^s = -j\omega\vec{A} + \frac{\nabla\nabla \bullet \vec{A}}{j\omega\mu\varepsilon}, \vec{H}^s = \frac{1}{\mu}\nabla \times \vec{A}$$

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- Since scattered electric and magnetic field are dependent on magnetic vector potential which is dependent on the equivalent current density, hence the fields are dependent on the equivalent current density $\vec{J}_{eq}$
- Let us analyze the dependence of fields on the $\vec{J}_{eq}$
- Suppose $\vec{J}_{eq}$ is an infinitesimal elementary source at $\vec{r}^{'}$ pointed in x direction so that

$$\vec{J}_{eq} = \delta\left(\vec{r} - \vec{r}^{'}\right)\hat{x}$$

- The corresponding magnetic vector potential is

$$\vec{A} = \mu_0 G_0\left(\vec{r}, \vec{r}^{'}\right)\hat{x} \quad \because \vec{A} = \mu_0 \int_{V^{'}} G_0\left(\vec{r}, \vec{r}^{'}\right)\vec{J}_{eq}\left(\vec{r}^{'}\right)dv^{'}$$

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- If $\vec{G}_{0x}\left(\vec{r},\vec{r}^{'}\right)$ is the electric field produced by the above mentioned elementary source, it must satisfy the wave equation

$$\nabla\times\nabla\times\vec{G}_{0x}\left(\vec{r},\vec{r}^{'}\right)-k_0^2\vec{G}_{0x}\left(\vec{r},\vec{r}^{'}\right)=-j\omega\mu_0\delta\left(\vec{r}-\vec{r}^{'}\right)$$

- whose solution is given by

$$\vec{E}^s=\vec{G}_{0x}=-j\omega\mu_0 G_0\left(\vec{r},\vec{r}^{'}\right)\hat{x}+\frac{\nabla\nabla\bullet G_0\left(\vec{r},\vec{r}^{'}\right)}{j\omega\varepsilon}\hat{x} \qquad \because \vec{E}^s=-j\omega\vec{A}+\frac{\nabla\nabla\bullet\vec{A}}{j\omega\mu\varepsilon}$$

$$\vec{A}=\mu_0 G_0\left(\vec{r},\vec{r}^{'}\right)\hat{x}$$

$$\Rightarrow\vec{G}_{0x}=-j\omega\mu_0\left(1+\frac{1}{k^2}\nabla\nabla\bullet\right)G_0\left(\vec{r},\vec{r}^{'}\right)\hat{x}$$

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- $\vec{G}_{0x}(\vec{r},\vec{r}')$ is referred to as a free space vector Green's function with a source pointed in the x-direction
- We could also find the free space vector Green's function $\vec{G}_{0y}(\vec{r},\vec{r}'),\vec{G}_{0z}(\vec{r},\vec{r}')$ for a source pointed in the y-direction and z-direction respectively

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- We could now introduce a dyadic function which will store these three free space vector Green's function as

$$\overleftrightarrow{G}_0\left(\vec{r},\vec{r}^{'}\right)=\vec{G}_{0x}\left(\vec{r},\vec{r}^{'}\right)\hat{x}+\vec{G}_{0y}\left(\vec{r},\vec{r}^{'}\right)\hat{y}+\vec{G}_{0z}\left(\vec{r},\vec{r}^{'}\right)\hat{z}$$

- This is called free space dyadic Green's function
- It is a solution of the dyadic differential equation

$$\nabla\times\nabla\times\overleftrightarrow{G}_0\left(\vec{r},\vec{r}^{'}\right)-k_0^2\overleftrightarrow{G}_0\left(\vec{r},\vec{r}^{'}\right)=-j\omega\mu_0\overleftrightarrow{I}\delta\left(\vec{r}-\vec{r}^{'}\right)$$

- where unit dyad is given by

$$\overleftrightarrow{I}=\hat{x}\hat{x}+\hat{y}\hat{y}+\hat{z}\hat{z}$$

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- The physical meaning is that $\overleftrightarrow{G}_0\left(\vec{r},\vec{r}^{'}\right)$ is the electric field at a point $\vec{r}$ due to an infinitesimal source at $\vec{r}^{'}$ in any arbitrary orientation
- Then the scattered electric field due to any arbitrary equivalent current density may be expressed as

$$\vec{E}^s = \int_{V^{'}} \overleftrightarrow{G}_0\left(\vec{r},\vec{r}^{'}\right) \bullet \vec{J}_{eq}\left(\vec{r}^{'}\right) dv^{'}$$

- where

$$\overleftrightarrow{G}_0\left(\vec{r},\vec{r}^{'}\right) = \vec{G}_{0x}\left(\vec{r},\vec{r}^{'}\right)\hat{x} + \vec{G}_{0y}\left(\vec{r},\vec{r}^{'}\right)\hat{y} + \vec{G}_{0z}\left(\vec{r},\vec{r}^{'}\right)\hat{z}$$

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- Since $$\vec{G}_{0w} = -j\omega\mu_0 \left(1 + \frac{1}{k^2}\nabla\nabla\bullet\right) G_0\left(\vec{r},\vec{r}^{'}\right)\hat{w}$$
- where w can take any value

$$w = \{x, y, z\}; \hat{w} = \{\hat{x}, \hat{y}, \hat{z}\}$$

- Note that $\vec{G}_0\left(\vec{r},\vec{r}^{'}\right)$ has singularity of the order $\left|\vec{r} - \vec{r}^{'}\right|^3$
- In other words, the integral diverges if the $\vec{r}$ is inside the source region

MoM Advances

- The scattered field inside the body may be expressed in terms of equivalent current density by using the free-space tensor/dyadic Green's function as

$$\vec{E}^{s}=\int_{V'}\overleftrightarrow{G}_{0}\left(\vec{r},\vec{r}'\right)\bullet\vec{J}_{eq}\left(\vec{r}'\right)dv'$$

- However, when the field point is inside the body, scattered field must be evaluated with special care because of singularity

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- This difficulty is overcome by excluding a small volume surrounding the field point first and letting the small volume approach zero
- Integral is now well defined as the limit obtained when the radius of the sphere approaches zero
- We shall call this limit the "principal value" of the integral

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- This process involves defining a principal value and adding a correction term (derived in https://ieeexplore.ieee.org/stamp/stamp.jsp?tp=&arnumber=1145064)

$$\vec{E}^s(\vec{r}) = PV\left(\int_{V'} \overleftrightarrow{G}_0(\vec{r},\vec{r}') \bullet \vec{J}_{eq}(\vec{r}')dv'\right) + \left[\vec{E}^s(\vec{r})\right]_{correction}$$

$$= PV\left(\int_{V'} \overleftrightarrow{G}_0(\vec{r},\vec{r}') \bullet \vec{J}_{eq}(\vec{r}')dv'\right) - \frac{\vec{J}_{eq}(\vec{r})}{j3\omega\varepsilon_0}$$

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- The total electric field inside the body is the sum of the incident field and scattered field

$$\vec{E}(\vec{r}) = \vec{E}^i(\vec{r}) + \vec{E}^s(\vec{r})$$

$$\therefore \vec{E}^i(\vec{r}) = \vec{E}(\vec{r}) - \vec{E}^s(\vec{r})$$

$$= \vec{E}(\vec{r}) - PV\left(\int_{V'} \vec{\vec{G}}_0(\vec{r},\vec{r}') \bullet \vec{J}_{eq}(\vec{r}')dv'\right) + \frac{\vec{J}_{eq}(\vec{r})}{j3\omega\varepsilon_0}$$

$$= \vec{E}(\vec{r}) - PV\left(\int_{V'} \vec{\vec{G}}_0(\vec{r},\vec{r}') \bullet \tau(\vec{r})\vec{E}(\vec{r})dv'\right) + \frac{\tau(\vec{r})\vec{E}(\vec{r})}{j3\omega\varepsilon_0}$$

$$\Rightarrow \vec{E}^i(\vec{r}) = \left(1 + \frac{\tau(\vec{r})}{j3\omega\varepsilon_0}\right)\vec{E}(\vec{r}) - PV\left(\int_{V'} \tau(\vec{r})\vec{E}(\vec{r}) \bullet \vec{\vec{G}}_0(\vec{r},\vec{r}')dv'\right)$$

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- Matrix Equation:
- The inner product $\vec{E}(\vec{r}) \bullet \overleftrightarrow{G}_0(\vec{r},\vec{r}')$ is given as

$$\vec{E}(\vec{r}) \bullet \overleftrightarrow{G}_0(\vec{r},\vec{r}') = \begin{bmatrix} G_{xx}(\vec{r},\vec{r}') & G_{xy}(\vec{r},\vec{r}') & G_{xz}(\vec{r},\vec{r}') \\ G_{yx}(\vec{r},\vec{r}') & G_{yy}(\vec{r},\vec{r}') & G_{yz}(\vec{r},\vec{r}') \\ G_{zx}(\vec{r},\vec{r}') & G_{zy}(\vec{r},\vec{r}') & G_{zz}(\vec{r},\vec{r}') \end{bmatrix} \bullet \begin{bmatrix} E_x(\vec{r}) \\ E_y(\vec{r}) \\ E_z(\vec{r}) \end{bmatrix}$$

- Denoting x_1=x, x_2=y, x_3=z, we have,

$$G_{x_p x_q}(\vec{r},\vec{r}') = -j\omega\mu_0\left(\delta_{pq} + \frac{1}{k^2}\frac{\partial^2}{\partial x_q x_p}\right)G_0(\vec{r},\vec{r}'); p,q = 1,2,3$$

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- Partition the body into N subvolumes or cells, each denoted by v_m (m=1,2,…,N) and
- assume $\tau(\vec{r}), \vec{E}(\vec{r})$ are constant within each cell
- If $\vec{r}_m$ is the centre of the m[th] cell, then

$$E^i_{x_p}(\vec{r}_m) = \left(1 + \frac{\tau(\vec{r})}{j3\omega\varepsilon_0}\right) E_{x_p}(\vec{r}_m) - \sum_{q=1}^{3}\left[\sum_{n=1}^{N} \tau(\vec{r}_n) PV\left(\int_{V_m} G_{x_p x_q}(\vec{r}_m, \vec{r}^{'}) dv^{'}\right)\right] E_{x_q}(\vec{r}_n)$$

$$\because \vec{E}^i(\vec{r}) = \left(1 + \frac{\tau(\vec{r})}{j3\omega\varepsilon_0}\right)\vec{E}(\vec{r}) - PV\left(\int_{V^{'}} \tau(\vec{r})\vec{E}(\vec{r}) \bullet \vec{\bar{G}}_0(\vec{r}, \vec{r}^{'}) dv^{'}\right)$$

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- If we let $\left[G_{x_p x_q}\right]$ be an N×N matrix with elements

$$G_{x_p x_q}^{mn} = \tau(\mathbf{r}_n) PV \left(\int_{v_n} G_{x_p x_q}(\mathbf{r}_m, \mathbf{r}') dv' \right) - \delta_{pq}\delta_{mn}\left[1 + \frac{\tau(\mathbf{r})}{3 j\omega\varepsilon_0}\right]$$

- where m,n=1,2,…,N and p,q=1,2,3

- let $\left[E_{x_p}\right]$ and $\left[E_{x_p}^i\right]$ be column matrices with elements

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$$\left[E_{x_p}\right] = \begin{bmatrix} E_{x_p}(\mathbf{r}_1) \\ \vdots \\ E_{x_p}(\mathbf{r}_N) \end{bmatrix} \qquad \left[E^i_{x_p}\right] = \begin{bmatrix} E^i_{x_p}(\mathbf{r}_1) \\ \vdots \\ E^i_{x_p}(\mathbf{r}_N) \end{bmatrix}$$

- We obtain 3N simultaneous equations for Ex, Ey and Ez at the centers of N cells by the point matching technique

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- These simultaneous equations can be written in matrix form as

$$\begin{bmatrix} [G_{xx}] & [G_{xy}] & [G_{xz}] \\ [G_{yx}] & [G_{yy}] & [G_{yz}] \\ [G_{zx}] & [G_{zy}] & [G_{zz}] \end{bmatrix} \begin{bmatrix} [E_x] \\ [E_y] \\ [E_z] \end{bmatrix} = - \begin{bmatrix} [E_x^i] \\ [E_y^i] \\ [E_z^i] \end{bmatrix}$$

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- Radar cross section:
  - A measure of the effective area of the scatterer
  - Function of both angle of incidence and angle of observation
  - Larger is the radar cross section, larger is scattering

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- Scattering from a conducting cylinder of infinite length
- *Electric field integral equation*
- On the conductor surface

$$E_{\tan}^{total} = E_{\tan}^{inc} + E_{\tan}^{scatt} = 0$$

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- *Incident wave*
  - It is a plane wave from infinity
  - Assume a TM^z wave with $H_z=0$

$$\vec{E}^{inc} = \hat{z}E_z(x, y)$$

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- *Scattered wave*
- The incident field induces an electric current

$$\hat{z}J_z(x,y)$$

- which produces the scattered field
- The *scalar wave equation* in this case is

$$\nabla^2 E_z^{scatt} + k^2 E_z^{scatt} = j\omega\mu_0 J_z$$

MoM Advances

- *2-D Green's functions*
- We can compute 2-D Green's function by
  - computing the field radiated by a line source
  - carrying a time-harmonic electric current of amplitude of I in the +z direction
- Since the current associated with the line source is infinitely long

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- flows in the z direction only the z component of the electric field will become non zero
- Like in cylindrical waveguide E_z can be expressed as

$$E_z(\rho,\phi,z)=\sum_{m=0}^{\infty}\left[A_m H_m^{(1)}(k_\rho\rho)+B_m H_m^{(2)}(k_\rho\rho)\right]$$

$$\left[C_m e^{jm\phi}+D_m e^{-jm\phi}\right]\left[E_m e^{jk_z z}+F_m e^{-jk_z z}\right]$$

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- where $H_m^{(1)}, H_m^{(2)}$ are the Hankel function of first and second kind

$$k_\rho^2 + k_z^2 = k^2$$

- Since the line source is rotationally symmetric, the fields do not vary in the ϕ or z direction
- So $k_z=0$ and $C_m=D_m=0$ for $m \neq 0$
- Hence $$k_\rho^2 = k^2$$

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- Note on *Hankel's function*
- Hankel's function are related to Bessel's functions of first and second kind as

$$H_m^{(1)}(k_\rho \rho) = J_m(k_\rho \rho) + jY_m(k_\rho \rho)$$
$$H_m^{(2)}(k_\rho \rho) = J_m(k_\rho \rho) - jY_m(k_\rho \rho)$$

- These relations are similar to Euler's theorem

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- Physically, Bessel's functions represent standing waves whereas Hankel's functions represent propagating waves
- When the argument is large, the Hankel function can be approximated by

$$H_m^{(1)}(k_\rho \rho) \cong \sqrt{\frac{-2j}{\pi k_\rho \rho}} e^{jk_\rho \rho - \frac{m\pi}{2}}$$

$$H_m^{(2)}(k_\rho \rho) \cong \sqrt{\frac{2j}{\pi k_\rho \rho}} e^{-jk_\rho \rho + \frac{m\pi}{2}} = j\sqrt{\frac{-2j}{\pi k_\rho \rho}} e^{-\left(jk_\rho \rho - \frac{m\pi}{2}\right)}$$

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- The amplitude of Hankel's function decays as

$$\sqrt{\frac{1}{k_\rho \rho}}$$

- when $k_\rho \rho$ becomes large
- The phase of the oscillation depends on order m like Bessel's functions

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- $H_m^{(1)}(k_\rho \rho)$ represents a wave propagating in the - ρ direction (an in-going wave)
- $H_m^{(2)}(k_\rho \rho)$ represents a wave propagating in the + ρ direction (an out-going wave)
- An incoming wave defies the principle of causality, hence $A_0=0$

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- Hence

$$E_z(k_\rho \rho) = B_0 H_0^{(2)}(k_\rho \rho)$$

- Using Maxwell's curl equation

$$\vec{H} = -\frac{1}{j\omega\mu}\frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ 0 & 0 & E_z(k_\rho \rho) \end{vmatrix}$$

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$$\vec{H} = \frac{1}{j\omega\mu}\hat{\phi}\frac{\partial\left[E_z\left(k_\rho\rho\right)\right]}{\partial\rho} = \frac{1}{j\omega\mu}\hat{\phi}\frac{\partial\left[B_0 H_0^{(2)}\left(k_\rho\rho\right)\right]}{\partial\rho}$$

$$\vec{H} = \frac{1}{j}\left(\frac{jk_\rho}{j\omega\mu}\right)\hat{\phi}B_0 H_0^{(2)'}\left(k_\rho\rho\right) = \frac{1}{j\eta}\hat{\phi}B_0 H_0^{(2)'}\left(k_\rho\rho\right)$$

For small $k_\rho\rho$

$$H_0^{(2)}\left(k_\rho\rho\right) \cong 1 - \frac{2j}{\pi}\ln\left(\frac{1.78107k_\rho\rho}{2}\right)$$

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$$H_0^{(2)'}\left(k_\rho \rho\right) \cong -\frac{2j}{\pi} \frac{\left(\frac{1.78107 k_\rho}{2}\right)}{\left(\frac{1.78107 k_\rho \rho}{2}\right)} = -\frac{2j}{\pi\rho}$$

$$\vec{H} = \frac{1}{j\eta} \hat{\phi} B_0 \left(-\frac{2j}{\pi\rho}\right)$$

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- Using Ampere's law

$$I=\oint \vec{H}\bullet d\vec{l}=-\frac{2jB_0}{j\eta\pi}\hat{\phi}\int_0^{2\pi}\frac{1}{\rho}\rho d\phi=-\frac{4B_0}{\eta}\Rightarrow B_0=-\frac{\eta I}{4}$$

- So $E_z\left(k_\rho\rho\right)=-\frac{\eta I}{4}H_0^{(2)}\left(k_\rho\rho\right)$
- For I=1, this is the Green's function for 2-D electric current sources in TM^z polarization