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- Radar cross section:
  - A measure of the effective area of the scatterer
  - Function of both angle of incidence and angle of observation
  - Larger is the radar cross section, larger is scattering

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- Scattering from a conducting cylinder of infinite length
- *Electric field integral equation*
- On the conductor surface

$$E_{\tan}^{total} = E_{\tan}^{inc} + E_{\tan}^{scatt} = 0$$

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- *Incident wave*
  - It is a plane wave from infinity
  - Assume a TM^z wave with $H_z=0$

$$\vec{E}^{inc} = \hat{z}E_z(x, y)$$

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- *Scattered wave*
- The incident field induces an electric current

$$\hat{z}J_z(x,y)$$

- which produces the scattered field
- The *scalar wave equation* in this case is

$$\nabla^2 E_z^{scatt} + k^2 E_z^{scatt} = j\omega\mu_0 J_z$$

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- *2-D Green's functions*
- We can compute 2-D Green's function by
  - computing the field radiated by a line source
  - carrying a time-harmonic electric current of amplitude of I in the +z direction
- Since the current associated with the line source is infinitely long

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- flows in the z direction only the z component of the electric field will become non zero
- Like in cylindrical waveguide E_z can be expressed as

$$E_z(\rho,\phi,z)=\sum_{m=0}^{\infty}\left[A_m H_m^{(1)}(k_\rho \rho)+B_m H_m^{(2)}(k_\rho \rho)\right]$$

$$\left[C_m e^{jm\phi}+D_m e^{-jm\phi}\right]\left[E_m e^{jk_z z}+F_m e^{-jk_z z}\right]$$

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- where $H_m^{(1)}, H_m^{(2)}$ are the Hankel function of first and second kind

$$k_\rho^2 + k_z^2 = k^2$$

- Since the line source is rotationally symmetric, the fields do not vary in the ϕ or z direction
- So k_z=0 and C_m=D_m=0 for m ≠0
- Hence $$k_\rho^2 = k^2$$

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- Note on *Hankel's function*
- Hankel's function are related to Bessel's functions of first and second kind as

$$H_m^{(1)}(k_\rho \rho) = J_m(k_\rho \rho) + jY_m(k_\rho \rho)$$
$$H_m^{(2)}(k_\rho \rho) = J_m(k_\rho \rho) - jY_m(k_\rho \rho)$$

- These relations are similar to Euler's theorem

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- Physically, Bessel's functions represent standing waves whereas Hankel's functions represent propagating waves
- When the argument is large, the Hankel function can be approximated by

$$H_m^{(1)}(k_\rho \rho) \cong \sqrt{\frac{-2j}{\pi k_\rho \rho}} e^{jk_\rho \rho - \frac{m\pi}{2}}$$

$$H_m^{(2)}(k_\rho \rho) \cong \sqrt{\frac{2j}{\pi k_\rho \rho}} e^{-jk_\rho \rho + \frac{m\pi}{2}} = j\sqrt{\frac{-2j}{\pi k_\rho \rho}} e^{-\left(jk_\rho \rho - \frac{m\pi}{2}\right)}$$

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- The amplitude of Hankel's function decays as

$$\sqrt{\frac{1}{k_\rho \rho}}$$

- when $k_\rho \rho$ becomes large
- The phase of the oscillation depends on order m like Bessel's functions

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- $H_m^{(1)}(k_\rho \rho)$ represents a wave propagating in the - ρ direction (an in-going wave)
- $H_m^{(2)}(k_\rho \rho)$ represents a wave propagating in the + ρ direction (an out-going wave)
- An incoming wave defies the principle of causality, hence $A_0=0$

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- Hence

$$E_z\left(k_\rho \rho\right) = B_0 H_0^{(2)}\left(k_\rho \rho\right)$$

- Using Maxwell's curl equation

$$\vec{H} = -\frac{1}{j\omega\mu}\frac{1}{\rho}\begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{z} \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ 0 & 0 & E_z\left(k_\rho \rho\right) \end{vmatrix}$$

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$$\vec{H} = \frac{1}{j\omega\mu}\hat{\phi}\frac{\partial\left[E_z\left(k_\rho\rho\right)\right]}{\partial\rho} = \frac{1}{j\omega\mu}\hat{\phi}\frac{\partial\left[B_0 H_0^{(2)}\left(k_\rho\rho\right)\right]}{\partial\rho}$$

$$\vec{H} = \frac{1}{j}\left(\frac{jk_\rho}{j\omega\mu}\right)\hat{\phi}B_0 H_0^{(2)'}\left(k_\rho\rho\right) = \frac{1}{j\eta}\hat{\phi}B_0 H_0^{(2)'}\left(k_\rho\rho\right)$$

For small $k_\rho\rho$

$$H_0^{(2)}\left(k_\rho\rho\right) \cong 1 - \frac{2j}{\pi}\ln\left(\frac{1.78107k_\rho\rho}{2}\right)$$

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$$H_0^{(2)'}\left(k_\rho \rho\right) \cong -\frac{2j}{\pi} \frac{\left(\frac{1.78107 k_\rho}{2}\right)}{\left(\frac{1.78107 k_\rho \rho}{2}\right)} = -\frac{2j}{\pi\rho}$$

$$\vec{H} = \frac{1}{j\eta}\hat{\phi} B_0 \left(-\frac{2j}{\pi\rho}\right)$$

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- Using Ampere's law

$$I = \oint \vec{H} \bullet d\vec{l} = -\frac{2jB_0}{j\eta\pi}\hat{\phi}\int_0^{2\pi}\frac{1}{\rho}\rho d\phi = -\frac{4B_0}{\eta} \Rightarrow B_0 = -\frac{\eta I}{4}$$

- So $E_z(k_\rho \rho) = -\frac{\eta I}{4} H_0^{(2)}(k_\rho \rho)$
- For I=1, this is the Green's function for 2-D electric current sources in TM^z polarization

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- The electric field can be found by convolving the Green's function with the current distribution

$$\vec{E}(\rho) = -\frac{\eta}{4}\int H_0^{(2)}\left(k_\rho\left|\vec{\rho}-\vec{\rho}'\right|\right)J_z\left(\vec{\rho}'\right)d\vec{\rho}'; \vec{\rho}' = x'\hat{x}+y'\hat{y}, \vec{\rho} = x\hat{x}+y\hat{y}$$

- Hence

$$E_z^{scatt}(\rho) = -\frac{\eta}{4}\int H_0^{(2)}\left(k_\rho\left|\rho-\rho'\right|\right)J_z\left(\rho'\right)dl'$$

$$\Rightarrow E_z^{inc}(\rho) = -E_z^{scatt}(\rho) = \frac{\eta}{4}\int H_0^{(2)}\left(k_\rho\left|\rho-\rho'\right|\right)J_z\left(\rho'\right)dl'$$

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- The incident plane wave field is assumed to be propagating normal to the z-axis, the axis of the cylinder

$$E_z^{inc} = E_0 e^{-j\vec{k}\bullet\vec{r}} = E_0 e^{-jk\left(x\cos\phi_i + y\sin\phi_i\right)}$$

- where ϕ_i is the angle of incidence

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- For the MoM solution,
- we can divide the contour into a number of segments and
- use pulse function expansion for unknown current as

$$J_z(\rho') \cong \sum_{n=1}^{N} \alpha_n p(\rho - \rho_n)$$

- where $p(\rho - \rho_n)$ is the pulse function centred at ρ_n

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- If we divide the contour in sufficiently large number of segments,
- the curved segment may be replaced by a flat segment
- Point matching at the mid-point ρ_m of segment reduces the integral to

$$-E_z^{inc}(\rho_m) = \frac{\eta}{4}\sum_{n=1}^{N}\alpha_n \int_{w_n} H_0^{(2)}\left(k_\rho\left|\rho_m - \rho'\right|\right)dl' ; m = 1,2,3,\cdots,M$$

- w_n is the size of the m[th] segment

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- The above expression

$$\begin{bmatrix} E_z^{inc}(\rho_1) \\ E_z^{inc}(\rho_2) \\ \vdots \\ E_z^{inc}(\rho_N) \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & \cdots & Z_{1N} \\ Z_{21} & Z_{22} & \cdots & Z_{2N} \\ \vdots & \ddots & \ddots & \vdots \\ Z_{N1} & Z_{N2} & \cdots & Z_{NN} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{bmatrix}$$

- In matrix form

$$\left[E_z^{inc}\right] = [Z][\alpha]$$

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- where off-diagonal elements are

$$Z_{mn} = \frac{\eta}{4} \int_{\rho_n - \frac{w_n}{2}}^{\rho_n + \frac{w_n}{2}} H_0^{(2)}\left(k_\rho \left|\rho_m - \rho'\right|\right) dl' \cong \frac{\eta}{4} w_n H_0^{(2)}\left(k_\rho \left|\rho_m - \rho_n\right|\right)$$

$$= \frac{\eta}{4} w_n H_0^{(2)}\left(k_\rho R_{mn}\right); R_{mn} = \sqrt{\left(x_m - x_n\right)^2 + \left(y_m - y_n\right)^2}$$

- Diagonal elements are

$$Z_{nn} = \frac{\eta}{4} \int H_0^{(2)}\left(k_\rho \left|\rho_m - \rho'\right|\right) dl' = \frac{2\eta}{4} \int_0^{\frac{w_n}{2}} \left[1 - \frac{2j}{\pi} \ln\left(\frac{1.78107 k_\rho \rho}{2}\right)\right] d\rho$$

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- Induced current in the cylinder J_z can be obtained from

$$[\alpha]=[Z]^{-1}\left[E^{inc}\right]$$

- where $E_m^{inc}=e^{-j\vec{k}\bullet\vec{r}}=e^{-jk(x_m\cos\phi_i+y_m\sin\phi_i)}$
- and (x_m,y_m) denotes the coordinates of the mid-point of the segment

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- Radar Cross Section:
- $\sigma = \lim_{\rho \to \infty} 2\pi\rho \frac{|\vec{E}^s|^2}{|\vec{E}^{inc}|^2}$
- where $E_m^{inc} = e^{-j\vec{k} \bullet \vec{r}} = e^{-jk(x_m \cos\phi_i + y_m \sin\phi_i)}$
- and (x_m, y_m) denotes the coordinates of the mid-point of the segment