FDTD: An Introduction

Prof. Rakhesh Singh Kshetrimayum

FDTD: An Introduction

- How do one visualize wave?
  - How can one observe wave propagation?
- Radio waves cann't be observed in nature
  - But one can write some EM codes (simulation) and
  - observe wave behavior
- One popular choice for this is Finite Difference Time Domain method
  - Also referred to as FDTD

FDTD: An Introduction

- What is FDTD?
- Let us start with 4 Maxwell's equations

1. $\nabla \bullet \vec{E} = \frac{\rho_v}{\varepsilon}$

2. $\nabla \bullet \vec{B} = 0$

3. $\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

4. $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

Maxwell curl equations (why?) → 3., 4.

FDTD: An Introduction

- Basically, we replace spatial & time derivatives in the two Maxwell's curl equations
  - by central finite difference approximation
- What is central finite difference approximation of derivatives?
- Consider a function f(x), its derivative at x_0 from central finite difference approximation is given by

$$\frac{df(x_0)}{dx} = f^{'}(x_0) \cong \frac{f\left(x_0 + \frac{\Delta x}{2}\right) - f\left(x_0 - \frac{\Delta x}{2}\right)}{\Delta x}$$

FDTD: An Introduction

- The time-dependent and source free ($\vec{J}=0$) Maxwell's curl equations in a medium with $\varepsilon=\varepsilon_0\varepsilon_r$ and $\mu = \mu_0\,\mu_r$ are

3 equations → $$\frac{\partial \vec{E}}{\partial t} = \frac{1}{\varepsilon_0 \varepsilon_r} \nabla \times \vec{H}; \frac{\partial \vec{H}}{\partial t} = -\frac{1}{\mu_0 \mu_r} \nabla \times \vec{E}$$ ← 3 equations

- For Cartesian coordinate systems,
  - expanding the curl equations,
  - equating the vector components,
- we have 6 equations

FDTD: An Introduction

- Expanding the first vector curl equation

$$\frac{\partial \vec{E}}{\partial t} = \frac{1}{\varepsilon_0 \varepsilon_r} \nabla \times \vec{H}$$

- we get 3 equations

$$\frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon_0 \varepsilon_r} \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right)$$

$$\frac{\partial E_y}{\partial t} = \frac{1}{\varepsilon_0 \varepsilon_r} \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right)$$

$$\frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon_0 \varepsilon_r} \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right)$$

FDTD: An Introduction

- Expanding the second vector curl equation

$$\frac{\partial \vec{H}}{\partial t} = -\frac{1}{\mu_0 \mu_r} \nabla \times \vec{E}$$

- We get 3 equations

$$\frac{\partial H_x}{\partial t} = \frac{1}{\mu_0 \mu_r}\left(\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y}\right)$$

$$\frac{\partial H_y}{\partial t} = \frac{1}{\mu_0 \mu_r}\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right)$$

$$\frac{\partial H_z}{\partial t} = \frac{1}{\mu_0 \mu_r}\left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right)$$

FDTD: An Introduction

As we know that FDTD is a time-domain solver

The question is how do we solve those 6 equations above?

1. 1-D FDTD update equations

- For 1-D case (a major simplification), we can consider
- (a) Linearly polarized wave along x-axis
  - exciting an electric field which has E_x only ($E_y = E_z = 0$)
- (b) propagation along z-axis
  - no variation in the x-y plane, i.e. $\partial/\partial x = 0$ and $\partial/\partial y = 0$

FDTD: An Introduction

$$\frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon_0 \varepsilon_r}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)$$

$$\frac{\partial E_y}{\partial t} = \frac{1}{\varepsilon_0 \varepsilon_r}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)$$

$$\frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon_0 \varepsilon_r}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)$$

$$\frac{\partial H_x}{\partial t} = \frac{1}{\mu_0 \mu_r}\left(\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y}\right)$$

$$\frac{\partial H_y}{\partial t} = \frac{1}{\mu_0 \mu_r}\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right)$$

$$\frac{\partial H_z}{\partial t} = \frac{1}{\mu_0 \mu_r}\left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right)$$

FDTD: An Introduction

- Then, the above 6 equations from 2 Maxwell's curl equations
- reduce to 2 equations for 1-D FDTD

$$\frac{\partial E_x}{\partial t} = -\frac{1}{\varepsilon_0 \varepsilon_r}\left(\frac{\partial H_y}{\partial z}\right) \qquad \frac{\partial H_y}{\partial t} = -\frac{1}{\mu_0 \mu_r}\left(\frac{\partial E_x}{\partial z}\right)$$

- We want to solve these equations at different locations and time in solution space (observe fields at different place and time)
- Hence we need to discretize both in time and space

FDTD: An Introduction

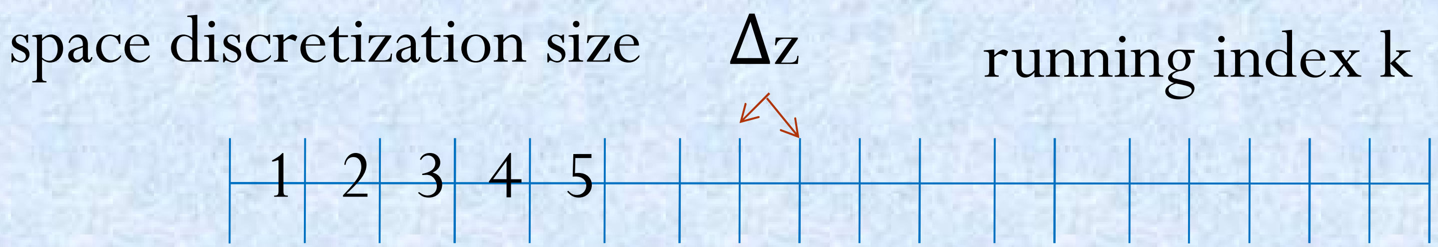


Fig. 1(a) Discretizations in 1-D space

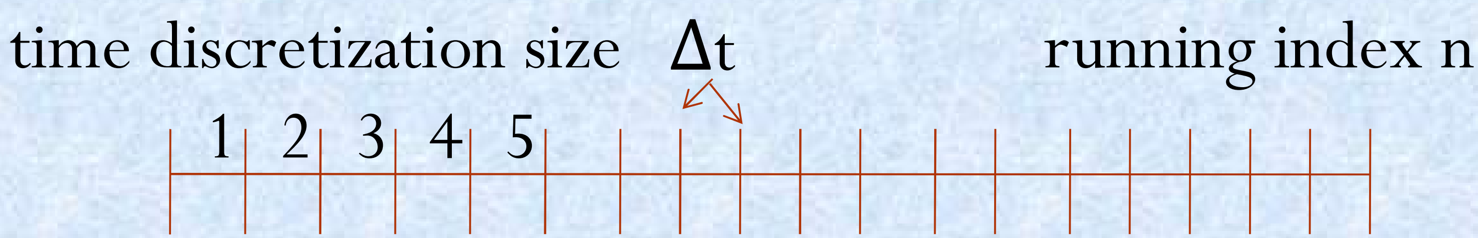


Fig. 1(b) Discretization in time

FDTD: An Introduction

- Notations:
  - n is the time index and
  - k is the spatial index,
- How to decide space and time discretization size?
- How do one determine the values of Δz and Δt?

FDTD: An Introduction

- Locate positions $z = k\Delta z$
- Usually Δz is taken as 10-15 per wavelength
  - → gives accurate results
- times $t = n\Delta t$ and
- Δt (dependent on Δz) chosen from Courant stability criterion
  - → gives stable FDTD solution (will be discussed later)

FDTD: An Introduction

- Let us discretize the first equation:

$$\frac{\partial E_x}{\partial t} = -\frac{1}{\varepsilon_0 \varepsilon_r}\left(\frac{\partial H_y}{\partial z}\right)$$

- the central difference approximations for both the temporal and spatial derivatives are obtained at
  - z = kΔz, (space discretization size and index)
  - t = nΔt (time discretization size and index)

$$\frac{E_x\left(k, n+\frac{1}{2}\right) - E_x\left(k, n-\frac{1}{2}\right)}{\Delta t} = -\frac{1}{\varepsilon_0 \varepsilon_r}\frac{H_y\left(k+\frac{1}{2}, n\right) - H_y\left(k-\frac{1}{2}, n\right)}{\Delta z}$$

FDTD: An Introduction

- For the second equation: $\dfrac{\partial H_y}{\partial t} = -\dfrac{1}{\mu_0 \mu_r}\left(\dfrac{\partial E_x}{\partial z}\right)$
- the central difference approximations for both the temporal and spatial derivatives are obtained at
- at (z + Δz/2, t + Δt/2)
- (increment all the time and space steps by 1/2):

$$\frac{H_y\left(k+\frac{1}{2},n+1\right)-H_y\left(k+\frac{1}{2},n\right)}{\Delta t} = -\frac{1}{\mu_0\mu_r}\frac{E_x\left(k+1,n+\frac{1}{2}\right)-E_x\left(k,n+\frac{1}{2}\right)}{\Delta z}$$

FDTD: An Introduction

- The above equations can be rearranged as a pair of 'computer update equations',
  - which can be repeatedly updated in loop,
  - to obtain the next instant time values of E_x (k, n+1/2) and H_y (k + ½, n+1)
    - from the previous instant time values as follows

$$E_x\left(k,n+\frac{1}{2}\right)=E_x\left(k,n-\frac{1}{2}\right)-\frac{1}{\varepsilon_0\varepsilon_r}\frac{\Delta t}{\Delta z}\left[H_y\left(k+\frac{1}{2},n\right)-H_y\left(k-\frac{1}{2},n\right)\right]$$

$$H_y\left(k+\frac{1}{2},n+1\right)=H_y\left(k+\frac{1}{2},n\right)-\frac{1}{\mu_0\mu_r}\frac{\Delta t}{\Delta z}\left[E_x\left(k+1,n+\frac{1}{2}\right)-E_x\left(k,n+\frac{1}{2}\right)\right]$$

FDTD: An Introduction

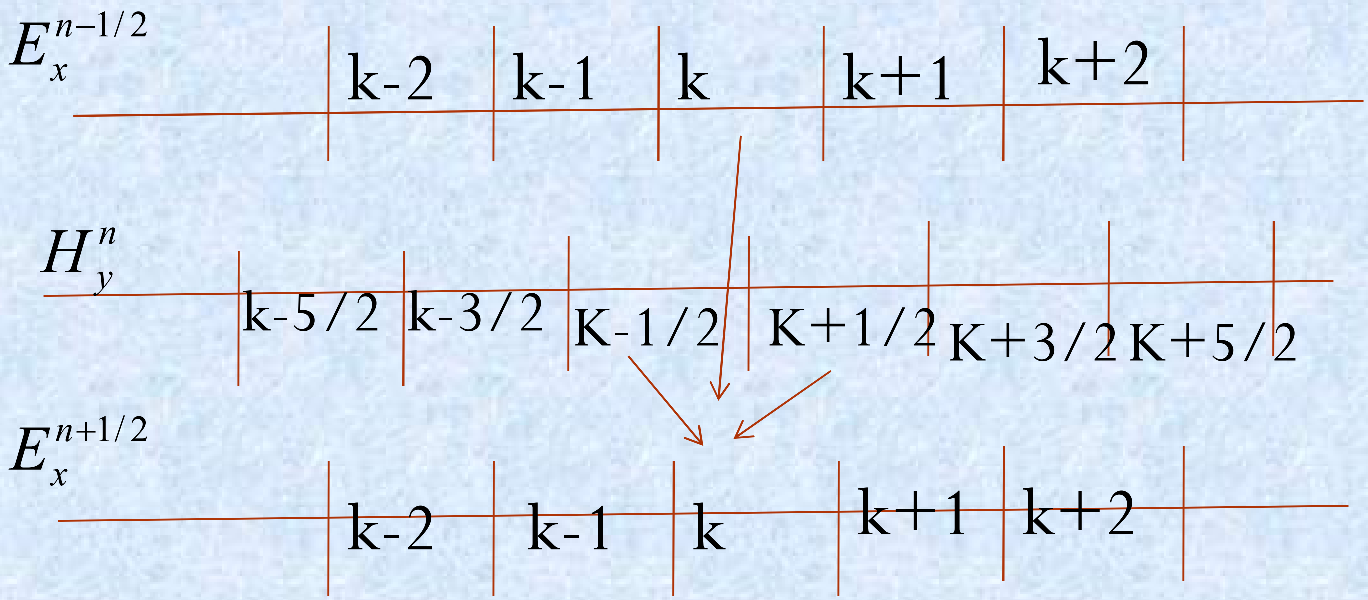


Fig. 2 Interleaving of E and H fields

FDTD: An Introduction

- Interleaving of the E and H fields in space and time in the FDTD formulation
- to calculate Ex(k),
  - the neighbouring values of Hy at k-1/2 and k+1/2 of the previous time instant are needed
- Similarly,
- to calculate Hy(k+1/2), for instance,
  - the neighbouring values of Ex at k and k+1 of the previous time instants are needed

FDTD: An Introduction

- In the above equations,
  - ε_0 and μ_0 differ by several orders of magnitude,
  - E_x and H_y will differ by several orders of magnitude
  - We also know that ratio of electric field and magnetic field for plane waves is 120π
- Numerical error is minimized by making the following change of variables as

$$E_x^{'} = \sqrt{\frac{\varepsilon}{\mu}} E_x$$

FDTD: An Introduction

- Hence,

$$\frac{1}{\sqrt{\frac{\varepsilon}{\mu}}}E_x^{'}\left(k,n+\frac{1}{2}\right)=\frac{1}{\sqrt{\frac{\varepsilon}{\mu}}}E_x^{'}\left(k,n-\frac{1}{2}\right)-\frac{1}{\varepsilon}\frac{\Delta t}{\Delta z}\left[H_y\left(k+\frac{1}{2},n\right)-H_y\left(k-\frac{1}{2},n\right)\right]$$

$$\Rightarrow E_x^{'}\left(k,n+\frac{1}{2}\right)=E_x^{'}\left(k,n-\frac{1}{2}\right)-\sqrt{\frac{\varepsilon}{\mu}}\frac{1}{\varepsilon}\frac{\Delta t}{\Delta z}\left[H_y\left(k+\frac{1}{2},n\right)-H_y\left(k-\frac{1}{2},n\right)\right]$$

$$\Rightarrow E_x^{'}\left(k,n+\frac{1}{2}\right)=E_x^{'}\left(k,n-\frac{1}{2}\right)-\frac{1}{\sqrt{\mu\varepsilon}}\frac{\Delta t}{\Delta z}\left[H_y\left(k+\frac{1}{2},n\right)-H_y\left(k-\frac{1}{2},n\right)\right]$$

$$\Rightarrow E_x^{'}\left(k,n+\frac{1}{2}\right)=E_x^{'}\left(k,n-\frac{1}{2}\right)+\frac{1}{\sqrt{\mu\varepsilon}}\frac{\Delta t}{\Delta z}\left[H_y\left(k-\frac{1}{2},n\right)-H_y\left(k+\frac{1}{2},n\right)\right]$$

FDTD: An Introduction

- Similarly,

$$H_y\left(k+\frac{1}{2},n+1\right)=H_y\left(k+\frac{1}{2},n\right)-\frac{1}{\sqrt{\frac{\varepsilon}{\mu}}}\frac{1}{\mu}\frac{\Delta t}{\Delta z}\left[E_x^{'}\left(k+1,n+\frac{1}{2}\right)-E_x^{'}\left(k,n+\frac{1}{2}\right)\right]$$

$$\Rightarrow H_y\left(k+\frac{1}{2},n+1\right)=H_y\left(k+\frac{1}{2},n\right)-\frac{1}{\sqrt{\mu\varepsilon}}\frac{\Delta t}{\Delta z}\left[E_x^{'}\left(k+1,n+\frac{1}{2}\right)-E_x^{'}\left(k,n+\frac{1}{2}\right)\right]$$

$$\Rightarrow H_y\left(k+\frac{1}{2},n+1\right)=H_y\left(k+\frac{1}{2},n\right)+\frac{1}{\sqrt{\mu\varepsilon}}\frac{\Delta t}{\Delta z}\left[E_x^{'}\left(k,n+\frac{1}{2}\right)-E_x^{'}\left(k+1,n+\frac{1}{2}\right)\right]$$

FDTD: An Introduction

Courant stability criteria:

- Courant stability criteria dictates the
  - relationship between the time increment Δt with respect to space increment Δz
  - in order to have a stable FDTD solution of the electromagnetic problems
- In isotropic media, an electromagnetic wave propagates a distance of one cell in time $\Delta t = \Delta z / v_p$,
  - where v_p is the phase velocity

FDTD: An Introduction

- This limits the maximum time step
- This equation implies that an EM wave cannot be allowed
  - to move more than a space cell during a time step
- Otherwise, FDTD will start diverging

FDTD: An Introduction

- If we choose $\Delta t > \Delta z / v_p$,
  - the distance moved by the EM wave over the time interval Δt will be more than Δz,
  - the EM wave will leave out the next node/cell and
  - FDTD cells are not causally interconnected and
- hence, it leads to instability

FDTD: An Introduction

Fig. 3 (a) 2-D discretizations

FDTD: An Introduction

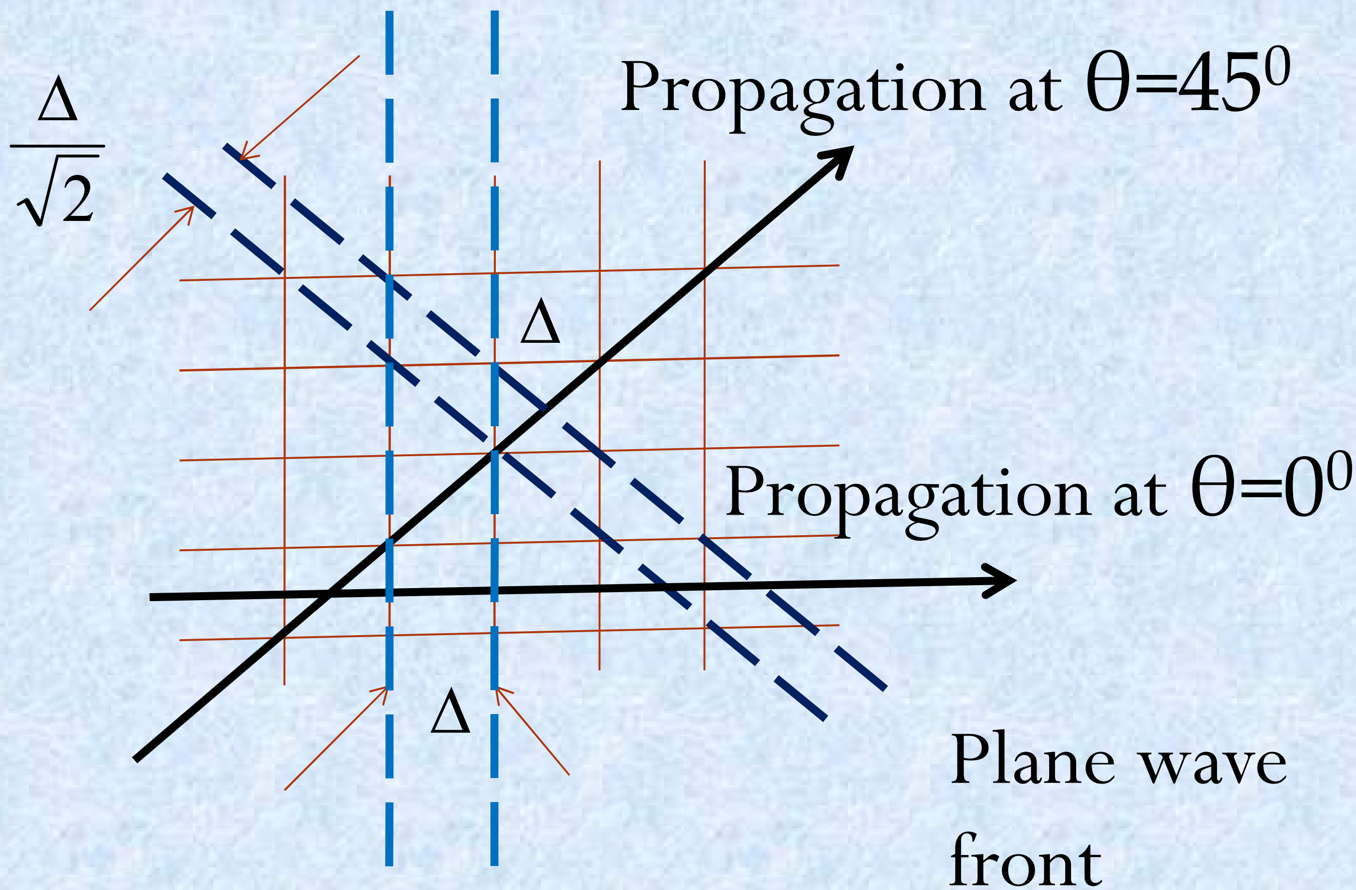


Fig. 3 (b) Plane wave front propagation along $\theta=0^0$ and $\theta=45^0$

FDTD: An Introduction

- Propagation at $\theta = 45^0$
- Wavefront jumps from one row of nodes to the next row,
- the spacing between the consecutive rows of nodes is $\frac{\Delta}{\sqrt{2}}$
- Similarly for 3-D case $\frac{\Delta}{\sqrt{3}}$

FDTD: An Introduction

- In the case of a 2-D simulation,
- we have to allow for the propagation in the diagonal direction,
- which brings the time requirement to $\Delta t = \Delta z/\sqrt{2}v_p$
- Obviously, three-dimensional simulation requires $\Delta t = \Delta z/\sqrt{3}v_p$
- We will use in all our simulations a time step Δt of

$$\Delta t = \frac{\Delta z}{2 \cdot v_p}$$

FDTD: An Introduction

- where v_p is the phase velocity,
- which satisfies the requirements in 1-D, 2-D and 3-D for all media ($\sqrt{2} \approx 1.414 < \sqrt{3} \approx 1.7321 < 2$)

- In 3D case, it may be more appropriate to modify the above stability criteria as

$$\Delta t = \frac{\min\left(\Delta x, \Delta y, \Delta z\right)}{2 \cdot v_p}$$

- Using the above relation, we may simplify for 1-D FDTD as

$$\frac{1}{\sqrt{\varepsilon\mu}} \frac{\Delta t}{\Delta z} = v_p \cdot \frac{\Delta t}{\Delta z} = \frac{v_p \cdot \Delta z}{2 \cdot v_p \cdot \Delta z} = \frac{1}{2}$$

FDTD: An Introduction

- Making use of this in the above two equations,
- we obtain the following equations

$$E_x'\left(k,n+\frac{1}{2}\right)=E_x'\left(k,n-\frac{1}{2}\right)+\frac{1}{2}\left[H_y\left(k-\frac{1}{2},n\right)-H_y\left(k+\frac{1}{2},n\right)\right]$$

$$H_y\left(k+\frac{1}{2},n+1\right)=H_y\left(k+\frac{1}{2},n\right)+\frac{1}{2}\left[E_x'\left(k,n+\frac{1}{2}\right)-E_x'\left(k+1,n+\frac{1}{2}\right)\right]$$

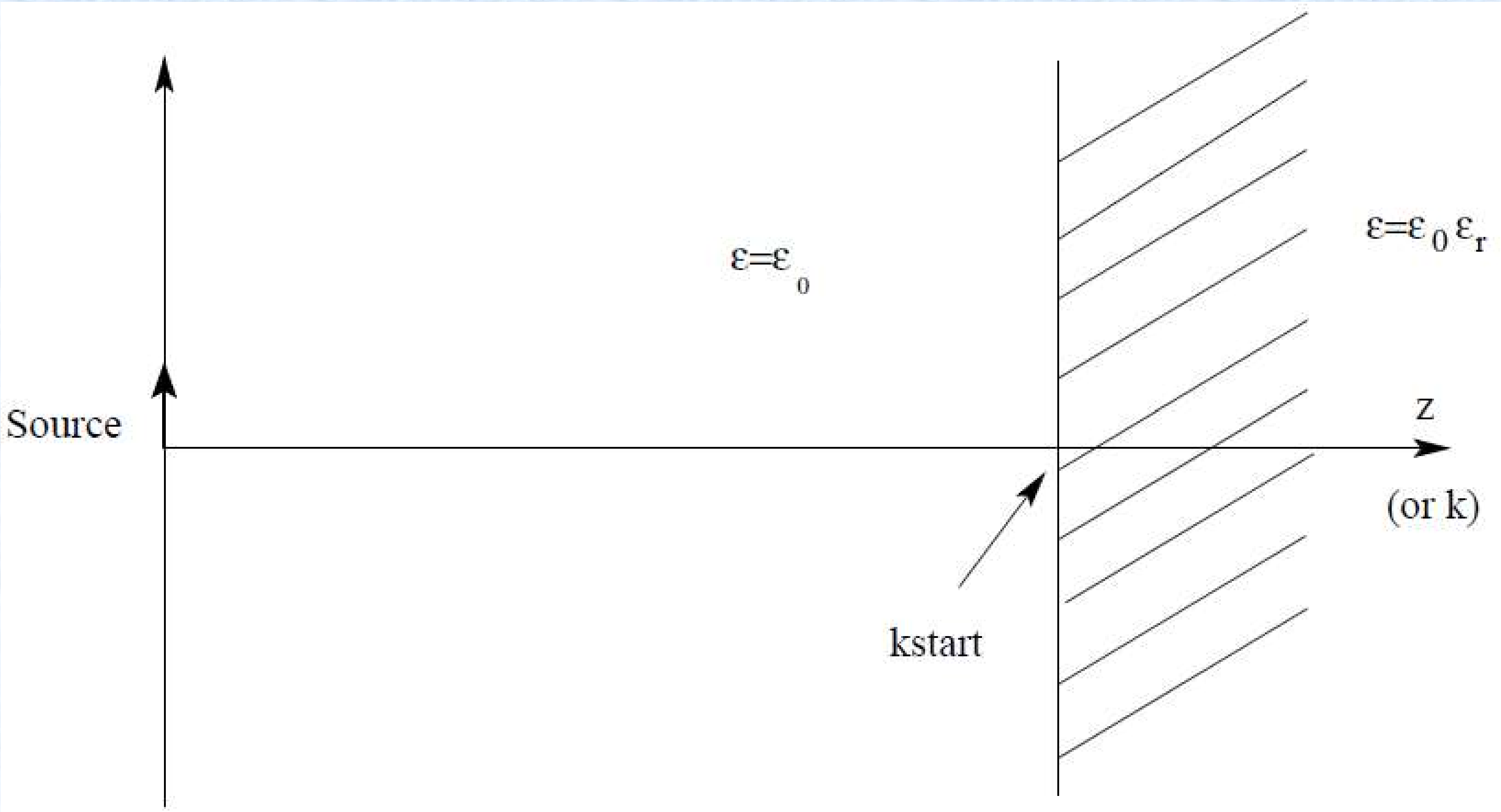
- Hence the computer update equations (for free space er(k)=1) are
- ex[k] = ex[k] + (0.5/er(k))*(hy[k-1] - hy[k])
- hy[k] = hy[k] + 0.5*(ex[k] - ex[k+1])

FDTD: An Introduction

  - k+1/2 and k-1/2 are replaced by k or k-1
  - Note that the n or n + 1/2 or n −1/2 in the superscripts do not appear
- *FDTD Simulation of Gaussian pulse propagation in free space (*fdtd_1d_1.m*)*
- *FDTD Simulation of Gaussian pulse hitting a dielectric medium (*fdtd1_dielectric.m*)*
- From theoretical analysis of EM wave hitting a dielectric surface

$$\tau = \frac{2\eta_2}{\eta_2 + \eta_1} = \frac{2\sqrt{\varepsilon_1}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}} \qquad \Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{\sqrt{\varepsilon_1} - \sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}}$$

FDTD: An Introduction



- Fig. EM wave hitting a dielectric surface

FDTD: An Introduction

- For ε_1=1.0 and ε_2=4.0, we have $\Gamma = -\frac{1}{3}; \tau = \frac{2}{3}$
- *FDTD simulation of Absorbing Boundary Condition*
  - (fdtd_1d_ABC_boundary.m) (**Programming Exercise 5**)
  - (fdtd_1d_no_boundary.m)
  - *%Define constants for ABC*
  - *ex_left_m1=0.0*
  - *ex_left_m2=0.0*
  - *ex_right_m1=0.0*
  - *ex_right_m2=0.0*