FDTD: An Introduction

- Wave propagates more slowly on the medium on the right and
  - the pulse length is shorter
- *FDTD simulation of TM wave propagation due to multiple sources* (fdtd_2d_TM_multi_source.m)
- Just like ABC, we can have **Mur's Boundary Condition**
- Consider Ez component of a TM wave

$$\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}-\frac{1}{v_p^2}\frac{\partial^2}{\partial t^2}\right]E_z=0$$

$$\Rightarrow\left[\frac{\partial^2}{\partial x^2}-\left(\frac{1}{v_p^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial y^2}\right)\right]E_z=0$$

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- It can be expressed as

$$\left[\frac{\partial}{\partial x}+\sqrt{\frac{1}{v_p^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial y^2}}\right]\left[\frac{\partial}{\partial x}-\sqrt{\frac{1}{v_p^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial y^2}}\right]E_z=0$$

$$\Rightarrow \Psi^+\Psi^-E_z=0;\Psi^+=\frac{\partial}{\partial x}+\sqrt{\frac{1}{v_p^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial y^2}};\Psi^-=\frac{\partial}{\partial x}-\sqrt{\frac{1}{v_p^2}\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial y^2}}$$

- This operator can be expressed as

$$\Psi^+=\frac{\partial}{\partial x}+\frac{1}{v_p}\frac{\partial}{\partial t}\sqrt{1-S^2};S\equiv\frac{v_p\dfrac{\partial}{\partial y}}{\dfrac{\partial}{\partial t}}$$

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- From Taylor's series expansion

$$\sqrt{1-S^2} = 1 - \frac{S^2}{2} - \frac{S^4}{8} - \frac{S^6}{16} - \cdots$$

- Simplest approximation, first-order Mur Boundary Condition gives one way wave equation

$$\Psi^+ E_z = \frac{\partial E_z}{\partial x} + \frac{1}{v_p}\frac{\partial E_z}{\partial t} \cong 0$$

$$\Psi^- E_z = \frac{\partial E_z}{\partial x} - \frac{1}{v_p}\frac{\partial E_z}{\partial t} \cong 0$$

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- How do you discretize it?

$$\frac{\partial E_z}{\partial t} = v_p \frac{\partial E_z}{\partial x} \rightarrow \frac{E_z\big|_{3/2}^{n+1} - E_z\big|_{3/2}^{n}}{\Delta t} = v_p \frac{E_z\big|_{2}^{n+1/2} - E_z\big|_{1}^{n+1/2}}{\Delta x}$$

- Note that in 1-D TM , Ez is defined at integer time and space step, hence, we take time and space average for available values

$$\frac{\left(E_z\big|_1^{n+1} + E_z\big|_2^{n+1}\right) - \left(E_z\big|_1^{n} + E_z\big|_2^{n}\right)}{2\Delta t}$$

$$= v_p \frac{\left(E_z\big|_2^{n+1} + E_z\big|_2^{n}\right) - \left(E_z\big|_1^{n+1} + E_z\big|_1^{n}\right)}{2\Delta x}$$

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- Simplifying

$$\left[\left(E_z\Big|_1^{n+1}+E_z\Big|_2^{n+1}\right)-\left(E_z\Big|_1^{n}+E_z\Big|_2^{n}\right)\right]$$

$$=\frac{\Delta t v_p}{\Delta x}\left[\left(E_z\Big|_2^{n+1}+E_z\Big|_2^{n}\right)-\left(E_z\Big|_1^{n+1}+E_z\Big|_1^{n}\right)\right]$$

- Rearranging

$$\left(1+\frac{v_p\Delta t}{\Delta x}\right)E_z\Big|_1^{n+1}=\left(1+\frac{v_p\Delta t}{\Delta x}\right)E_z\Big|_2^{n}-\left(1-\frac{v_p\Delta t}{\Delta x}\right)E_z\Big|_2^{n+1}+\left(1-\frac{v_p\Delta t}{\Delta x}\right)E_z\Big|_1^{n}$$

$$\Rightarrow E_z\Big|_1^{n+1}=E_z\Big|_2^{n}+\left(\frac{v_p\Delta t-\Delta x}{v_p\Delta t+\Delta x}\right)\left(E_z\Big|_2^{n+1}-E_z\Big|_1^{n}\right)$$

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- For CFL, it reduces to

$$E_z\Big|_1^{n+1} = E_z\Big|_2^n ; \frac{v_p \Delta t}{\Delta x} = 1$$

- Similarly for the rightmost edge

$$E_z\Big|_{i\max}^{n+1} = E_z\Big|_{i\max-1}^n + \left(\frac{v_p \Delta t - \Delta x}{v_p \Delta t + \Delta x}\right)\left(E_z\Big|_{i\max-1}^{n+1} - E_z\Big|_{i\max}^n\right)$$

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Second-order Mur Boundary Condition keeps two terms of Taylor series expansion

$$\sqrt{1-S^2} \cong 1-\frac{S^2}{2}$$

- Therefore,

$$\Psi^+ = \frac{\partial}{\partial x} + \frac{1}{v_p}\frac{\partial}{\partial t}\left(1-\frac{S^2}{2}\right)$$

$$\Rightarrow \Psi^+ = \frac{\partial}{\partial x} + \frac{1}{v_p}\frac{\partial}{\partial t} - \frac{1}{2}\frac{1}{v_p}\frac{\partial}{\partial t}\left(\frac{v_p \frac{\partial}{\partial y}}{\frac{\partial}{\partial t}}\right)^2$$

$$\Rightarrow \Psi^+ = \frac{\partial}{\partial x} + \frac{1}{v_p}\frac{\partial}{\partial t} - \frac{v_p}{2}\left(\frac{\frac{\partial^2}{\partial y^2}}{\frac{\partial}{\partial t}}\right)$$

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- Hence,

$$\Psi^{+}E_z = \frac{\partial E_z}{\partial x} + \frac{1}{v_p}\frac{\partial E_z}{\partial t} - \frac{v_p}{2}\left(\frac{\frac{\partial^2 E_z}{\partial y^2}}{\frac{\partial}{\partial t}}\right) = 0$$

$$\Rightarrow \frac{\partial^2 E_z}{\partial x \partial t} + \frac{1}{v_p}\frac{\partial^2 E_z}{\partial t^2} - \frac{v_p}{2}\frac{\partial^2 E_z}{\partial y^2} = 0$$

- Similarly,

$$\Psi^{-}E_z = 0$$

$$\Rightarrow \frac{\partial^2 E_z}{\partial x \partial t} - \frac{1}{v_p}\frac{\partial^2 E_z}{\partial t^2} + \frac{v_p}{2}\frac{\partial^2 E_z}{\partial y^2} = 0$$

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- How to discretize it?

$$\left.\frac{\partial^2 E_z}{\partial x \partial t}\right|_{3/2,j}^{n} = \frac{1}{2\Delta t}\left(\left.\frac{\partial E_z}{\partial x}\right|_{3/2,j}^{n+1} - \left.\frac{\partial E_z}{\partial x}\right|_{3/2,j}^{n-1}\right)$$

$$= \frac{1}{2\Delta t}\left(\frac{E_z|_{2,j}^{n+1} - E_z|_{1,j}^{n+1}}{\Delta x} - \frac{E_z|_{2,j}^{n-1} - E_z|_{1,j}^{n-1}}{\Delta x}\right)$$

$$\left.\frac{\partial^2 E_z}{\partial y^2}\right|_{3/2,j}^{n} \cong \frac{1}{2}\left(\left.\frac{\partial^2 E_z}{\partial y^2}\right|_{2,j}^{n} + \left.\frac{\partial^2 E_z}{\partial y^2}\right|_{1,j}^{n}\right)$$

$$= \frac{1}{2}\left(\frac{E_z|_{2,j+1}^{n} - 2E_z|_{2,j}^{n} + E_z|_{2,j-1}^{n}}{(\Delta y)^2} - \frac{E_z|_{1,j+1}^{n} - 2E_z|_{1,j}^{n} + E_z|_{1,j-1}^{n}}{(\Delta y)^2}\right)$$

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- Similarly

$$\left.\frac{\partial^2 E_z}{\partial t^2}\right|_{3/2,j}^{n} \cong \frac{1}{2}\left(\left.\frac{\partial^2 E_z}{\partial t^2}\right|_{2,j}^{n} + \left.\frac{\partial^2 E_z}{\partial t^2}\right|_{1,j}^{n}\right)$$

$$= \frac{1}{2}\left(\frac{E_z|_{2,j}^{n+1} - 2E_z|_{2,j}^{n} + E_z|_{2,j}^{n-1}}{(\Delta t)^2} - \frac{E_z|_{1,j}^{n+1} - 2E_z|_{1,j}^{n} + E_z|_{1,j}^{n-1}}{(\Delta t)^2}\right)$$

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- Finally

$$E_z\Big|_{1,j}^{n+1} = -E_z\Big|_{2,j}^{n-1} - \left(\frac{\Delta x - v_p \Delta t}{v_p \Delta t + \Delta x}\right)\left(E_z\Big|_{2,j}^{n+1} + E_z\Big|_{1,j}^{n-1}\right) + \left(\frac{2\Delta x}{v_p \Delta t + \Delta x}\right)\left(E_z\Big|_{1,j}^{n} + E_z\Big|_{2,j}^{n}\right)$$

$$+\left(\frac{\Delta x\left(v_p \Delta t\right)^2}{2\left(\Delta y\right)^2\left(v_p \Delta t + \Delta x\right)}\right)\left(E_z\Big|_{1,j+1}^{n} - 2E_z\Big|_{1,j}^{n} + E_z\Big|_{1,j-1}^{n} + E_z\Big|_{2,j+1}^{n} - 2E_z\Big|_{2,j}^{n} + E_z\Big|_{2,j-1}^{n}\right)$$

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3. 3-D FDTD

- From two Maxwell's curl equations,
  - we need to consider all 6 equations for 3-D case

$$\frac{\partial \vec{E}}{\partial t} = \frac{1}{\varepsilon_0 \varepsilon_r} \nabla \times \vec{H}; \frac{\partial \vec{H}}{\partial t} = -\frac{1}{\mu_0 \mu_r} \nabla \times \vec{E}$$

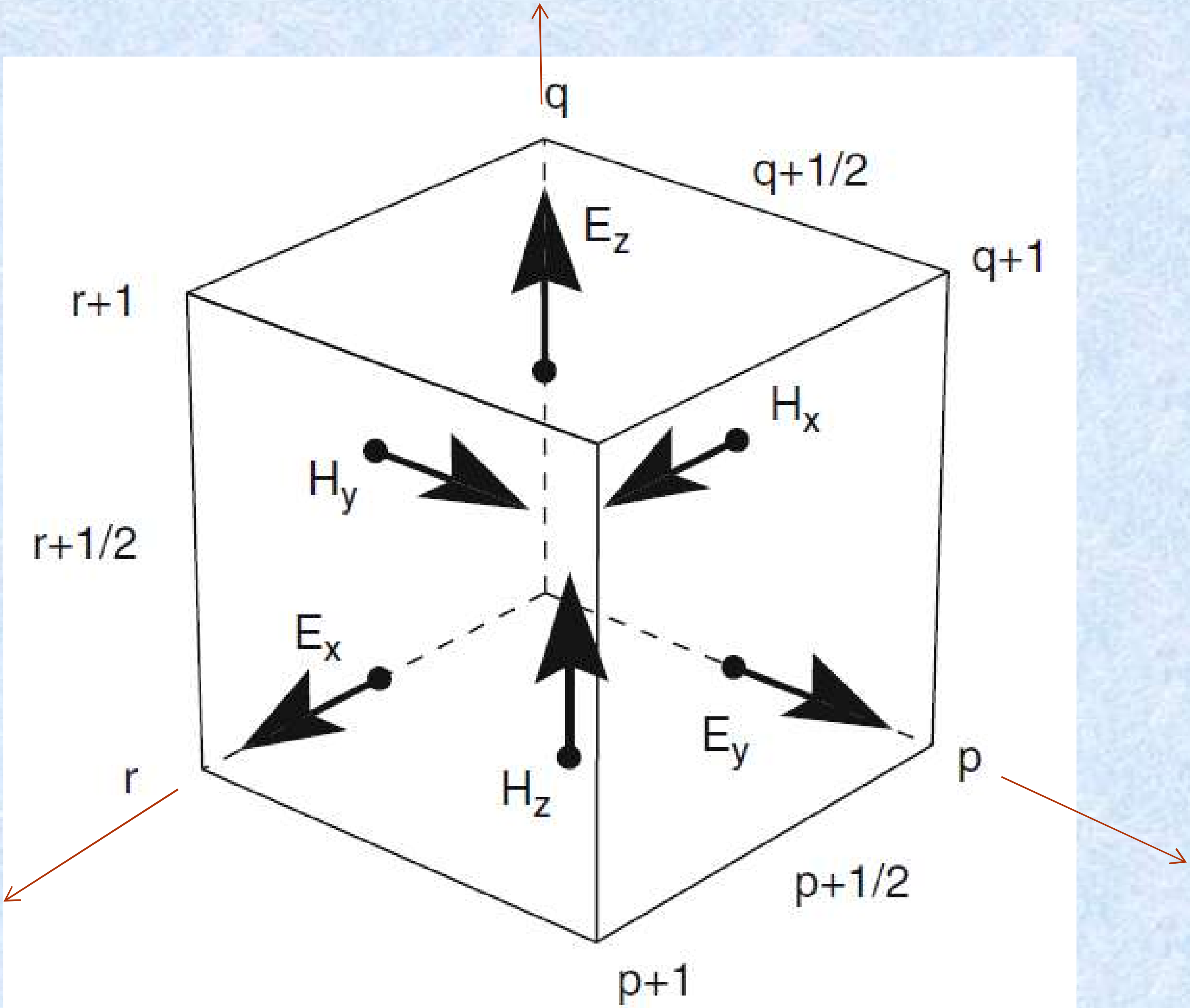

Fig. 4 Yee's FDTD grid

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Yee's FDTD grid

- Unlike 1-D and 2-D FDTD cases,
- Electric fields are calculated at "integer" time-steps
  - and magnetic field at "half-integer" time-steps
- Electric field components are placed at mid-points of the corresponding edges
- E.g.
  - Ex is placed at midpoints of edges oriented along x-direction
  - Ex is on half-grid in x and the integer grids in y & z

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- The magnetic field components are
  - placed at the centers of the faces of the cubes and
  - oriented normal to the faces
- E.g.
- Hx components are placed at the centers of the faces in the yz-plane
- Hence Hx is
  - on the integer grid in x and
  - on the half-grid in y & z

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Electric fields are calculated at "integer" time-steps
and magnetic field at "half-integer" time-steps

Ex is placed at midpoints of edges oriented along x-direction

$$\frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon_r \varepsilon_0}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)$$

Hz is placed at center of faces in xy-plane

$$\frac{E_x^{n+1}(p+1/2,q,r) - E_x^{n}(p+1/2,q,r)}{\Delta t} =$$

$$\frac{1}{\varepsilon_r \varepsilon_0}\left[\frac{H_z^{n+1/2}(p+1/2,q+1/2,r) - H_z^{n+1/2}(p+1/2,q-1/2,r)}{\Delta y}\right]$$

$$-\frac{1}{\varepsilon_r \varepsilon_0}\left[\frac{H_y^{n+1/2}(p+1/2,q,r+1/2) - H_y^{n+1/2}(p+1/2,q,r-1/2)}{\Delta z}\right]$$

Hy is placed at center of faces in xz-plane

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$$\frac{\partial E_y}{\partial t} = \frac{1}{\varepsilon_r \varepsilon_0}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)$$

$$\frac{E_y^{n+1}(p,q+1/2,r) - E_y^{n}(p,q+1/2,r)}{\Delta t} =$$

$$\frac{1}{\varepsilon_r \varepsilon_0}\left[\frac{H_x^{n+1/2}(p,q+1/2,r+1/2) - H_x^{n+1/2}(p,q+1/2,r-1/2)}{\Delta z}\right]$$

$$-\frac{1}{\varepsilon_r \varepsilon_0}\left[\frac{H_z^{n+1/2}(p+1/2,q+1/2,r) - H_z^{n+1/2}(p-1/2,q+1/2,r)}{\Delta z}\right]$$

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$$\frac{\partial E_z}{\partial t}=\frac{1}{\varepsilon_r\varepsilon_0}\left(\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}\right)$$

$$\frac{E_z^{n+1}(p,q,r+1/2)-E_z^{n}(p,q,r+1/2)}{\Delta t}=$$

$$\frac{1}{\varepsilon_r\varepsilon_0}\left[\frac{H_y^{n+1/2}(p+1/2,q,r+1/2)-H_y^{n+1/2}(p-1/2,q,r+1/2)}{\Delta x}\right]$$

$$-\frac{1}{\varepsilon_r\varepsilon_0}\left[\frac{H_x^{n+1/2}(p,q+1/2,r+1/2)-H_x^{n+1/2}(p,q-1/2,r+1/2)}{\Delta y}\right]$$