FDTD: An Introduction

$$\frac{\partial H_x}{\partial t} = -\frac{1}{\mu_0}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right)$$

$$\frac{H_x^{n+1/2}\left(p, q+1/2, r+1/2\right) - H_x^{n-1/2}\left(p, q+1/2, r+1/2\right)}{\Delta t} =$$

$$\frac{1}{\mu_0}\left[\frac{E_y^n\left(p, q+1/2, r+1\right) - E_y^n\left(p, q+1/2, r\right)}{\Delta z}\right]$$

$$-\frac{1}{\mu_0}\left[\frac{E_z^n\left(p, q+1, r+1/2\right) - E_z^n\left(p, q, r+1/2\right)}{\Delta y}\right]$$

FDTD: An Introduction

$$\frac{\partial H_y}{\partial t} = -\frac{1}{\mu_0}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)$$

$$\frac{H_y^{n+1/2}(p+1/2, q, r+1/2) - H_y^{n-1/2}(p+1/2, q, r+1/2)}{\Delta t} =$$

$$\frac{1}{\mu_0}\left[\frac{E_z^n(p+1, q, r+1/2) - E_z^n(p, q, r+1/2)}{\Delta x}\right]$$

$$-\frac{1}{\mu_0}\left[\frac{E_x^n(p+1/2, q, r+1) - E_x^n(p+1/2, q, r)}{\Delta z}\right]$$

FDTD: An Introduction

$$\frac{\partial H_z}{\partial t} = -\frac{1}{\mu_0}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)$$

$$\frac{H_z^{n+1/2}(p+1/2, q+1/2, r) - H_z^{n-1/2}(p+1/2, q+1/2, r)}{\Delta t} =$$

$$\frac{1}{\mu_0}\left[\frac{E_x^n(p+1/2, q+1/2, r) - E_x^n(p+1/2, q, r)}{\Delta y}\right]$$

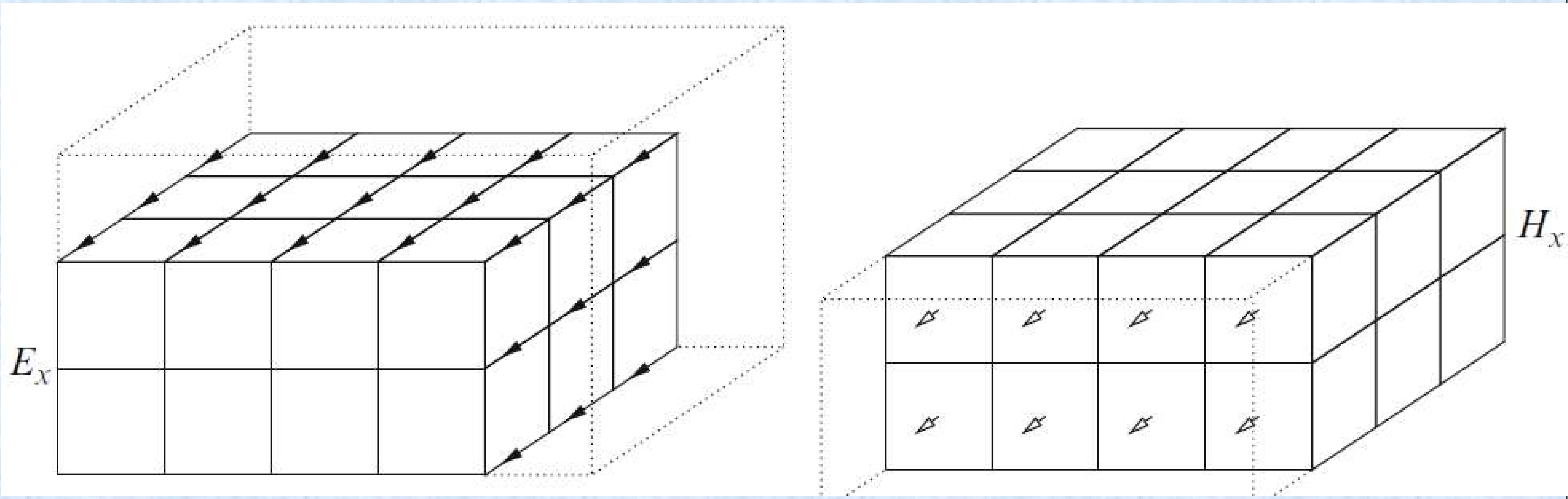
$$-\frac{1}{\mu_0}\left[\frac{E_y^n(p+1, q+1/2, r) - E_y^n(p, q+1/2, r)}{\Delta x}\right]$$

FDTD: An Introduction

- *FDTD simulation of cubical cavity (fdtd_3D_demo.m)*
- We will use FDTD to find
  - the resonant frequencies of an air-filled cubical cavity with metal walls
- Peaks in the frequency response indicate
  - the presence of resonant modes in the cavity
- An initial Hz field excites only
  - the TE modes
- The first four eigenfrequencies
  - TE_{101}, TE_{011}/TE_{201}, TE_{111} and TE_{102} modes

FDTD: An Introduction

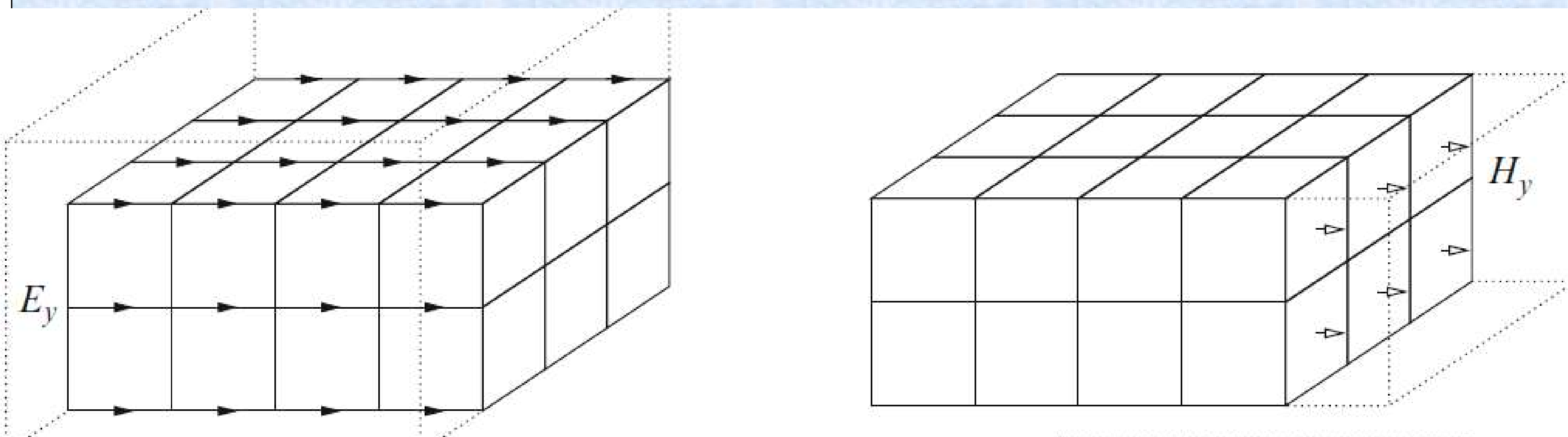
- Field components placed on a grid with 3×4×2 cells



- For E_x (3×5×3) and H_x (4×4×2)

FDTD: An Introduction

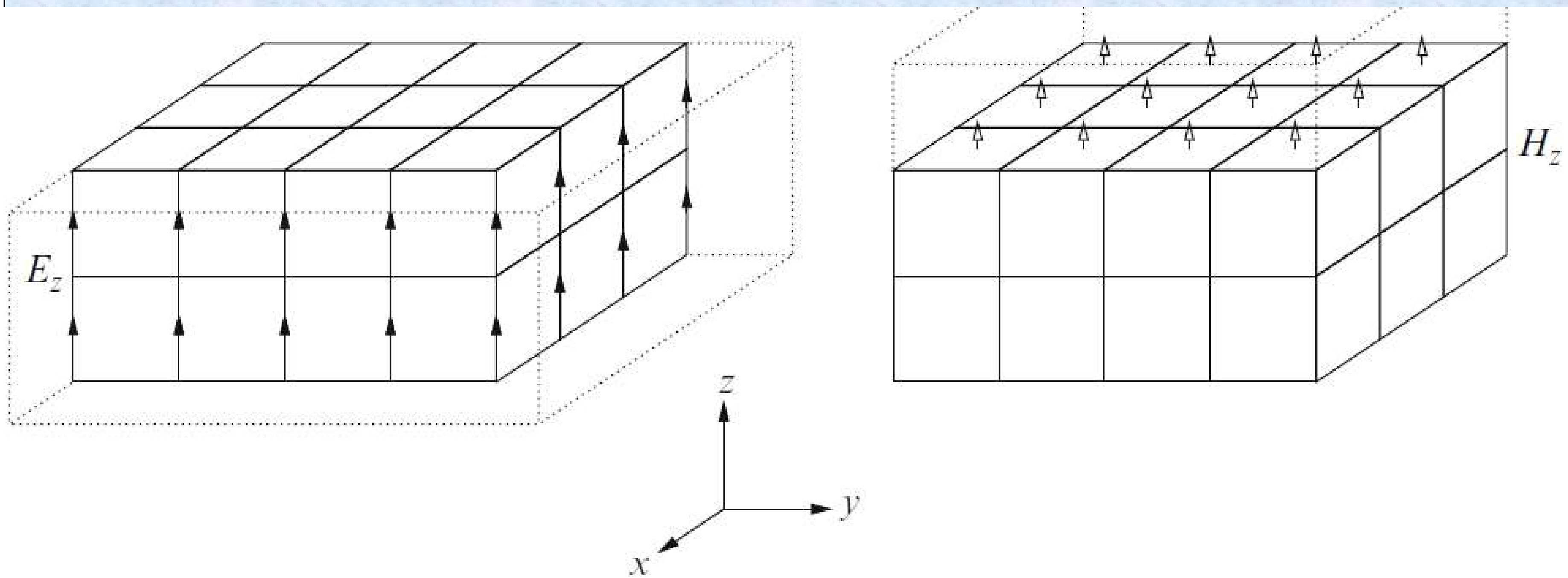
- Field components placed on a grid with 3×4×2 cells



- For E_y (4×4×3) and H_y (3×5×2)

FDTD: An Introduction

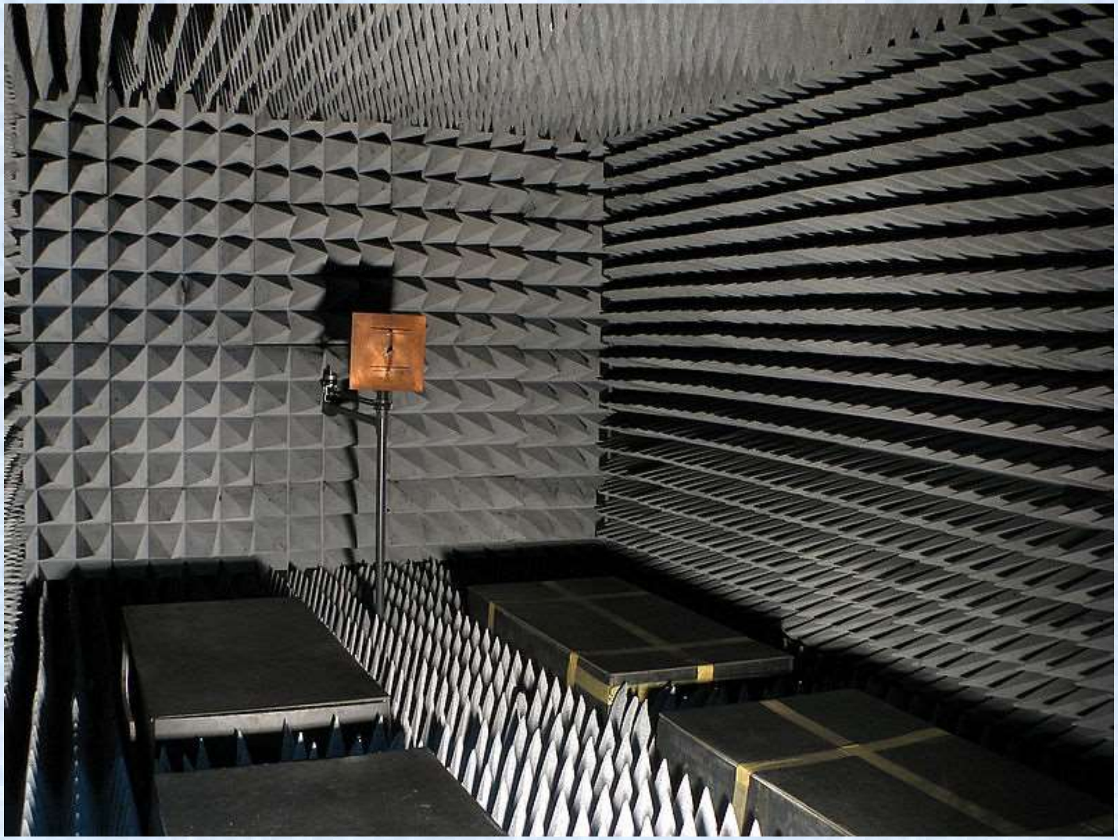
- Field components placed on a grid with 3×4×2 cells



- For E_z (4×5×2) and H_z (3×4×3)

FDTD: An Introduction

- Artificial absorbers on the walls of anechoic chamber



FDTD: An Introduction

4. Perfectly Matched Layers

- Berenger's PML is an artificial material
- that is theoretically designed to create no reflections regardless of the
  - frequency,
  - polarization and
  - angle of incidence of a plane wave upon its interface

FDTD: An Introduction

- PML could be analyzed in stretched coordinate systems
- Source-free Maxwell's equations

$$\nabla_{stretched} \times \vec{E} = -j\omega\mu\vec{H}; \nabla_{stretched} \times \vec{H} = j\omega\varepsilon\vec{E};$$

$$\nabla_{stretched} \bullet \left(\varepsilon\vec{E}\right) = 0; \nabla_{stretched} \bullet \left(\mu\vec{H}\right) = 0;$$

$$\nabla_{stretched} = \hat{x}\frac{1}{s_x(x)}\frac{\partial}{\partial x} + \hat{y}\frac{1}{s_y(y)}\frac{\partial}{\partial y} + \hat{z}\frac{1}{s_z(z)}\frac{\partial}{\partial z}$$

- Consider plane waves

$$\vec{E} = \vec{E}_0 e^{-j\vec{k}\bullet\vec{r}}; \vec{H} = \vec{H}_0 e^{-j\vec{k}\bullet\vec{r}}$$

$$\vec{k} \bullet \vec{r} = k_x x + k_y y + k_z z$$

FDTD: An Introduction

- Substituting these into Maxwell's equations, we have,

$$\vec{k}_{stretched} \times \vec{E} = \omega\mu\vec{H}; \vec{k}_{stretched} \times \vec{H} = -\varepsilon\vec{E};$$

$$\vec{k}_{stretched} \bullet \left(\varepsilon\vec{E}\right) = 0; \vec{k}_{stretched} \bullet \left(\mu\vec{H}\right) = 0;$$

$$\vec{k}_{stretched} = \hat{x}\frac{k_x}{s_x} + \hat{y}\frac{k_y}{s_y} + \hat{z}\frac{k_z}{s_z}$$

$$\vec{k}_{stretched} \bullet \vec{k}_{stretched} = \left(\frac{k_x}{s_x}\right)^2 + \left(\frac{k_y}{s_y}\right)^2 + \left(\frac{k_z}{s_y}\right)^2 = k^2$$

- Solution to the above equation is given by

$$k_x = ks_x \sin\theta\cos\phi, k_y = ks_y \sin\theta\sin\phi, k_z = ks_z \cos\theta$$

FDTD: An Introduction

- If s_x is a complex number with a negative imaginary part,
  - the wave will be attenuated in the x direction

$$e^{-jk_x x} = e^{-jks_x \sin\theta\cos\phi x} = e^{-jk\left(s' - js''\right)\sin\theta\cos\phi x}, s' \geq 1, s'' > 0$$

- Wave impedance is independent of coordinate stretching

$$\eta = \frac{\left|\vec{E}\right|}{\left|\vec{H}\right|} = \frac{\omega\mu}{\left|\vec{k}_{stretched}\right|} = \frac{\omega\mu}{\left|\vec{k}\right|} = \sqrt{\frac{\mu}{\varepsilon}}$$

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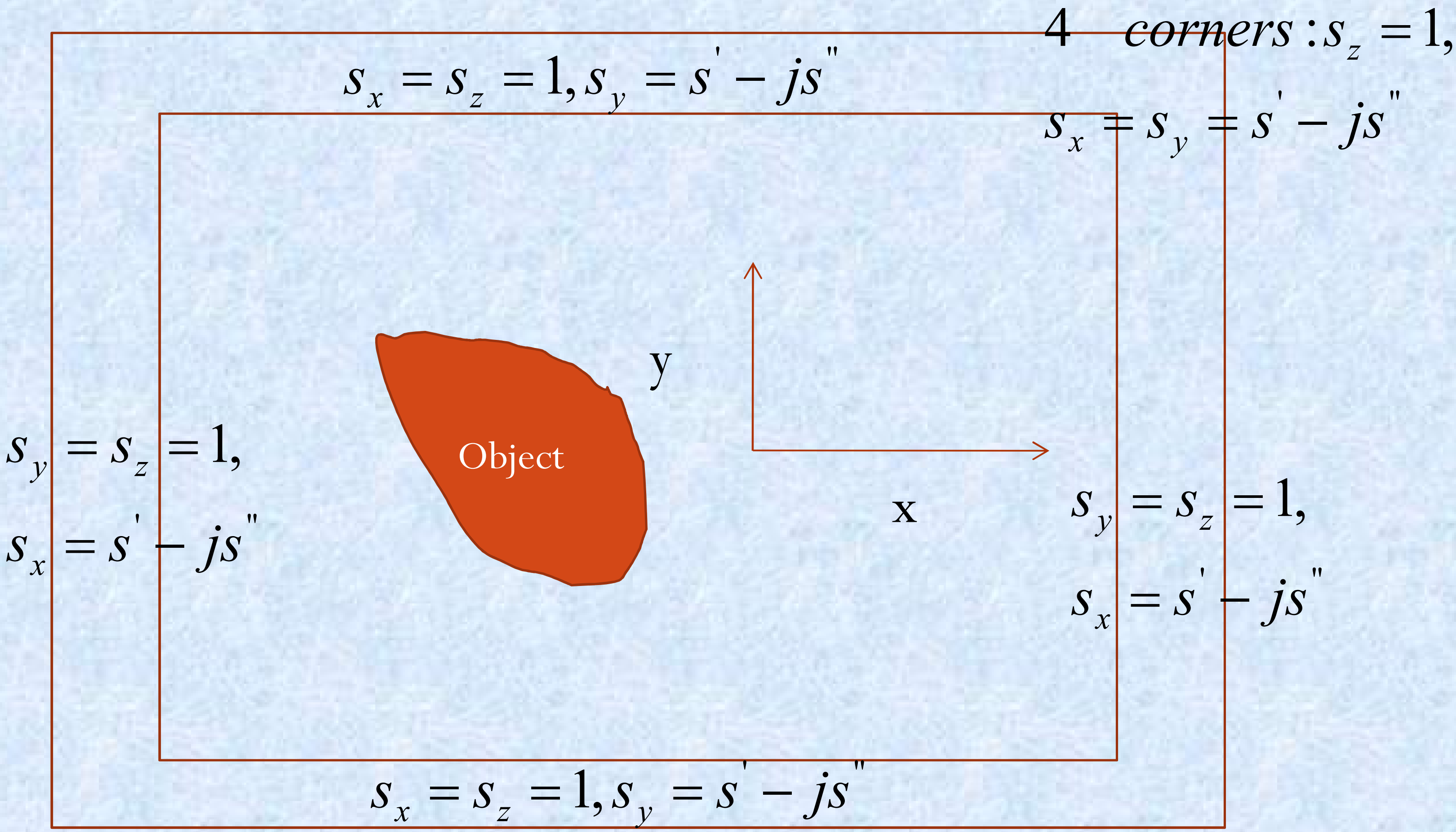

Fig. 5 Computational domain truncated using the conductor backed PMLs

FDTD: An Introduction

$$
\begin{aligned}
&\nabla_{stretched} \times \vec{E} \\
&= \frac{1}{s_x}\frac{\partial}{\partial x}\left(\hat{x}\times\vec{E}\right) + \frac{1}{s_y}\frac{\partial}{\partial y}\left(\hat{y}\times\vec{E}\right) + \frac{1}{s_z}\frac{\partial}{\partial z}\left(\hat{z}\times\vec{E}\right) \\
&= -j\omega\mu\vec{H}
\end{aligned}
$$

LHS has three terms, we can split the magnetic field into three parts

FDTD: An Introduction

Splitting the magnetic field and equating LHS and RHS

$$\vec{H} = \vec{H}_{sx} + \vec{H}_{sy} + \vec{H}_{sz}$$

$$\frac{1}{s_x}\frac{\partial}{\partial x}\left(\hat{x}\times\vec{E}\right) = -j\omega\mu\vec{H}_{sx}$$

$$\frac{1}{s_y}\frac{\partial}{\partial y}\left(\hat{y}\times\vec{E}\right) = -j\omega\mu\vec{H}_{sy}$$

$$\frac{1}{s_z}\frac{\partial}{\partial z}\left(\hat{z}\times\vec{E}\right) = -j\omega\mu\vec{H}_{sz}$$

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$$Choosing \quad s_x = 1 - j\frac{\sigma_x}{\omega\varepsilon}, s_y = 1 - j\frac{\sigma_y}{\omega\varepsilon}, s_z = 1 - j\frac{\sigma_z}{\omega\varepsilon}$$

$$\frac{\partial}{\partial x}\left(\hat{x}\times\vec{E}\right) = -j\omega\mu\left(1 - j\frac{\sigma_x}{\omega\varepsilon}\right)\vec{H}_{sx} = -\mu\frac{\partial\vec{H}_{sx}}{\partial t} - \frac{\sigma_x\mu}{\varepsilon}\vec{H}_{sx}$$

$$\frac{\partial}{\partial y}\left(\hat{y}\times\vec{E}\right) = -j\omega\mu s_y\vec{H}_{sy} = -\mu\frac{\partial\vec{H}_{sy}}{\partial t} - \frac{\sigma_y\mu}{\varepsilon}\vec{H}_{sy}$$

$$\frac{\partial}{\partial z}\left(\hat{z}\times\vec{E}\right) = -j\omega\mu s_z\vec{H}_{sz} = -\mu\frac{\partial\vec{H}_{sz}}{\partial t} - \frac{\sigma_z\mu}{\varepsilon}\vec{H}_{sz}$$

May be used for update equations for magnetic fields

FDTD: An Introduction

Similarly, Splitting the electric field

$$\nabla_{stretched} \times \vec{H} = j\omega\varepsilon\vec{E}; \qquad \vec{E} = \vec{E}_{sx} + \vec{E}_{sy} + \vec{E}_{sz}$$

$$\frac{\partial}{\partial x}\left(\hat{x} \times \vec{H}\right) = \varepsilon\frac{\partial \vec{E}_{sx}}{\partial t} + \sigma_x \vec{E}_{sx}$$

$$\frac{\partial}{\partial y}\left(\hat{y} \times \vec{H}\right) = \varepsilon\frac{\partial \vec{E}_{sy}}{\partial t} + \sigma_y \vec{E}_{sy}$$

$$\frac{\partial}{\partial z}\left(\hat{z} \times \vec{H}\right) = \varepsilon\frac{\partial \vec{E}_{sz}}{\partial t} + \sigma_z \vec{E}_{sz}$$

May be used for update equations for electric fields

FDTD: An Introduction

- For instance, consider 2-D TM^z problem
- Ex = 0, Ey = 0, Hz = 0, $\frac{\partial}{\partial z}=0$

$$\frac{1}{s_x}\frac{\partial}{\partial x}\left(\hat{x}\times\vec{E}\right)=-j\omega\mu\vec{H}_{sx}$$

$$\frac{1}{s_y}\frac{\partial}{\partial y}\left(\hat{y}\times\vec{E}\right)=-j\omega\mu\vec{H}_{sy}$$

$$\frac{1}{s_z}\frac{\partial}{\partial z}\left(\hat{z}\times\vec{E}\right)=-j\omega\mu\vec{H}_{sz}$$

$$\vec{E}=E_z\hat{z};\vec{H}=H_x\hat{x}+H_y\hat{y}$$

$$\hat{x}\times\vec{E}\quad \textit{will} \quad \textit{be} \quad \textit{along} \quad \hat{y} \quad \textit{direction}$$

- Hence, $\vec{H}_{sx}=\hat{y}H_y$
- Similarly,

$$\vec{H}_{sy}=\hat{x}H_x,\vec{H}_{sz}=0$$

FDTD: An Introduction

$$\frac{\partial}{\partial x}\left(\hat{x}\times\vec{E}\right)=\frac{\partial}{\partial x}\left(\hat{x}\times E_z\hat{z}\right)=-\frac{\partial E_z}{\partial x}\hat{y}$$

$$=-j\omega\mu s_x\vec{H}_{sx}=-j\omega\mu s_y H_y\hat{y}=\left(-\mu\frac{\partial H_y}{\partial t}-\frac{\sigma_x\mu}{\varepsilon}H_y\right)\hat{y}$$

$$\frac{\partial}{\partial y}\left(\hat{y}\times\vec{E}\right)=\frac{\partial}{\partial y}\left(\hat{y}\times E_z\hat{z}\right)=\frac{\partial E_z}{\partial y}\hat{x}$$

$$=-j\omega\mu s_y\vec{H}_{sy}=-j\omega\mu s_y H_x\hat{x}=\left(-\mu\frac{\partial H_x}{\partial t}-\frac{\sigma_y\mu}{\varepsilon}H_x\right)\hat{x}$$

$$\frac{\partial}{\partial z}\left(\hat{z}\times\vec{E}\right)=-j\omega\mu s_z\vec{H}_{sz}=0$$

FDTD: An Introduction

- Split field for magnetic fields for 2-D TM^z problem

$$\varepsilon \frac{\partial E_z}{\partial x} = \mu\varepsilon \frac{\partial H_y}{\partial t} + \sigma_x \mu H_y$$

$$\varepsilon \frac{\partial E_z}{\partial y} = -\mu\varepsilon \frac{\partial H_x}{\partial t} - \sigma_y \mu H_x$$

- Split fields for electric fields (TM^z, electric field has only z components)

$$\hat{x} \times \vec{H}, \hat{y} \times \vec{H} \quad will \quad be \quad along \quad \hat{z} \quad direction$$

$$\because \vec{E} = E_z \hat{z}; \vec{H} = H_x \hat{x} + H_y \hat{y}$$

FDTD: An Introduction

$$\frac{\partial}{\partial x}\left(\hat{x}\times\vec{H}\right)=\frac{\partial H_y}{\partial x}\hat{z}=\varepsilon\frac{\partial \vec{E}_{sx}}{\partial t}+\sigma_x\vec{E}_{sx}=\left(\varepsilon\frac{\partial \vec{E}_{sx,z}}{\partial t}+\sigma_x\vec{E}_{sx,z}\right)\hat{z}$$

$$\frac{\partial}{\partial y}\left(\hat{y}\times\vec{H}\right)=-\frac{\partial H_x}{\partial y}\hat{z}=\varepsilon\frac{\partial \vec{E}_{sy}}{\partial t}+\sigma_y\vec{E}_{sy}=\left(\varepsilon\frac{\partial \vec{E}_{sy,z}}{\partial t}+\sigma_y\vec{E}_{sy,z}\right)\hat{z}$$

$$\frac{\partial}{\partial z}\left(\hat{z}\times\vec{H}\right)=\varepsilon\frac{\partial \vec{E}_{sz}}{\partial t}+\sigma_z\vec{E}_{sz}=0$$

$$\because \vec{E}=E_z\hat{z};\vec{H}=H_x\hat{x}+H_y\hat{y}$$

FDTD: An Introduction

- Hence, Split field for electric fields for 2-D TM^z problem

$$\frac{\partial H_y}{\partial x} = \varepsilon \frac{\partial E_{sx,z}}{\partial t} + \sigma_x E_{sx,z}$$

$$\frac{\partial H_x}{\partial y} = -\varepsilon \frac{\partial E_{sy,z}}{\partial t} - \sigma_y E_{sy,z}$$

$$E_z = E_{sx,z} + E_{sy,z}$$