FDTD: An Introduction

- Convention:
- Integer time and space step for E
- Half time step for H
- Half space step for H
- Hx(i,j+1/2), Hx(i,j-1/2)
- Hy(i+1/2,j), Hx(i-1/2,j)

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- Update equation for Hy

$$\varepsilon\frac{\partial H_y}{\partial t}=\frac{\varepsilon}{\mu}\frac{\partial E_z}{\partial x}-\sigma_x H_y$$

$$\Rightarrow\frac{\varepsilon}{\Delta t}\left\{H_y^{n+1/2}(i+1/2,j)-H_y^{n-1/2}(i+1/2,j)\right\}$$

$$=\frac{\varepsilon}{\mu\Delta x}\left\{E_z^n(i+1,j)-E_z^n(i,j)\right\}-\frac{\sigma_x}{2}\left\{H_y^{n+1/2}(i+1/2,j)+H_y^{n-1/2}(i+1/2,j)\right\}$$

$$\Rightarrow\left(\frac{\varepsilon}{\Delta t}+\frac{\sigma_x}{2}\right)\left\{H_y^{n+1/2}(i+1/2,j)\right\}$$

$$=\frac{\varepsilon}{\mu\Delta x}\left\{E_z^n(i+1,j)-E_z^n(i,j)\right\}+\left(\frac{\varepsilon}{\Delta t}-\frac{\sigma_x}{2}\right)\left\{H_y^{n-1/2}(i+1/2,j)\right\}$$

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- Finally,

$$\Rightarrow H_y^{n+1/2}(i+1/2,j)$$

$$=\frac{1}{\left(\frac{\varepsilon}{\Delta t}+\frac{\sigma_x}{2}\right)}\left[\frac{\varepsilon}{\mu\Delta x}\left\{E_z^n(i+1,j)-E_z^n(i,j)\right\}+\left(\frac{\varepsilon}{\Delta t}-\frac{\sigma_x}{2}\right)\left\{H_y^{n-1/2}(i+1/2,j)\right\}\right]$$

$$=\frac{1}{\beta_x(i+1/2,j)}\left[\frac{\varepsilon}{\mu\Delta x}\left\{E_z^n(i+1,j)-E_z^n(i,j)\right\}+\alpha_x(i+1/2,j)\left\{H_y^{n-1/2}(i+1/2,j)\right\}\right];$$

$$\beta_x=\frac{\varepsilon}{\Delta t}+\frac{\sigma_x}{2},\alpha_x=\left(\frac{\varepsilon}{\Delta t}-\frac{\sigma_x}{2}\right)$$

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- Similarly update equation for Hx

$$\sigma_y H_x - \frac{\varepsilon}{\mu}\frac{\partial E_z}{\partial y} = \varepsilon \frac{\partial H_x}{\partial t}$$

$$\Rightarrow H_x^{n+1/2}(i, j+1/2)$$

$$= \frac{1}{\left(\frac{\varepsilon}{\Delta t} + \frac{\sigma_y}{2}\right)}\left[-\frac{\varepsilon}{\mu \Delta y}\left\{E_z^n(i, j+1) - E_z^n(i, j)\right\} + \left(\frac{\varepsilon}{\Delta t} - \frac{\sigma_y}{2}\right)\left\{H_x^{n-1/2}(i, j+1/2)\right\}\right]$$

$$= \frac{1}{\beta_y(i, j+1/2)}\left[-\frac{\varepsilon}{\mu \Delta y}\left\{E_z^n(i, j+1) - E_z^n(i, j)\right\} + \alpha_y(i, j+1/2)\left\{H_x^{n-1/2}(i, j+1/2)\right\}\right];$$

$$\beta_y = \frac{\varepsilon}{\Delta t} + \frac{\sigma_y}{2}, \alpha_y = \left(\frac{\varepsilon}{\Delta t} - \frac{\sigma_y}{2}\right)$$

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- Update equation for split electric field may be obtained similarly

$$\varepsilon \frac{\partial E_{sx,z}}{\partial t} = \frac{\partial H_y}{\partial x} - \sigma_x E_{sx,z}$$

$$\varepsilon \frac{\partial E_{sy,z}}{\partial t} = -\frac{\partial H_x}{\partial y} + \sigma_y E_{sy,z}$$

$$E_z = E_{sx,z} + E_{sy,z}$$

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$$E_{sx,z}^{n+1}(i,j) =$$

$$\frac{1}{\beta_x(i,j)}\left[E_{sx,z}^{n}(i,j)\alpha_x(i,j)+\frac{1}{\Delta x}\left\{H_y^{n+1/2}(i+1/2,j)-H_y^{n+1/2}(i-1/2,j)\right\}\right]$$

$$E_{sy,z}^{n+1}(i,j) =$$

$$\frac{1}{\beta_y(i,j)}\left[E_{sy,z}^{n}(i,j)\alpha_y(i,j)-\frac{1}{\Delta y}\left\{H_x^{n+1/2}(i,j+1/2)-H_x^{n+1/2}(i,j-1/2)\right\}\right]$$

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- Reflectionless theoretically,
  - may not be true for numerical simulations
- Sudden change of conductivity from 0 to that of the PML layer,
  - undesirable numerical reflections may occur
- Technique:
  - gradually change the value of $\sigma_{x,y,z}$ within the PML

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- For examples,
- we can set the conductivity as an
- m^{th} order polynomial (m=2 or 3 is a good choice)

$$\sigma_{x,y,z} = \sigma_{\max}\left(\frac{l}{L}\right)^m, m = 1,2,\cdots$$

- l is the distance from the PML surface,
- L is the thickness of the PML and
- σ_{max} is the maximum conductivity inside the PML

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- **Uniaxial PML**
- Assume a diagonally anisotropic medium and write down Maxwell's equation in this medium

$$\overleftrightarrow{\Lambda} = diag(a,b,c)$$

$$\nabla \times \vec{E} = -j\omega\mu_0 \overleftrightarrow{\Lambda}\vec{H}; \nabla \times \vec{H} = j\omega\varepsilon_0 \overleftrightarrow{\Lambda}\vec{E};$$

$$\nabla \bullet \vec{D} = \nabla \bullet \left(\overleftrightarrow{\Lambda}\vec{E}\right) = 0; \nabla \bullet \vec{B} = \nabla \bullet \left(\overleftrightarrow{\Lambda}\vec{H}\right) = 0;$$

$$\overleftrightarrow{\Lambda}_1 = \frac{1}{abc} diag(a,b,c) = diag\left(\frac{1}{bc}, \frac{1}{ac}, \frac{1}{ab}\right)$$

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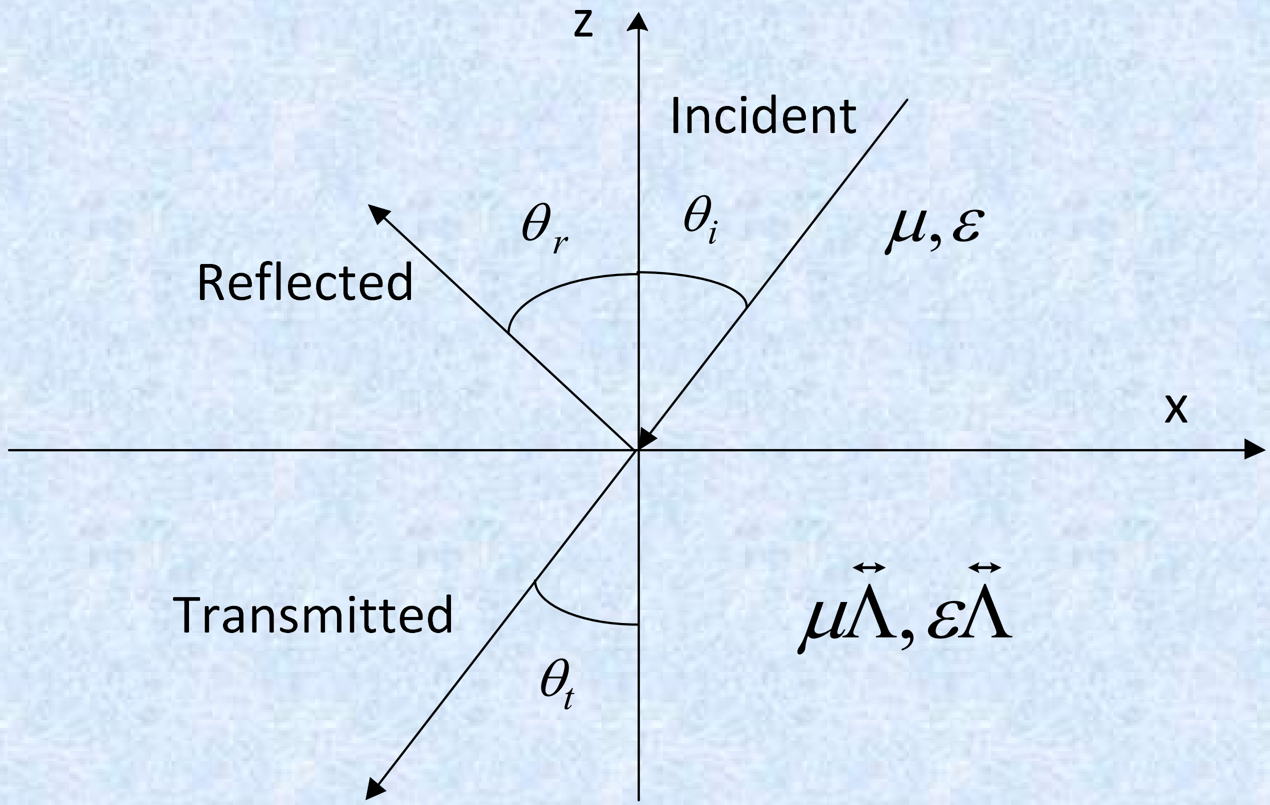

Fig. Plane wave incident on the surface of an anisotropic medium

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$$\nabla \times \vec{E} = -j\omega\mu_0 \overleftrightarrow{\Lambda}_1 \vec{H} \Rightarrow \left(\overleftrightarrow{\Lambda}_1\right)^{-1} \nabla \times \vec{E} = -j\omega\mu_0 \vec{H}$$

$$\nabla \times \vec{H} = j\omega\varepsilon_0 \overleftrightarrow{\Lambda} \vec{E} \Rightarrow \left(\overleftrightarrow{\Lambda}_1\right)^{-1} \nabla \times \vec{H} = j\omega\varepsilon_0 \vec{E}$$

$$\therefore \left(\overleftrightarrow{\Lambda}_1\right)^{-1} \nabla \times \left(\overleftrightarrow{\Lambda}_1\right)^{-1} \left(\nabla \times \vec{E}\right) = \omega^2 \mu_0 \varepsilon_0 \vec{E} = k_0^2 \vec{E}$$

$$\nabla \bullet \vec{D} = \nabla \bullet \left(\overleftrightarrow{\Lambda}_1 \vec{E}\right) = 0; \nabla \bullet \vec{B} = \nabla \bullet \left(\overleftrightarrow{\Lambda}_1 \vec{H}\right) = 0;$$

$$\overleftrightarrow{\Lambda}_1 = \frac{1}{abc} diag\left(a,b,c\right) = diag\left(\frac{1}{bc}, \frac{1}{ac}, \frac{1}{ab}\right)$$

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- For spatially constant case,

$$\overleftrightarrow{\Lambda}_1^{-1}\left(\nabla\times\left(\overleftrightarrow{\Lambda}_1^{-1}\left(\nabla\times\vec{E}\right)\right)\right)=\omega^2\mu_0\varepsilon_0\vec{E}=k_0^2\vec{E}$$

Or,- $$\left(\nabla\bullet\left(\overleftrightarrow{\Lambda}_1\nabla\right)\right)\vec{E}+\nabla\left(\nabla\bullet\left(\overleftrightarrow{\Lambda}_1\vec{E}\right)\right)=k_0^2\vec{E}$$

$$\because\nabla\bullet\left(\overleftrightarrow{\Lambda}_1\vec{E}\right)=0$$

Or,- $$\left(\nabla\bullet\left(\overleftrightarrow{\Lambda}_1\nabla\right)\right)\vec{E}=k_0^2\vec{E}$$

- For plane wave propagation, we have

$$-\left(\left(-j\vec{k}\right)\bullet\left(\overleftrightarrow{\Lambda}_1\left(-j\vec{k}\right)\right)\right)\vec{E}=\vec{k}\bullet\left(\overleftrightarrow{\Lambda}_1\vec{k}\right)\vec{E}=k_0^2\vec{E}$$

$$\Rightarrow\frac{k_x^2}{bc}+\frac{k_y^2}{ac}+\frac{k_z^2}{ab}=k^2$$

Digression: Plane wave reflection from media interface at oblique incidence

- We will consider the problem of a plane wave
  - obliquely incident on a plane interface
  - between two lossy conducting regions
- We will first consider two particular cases of this problem as follows:
  - the electric field is in the xz plane (parallel polarization)
  - the electric field is in normal to the xz plane (perpendicular polarization)

Digression: Plane wave reflection from media interface at oblique incidence

- Any arbitrary incident plane wave can be expressed
  - as a linear combination of these two principal polarizations
- The plane of incidence is that plane containing
  - the normal vector to the interface and
  - the direction of propagation vector of the incident wave

Digression: Plane wave reflection from media interface at oblique incidence

- For Fig., this is the xz plane
- For perpendicular polarization (TE),
  - electric field is perpendicular to the plane of incidence
- For parallel polarization (TM),
  - electric field is parallel to the plane of incidence

Digression: Plane wave reflection from media interface at oblique incidence

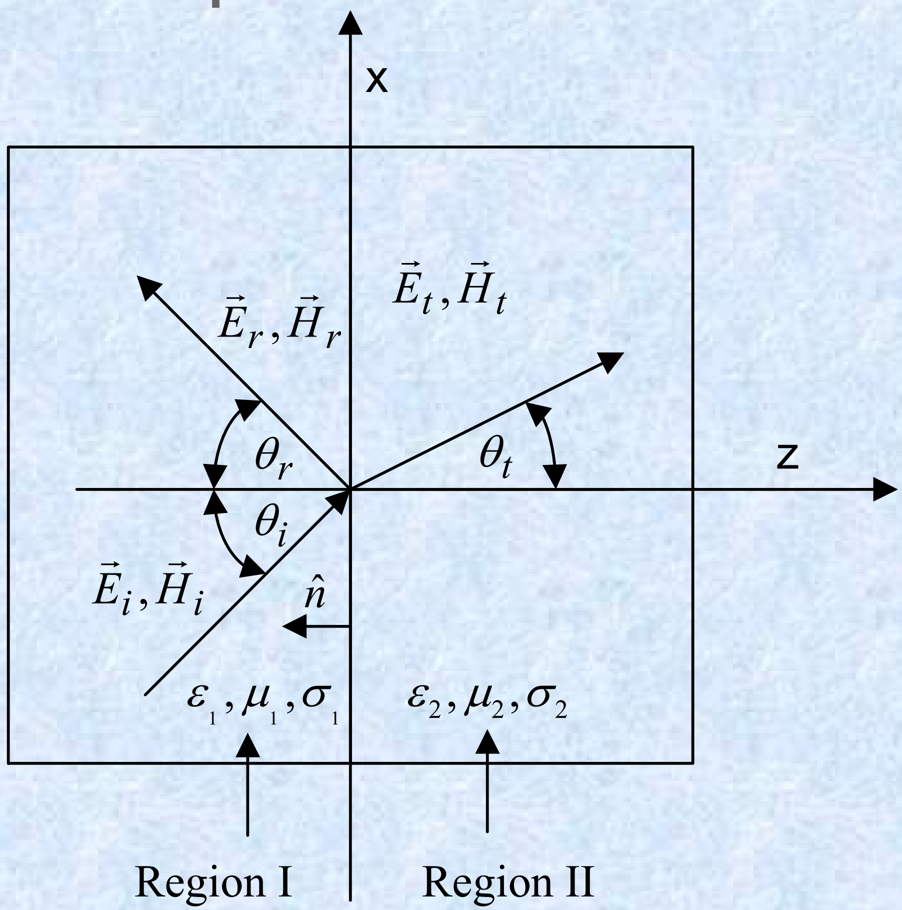

Fig. Oblique incidence of plane EM wave at a media interface

Digression: Plane wave reflection from media interface at oblique incidence

Perpendicular polarization (TE):

- In this case, electric field vector is perpendicular to the xz plane,
- Hence, it will have component along the y-axis
- Since the electric field is transversal to the plane of incidence
- They are also known transverse electric (TE) waves

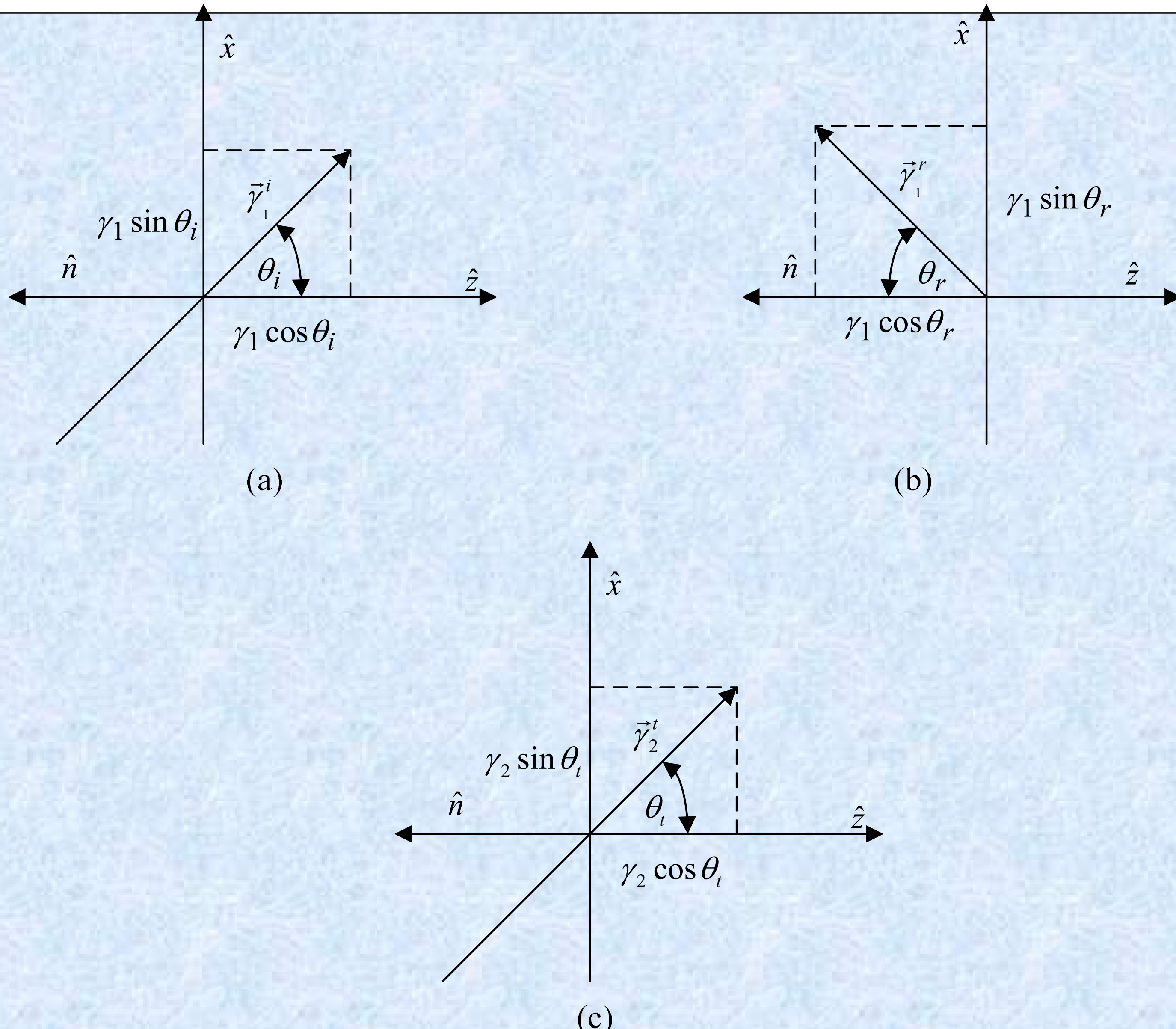

Fig. Wave propagation vector for (a) incident (b) reflected and (c) transmitted EM waves at oblique incidence

Digression: Plane wave reflection from media interface at oblique incidence

- Let us assume that the incident wave propagates in the first quadrant of xz plane without loss of generality and
- $\vec{\gamma}_1{}^{i}$ (incident propagation vector) makes an angle θ_i with the normal (see Fig. 6.6 (a))

$$\vec{\gamma}_1^i \bullet \vec{z}' = (\gamma_1 \cos\theta_i \hat{z} + \gamma_1 \sin\theta_i \hat{x}) \bullet (z\hat{z} + x\hat{x}) = \gamma_1 \cos\theta_i z + \gamma_1 \sin\theta_i x = \gamma_1 (z\cos\theta_i + x\sin\theta_i)$$

$$\vec{E}_i = E_0 e^{-\gamma_1 (z\cos\theta_i + x\sin\theta_i)} \hat{y}$$

$$\because \nabla \times \vec{E}_i = -j\omega\mu_1 \vec{H}_i \Rightarrow \vec{H}_i = \frac{\nabla \times \vec{E}_i}{-j\omega\mu_1}$$

Digression: Plane wave reflection from media interface at oblique incidence

$$= \frac{1}{-j\omega\mu_1}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_0 e^{-\gamma_1(z\cos\theta_i + x\sin\theta_i)} & 0 \end{vmatrix} = \frac{E_0}{-j\omega\mu_1}\left\{\left(-\frac{\partial e^{-\gamma_1(z\cos\theta_i + x\sin\theta_i)}}{\partial z}\hat{x}\right) + \left(\frac{\partial e^{-\gamma_1(z\cos\theta_i + x\sin\theta_i)}}{\partial x}\hat{z}\right)\right\}$$

$$= \frac{E_0}{j\omega\mu_1}\left\{\left(\frac{\partial e^{-\gamma_1(z\cos\theta_i + x\sin\theta_i)}}{\partial z}\hat{x}\right) - \left(\frac{\partial e^{-\gamma_1(z\cos\theta_i + x\sin\theta_i)}}{\partial x}\hat{z}\right)\right\} = \frac{E_0\gamma_1}{j\omega\mu_1} e^{-\gamma_1(z\cos\theta_i + x\sin\theta_i)}\left\{-\cos\theta_i \hat{x} + \hat{z}\sin\theta_i\right\}$$

$$= \frac{E_0}{\eta_1} e^{-\gamma_1(z\cos\theta_i + x\sin\theta_i)}\left(-\hat{x}\cos\theta_i + \hat{z}\sin\theta_i\right)$$

Digression: Plane wave reflection from media interface at oblique incidence

- Let us assume that the reflected wave propagates in the second quadrant of xz plane and
- $\vec{\gamma}_1^{\,r}$ (reflected propagation vector) makes an angle θ_r with the normal (see Fig. 6.6 (b))

$$\vec{\gamma}_1^{\,r} \bullet \vec{z}' = \left(-\gamma_1 \cos\theta_r \hat{z} + \gamma_1 \sin\theta_r \hat{x}\right) \bullet \left(z\hat{z} + x\hat{x}\right) = -\gamma_1 \cos\theta_r z + \gamma_1 \sin\theta_r x = \gamma_1 \left(-z\cos\theta_r + x\sin\theta_r\right)$$

$$\vec{E}_r = E_0 \Gamma_{TE} e^{-\gamma_1\left(-z\cos\theta_r + x\sin\theta_r\right)} \hat{y}$$

Digression: Plane wave reflection from media interface at oblique incidence

- Note that $\vec{\gamma}_1^{\,r}$ and $\vec{\gamma}_1^{\,i}$ will have the same magnitude
  - since both the waves are still in the same region I,
  - only their direction changes
- Since the Poynting vector must be negative like the previous case of normal incidence,

$$\vec{H}_r = \frac{E_0}{\eta_1}\Gamma_{TE} e^{-\gamma_1(-z\cos\theta_r + x\sin\theta_r)}\left(\hat{x}\cos\theta_r + \hat{z}\sin\theta_r\right)$$

- You could also use the Maxwell's curl equation below to find this

$$\vec{H}_r = \frac{\nabla\times\vec{E}_r}{-j\omega\mu_1}$$

Digression: Plane wave reflection from media interface at oblique incidence

- The transmitted fields will have similar expression with the incident fields except
  - that now the θ_i should be replaced by θ_t (angle that transmitted propagation vector makes with the normal),
  - γ_1 should be replaced by γ_2 (wave is in region II now) and
  - multiplication by (transmission coefficient)
- The transmitted fields are $\vec{E}_t = \hat{y} E_0 \tau_{TE} e^{-\gamma_2 (z\cos\theta_t + x\sin\theta_t)}$

$$\vec{H}_t = \frac{\nabla \times \vec{E}_t}{-j\omega\mu_2} = \frac{E_0 \tau_{TE}}{\eta_2} e^{-\gamma_2 (z\cos\theta_t + x\sin\theta_t)} \left(-\hat{x}\cos\theta_t + \hat{z}\sin\theta_t\right)$$

Digression: Plane wave reflection from media interface at oblique incidence

Table Fields in two regions (oblique incidence: perpendicular polarization)

Region I (lossy medium 1)	Region II (lossy medium 2)
$\vec{E}_i = E_0 e^{-\gamma_1(z\cos\theta_i + x\sin\theta_i)}\hat{y}$ $\vec{H}_i = \frac{E_0}{\eta_1} e^{-\gamma_1(z\cos\theta_i + x\sin\theta_i)}(-\hat{x}\cos\theta_i + \hat{z}\sin\theta_i)$ $\vec{E}_r = E_0\Gamma_{TE} e^{-\gamma_1(-z\cos\theta_r + x\sin\theta_r)}\hat{y}$ $\vec{H}_r = \frac{E_0\Gamma_{TE}}{\eta_1} e^{-\gamma_1(z\cos\theta_i + x\sin\theta_i)}(\hat{x}\cos\theta_r + \hat{z}\sin\theta_r)$	$\vec{E}_t = \hat{y}E_0\tau_{TE} e^{-\gamma_2(z\cos\theta_t + x\sin\theta_t)}$ $\vec{H}_t = \frac{E_0\tau_{TE}}{\eta_2} e^{-\gamma_2(z\cos\theta_t + x\sin\theta_t)}(-\hat{x}\cos\theta_t + \hat{z}\sin\theta_t)$

Digression: Plane wave reflection from media interface at oblique incidence

- Equating the tangential components of electric field
  - (electric field has only E_y component and it is tangential at the interface z=0) and
- magnetic field
  - (magnetic field has two components: H_x and H_z and only H_x is tangential at the interface z=0)
- at z=0 gives

$$e^{-\gamma_1 x \sin\theta_i} + \Gamma_{TE} e^{-\gamma_1 x \sin\theta_r} = \tau_{TE} e^{-\gamma_2 x \sin\theta_t}$$

$$\frac{-1}{\eta_1}\cos\theta_i e^{-\gamma_1 x \sin\theta_i} + \frac{\Gamma_{TE}}{\eta_1}\cos\theta_r e^{-\gamma_1 x \sin\theta_r} = -\frac{\tau_{TE}}{\eta_2}\cos\theta_t e^{-\gamma_2 x \sin\theta_t}$$

Digression: Plane wave reflection from media interface at oblique incidence

- If E_x and H_y are to be continuous at the interface z = 0 for all x,
- then, this x variation must be the same on both sides of the equations (also known as *phase matching condition*)

$$\gamma_1 \sin\theta_i = \gamma_1 \sin\theta_r = \gamma_2 \sin\theta_t$$

$$\Rightarrow \theta_i = \theta_r ; \gamma_1 \sin\theta_i = \gamma_2 \sin\theta_t$$

Digression: Plane wave reflection from media interface at oblique incidence

- The first is Snell's law of reflection
  - which states that the angle of incidence equals the angle of reflection
- The second result is the Snell's law of refraction
  - (refraction is the change in direction of a wave due to change in velocity from one medium to another medium)
- Also note that refractive index of a medium is defined as

$$n = \frac{c}{v_p} = \frac{\sqrt{\mu_r \varepsilon_r \mu_0 \varepsilon_0}}{\sqrt{\mu_0 \varepsilon_0}} = \sqrt{\mu_r \varepsilon_r}$$

Digression: Plane wave reflection from media interface at oblique incidence

- hence, for a lossless dielectric media,

$$\frac{\sin\theta_i}{\sin\theta_t}=\frac{\gamma_2}{\gamma_1}=\frac{\beta_2}{\beta_1}=\frac{\sqrt{\mu_2\varepsilon_2}}{\sqrt{\mu_1\varepsilon_1}}=\frac{v_1}{v_2}=\frac{\sqrt{\varepsilon_2}}{\sqrt{\varepsilon_1}}=\frac{n_2}{n_1}$$

- Now we can simplify above two equations by applying Snell's two laws as follows

$$1+\Gamma_{TE}=\tau_{TE}$$

$$-\frac{\cos\theta_i}{\eta_1}+\Gamma_{TE}\frac{\cos\theta_r}{\eta_1}=-\frac{\tau_{TE}}{\eta_2}\cos\theta_t$$

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- $\frac{k_x^2}{bc}+\frac{k_y^2}{ac}+\frac{k_z^2}{ab}=k^2$ whose solution is

$$k_x = k\sqrt{bc}\sin\theta\cos\phi, k_y = k\sqrt{ac}\sin\theta\sin\phi, k_z = k\sqrt{ab}\cos\theta$$

- For incident and reflected fields, it is still the same
- For xz plane (region II), transmitted fields, we have uniaxial medium hence, $\varphi = 0^0; \theta = \theta_t; k_x = k\sqrt{bc}\sin\theta_t, k_z = k\sqrt{ab}\cos\theta_t$
- For TE case, $\vec{E}^i = \hat{y}E_0 e^{-jk(x\sin\theta_i + z\cos\theta_i)}$

$$\vec{E}^r = \hat{y}\Gamma_{TE}E_0 e^{-jk(x\sin\theta_r - z\cos\theta_r)} \quad \vec{E}_t = \hat{y}E_0\tau_{TE}e^{-\gamma_2(z\cos\theta_t + x\sin\theta_t)}$$

$$\vec{E}^t = \hat{y}\tau_{TE}E_0 e^{-jk\left(\sqrt{bc}\,x\sin\theta_t + z\sqrt{ab}\cos\theta_t\right)}$$

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- From Maxwell's curl equations,

$$\vec{H}^i = \left(-\hat{x}\cos\theta_i + \hat{z}\sin\theta_i\right)\frac{E_0}{\eta}e^{-jk\left(x\sin\theta_i + z\cos\theta_i\right)}$$

$$\vec{H}^r = \left(\hat{x}\cos\theta_r + \hat{z}\sin\theta_r\right)\frac{E_0}{\eta}\Gamma_{TE}e^{-jk\left(x\sin\theta_r - z\cos\theta_r\right)}$$

$$\vec{H}^t = \left(-\hat{x}\sqrt{\frac{b}{a}}\cos\theta_t + \hat{z}\sqrt{\frac{b}{c}}\sin\theta_t\right)\tau_{TE}\frac{E_0}{\eta}e^{-jk\left(\sqrt{bc}\,x\sin\theta_t + z\sqrt{ab}\cos\theta_t\right)}$$

$$\vec{H}_t = \frac{E_0\tau_{TE}}{\eta_2}e^{-\gamma_2\left(z\cos\theta_t + x\sin\theta_t\right)}\left(-\hat{x}\cos\theta_t + \hat{z}\sin\theta_t\right)$$

FDTD: An Introduction

- Tangential components are x- and y-components at z=0 interface

$$\Rightarrow \theta_i = \theta_r ; \gamma_1 \sin\theta_i = \gamma_2 \sin\theta_t$$

$$\sin\theta_i = \sin\theta_r = \sqrt{bc}\sin\theta_t$$

$$\vec{E}^t = \hat{y}\tau_{TE}E_0 e^{-jk\left(\sqrt{bc}x\sin\theta_t + z\sqrt{ab}\cos\theta_t\right)}$$

$$1 + \Gamma_{TE} = \tau_{TE}$$

$$-\frac{\cos\theta_i}{\eta_1} + \Gamma_{TE}\frac{\cos\theta_r}{\eta_1} = -\frac{\tau_{TE}}{\eta_2}\cos\theta_t$$

$$\cos\theta_i - \Gamma_{TE}\cos\theta_r = \tau_{TE}\sqrt{\frac{b}{a}}\cos\theta_t$$

$$\therefore \Gamma_{TE} = \frac{\cos\theta_i - \sqrt{\frac{b}{a}}\cos\theta_t}{\cos\theta_i + \sqrt{\frac{b}{a}}\cos\theta_t} = \Gamma_{TM}$$

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$$Choose \quad \sqrt{bc} = 1, \quad then \quad \theta_i = \theta_t$$

- If we take b/a=1, then b=a, then

$$\Gamma_{TE/TM} = 0$$

- Or in other words, a=b=1/c
- Anisotropic medium will be reflectionless