FDTD: An Introduction

- In this case, one can attenuate wave in the z-direction through the choice of a, hence we will call a=s_z
- Then

$$\overleftrightarrow{\Lambda}_z = \begin{bmatrix} s_z & 0 & 0 \\ 0 & s_z & 0 \\ 0 & 0 & \dfrac{1}{s_z} \end{bmatrix}$$

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- Similarly,

$$\overleftrightarrow{\Lambda}_y = \begin{bmatrix} s_y & 0 & 0 \\ 0 & \frac{1}{s_y} & 0 \\ 0 & 0 & s_y \end{bmatrix}, \overleftrightarrow{\Lambda}_x = \begin{bmatrix} \frac{1}{s_x} & 0 & 0 \\ 0 & s_x & 0 \\ 0 & 0 & s_x \end{bmatrix}$$

- Combining

$$\overleftrightarrow{\Lambda} = \overleftrightarrow{\Lambda}_x \overleftrightarrow{\Lambda}_y \overleftrightarrow{\Lambda}_z = \begin{bmatrix} \frac{s_y s_z}{s_x} & 0 & 0 \\ 0 & \frac{s_x s_z}{s_y} & 0 \\ 0 & 0 & \frac{s_x s_y}{s_z} \end{bmatrix}$$

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- Two Maxwell's curl equations

$$\nabla \times \vec{H} = j\omega\varepsilon_0 \overleftrightarrow{\Lambda}\vec{E}; \nabla \times \vec{E} = -j\omega\mu_0 \overleftrightarrow{\Lambda}\vec{H};$$

$$\overleftrightarrow{\Lambda} = \overleftrightarrow{\Lambda}_x \overleftrightarrow{\Lambda}_y \overleftrightarrow{\Lambda}_z = \begin{bmatrix} \frac{s_y s_z}{s_x} & 0 & 0 \\ 0 & \frac{s_x s_z}{s_y} & 0 \\ 0 & 0 & \frac{s_x s_y}{s_z} \end{bmatrix}$$

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- Expanding curl on LHS of $\nabla \times \vec{H} = j\omega\varepsilon_0 \overleftrightarrow{\Lambda}\vec{E}$;

$$\begin{bmatrix} \dfrac{\partial H_z}{\partial y} - \dfrac{\partial H_y}{\partial z} \\ \dfrac{\partial H_x}{\partial z} - \dfrac{\partial H_z}{\partial x} \\ \dfrac{\partial H_y}{\partial x} - \dfrac{\partial H_x}{\partial y} \end{bmatrix} = j\omega\varepsilon \begin{bmatrix} \dfrac{s_y s_z}{s_x} & 0 & 0 \\ 0 & \dfrac{s_x s_z}{s_y} & 0 \\ 0 & 0 & \dfrac{s_x s_y}{s_z} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

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- Auxiliary field vectors $D_x = \varepsilon \frac{s_z}{s_x} E_x ; D_y = \varepsilon \frac{s_x}{s_y} E_y ; D_z = \varepsilon \frac{s_y}{s_z} E_z$

$$\begin{bmatrix} \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \\ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \end{bmatrix} = j\omega \begin{bmatrix} s_y & 0 & 0 \\ 0 & s_z & 0 \\ 0 & 0 & s_x \end{bmatrix} \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix}$$

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- Choosing

$$s_x = k_x + \frac{\sigma_x}{j\omega\varepsilon},$$

$$s_y = k_y + \frac{\sigma_y}{j\omega\varepsilon},$$

$$s_z = k_z + \frac{\sigma_z}{j\omega\varepsilon}$$

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- Substituting s_x, s_y and s_z, we have,

$$\begin{bmatrix} \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \\ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \end{bmatrix} = j\omega \begin{bmatrix} k_y + \frac{\sigma_y}{j\omega\varepsilon} & 0 & 0 \\ 0 & k_z + \frac{\sigma_z}{j\omega\varepsilon} & 0 \\ 0 & 0 & k_x + \frac{\sigma_x}{j\omega\varepsilon} \end{bmatrix} \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix}$$

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- Expanding the RHS, we have,

$$RHS = j\omega \begin{bmatrix} k_y & 0 & 0 \\ 0 & k_z & 0 \\ 0 & 0 & k_x \end{bmatrix} \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} + \begin{bmatrix} \frac{\sigma_y}{\varepsilon} & 0 & 0 \\ 0 & \frac{\sigma_z}{\varepsilon} & 0 \\ 0 & 0 & \frac{\sigma_x}{\varepsilon} \end{bmatrix} \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix}$$

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- Note that for time harmonic fields $\frac{\partial}{\partial t} = j\omega$

$$\begin{bmatrix} \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \\ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \end{bmatrix} = \frac{\partial}{\partial t}\begin{bmatrix} k_y & 0 & 0 \\ 0 & k_z & 0 \\ 0 & 0 & k_x \end{bmatrix}\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} + \frac{1}{\varepsilon}\begin{bmatrix} \sigma_y & 0 & 0 \\ 0 & \sigma_z & 0 \\ 0 & 0 & \sigma_x \end{bmatrix}\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix}$$

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- We can further simplify

$$\begin{bmatrix} \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \\ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \end{bmatrix} = \begin{bmatrix} k_y & 0 & 0 \\ 0 & k_z & 0 \\ 0 & 0 & k_x \end{bmatrix} \begin{bmatrix} \frac{\partial D_x}{\partial t} \\ \frac{\partial D_y}{\partial t} \\ \frac{\partial D_z}{\partial t} \end{bmatrix} + \begin{bmatrix} \frac{\sigma_y}{\varepsilon} & 0 & 0 \\ 0 & \frac{\sigma_z}{\varepsilon} & 0 \\ 0 & 0 & \frac{\sigma_x}{\varepsilon} \end{bmatrix} \begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix}$$

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- In component wise form,

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = k_y \frac{\partial}{\partial t}\left(D_x\right) + \frac{\sigma_y}{\varepsilon} D_x;$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = k_z \frac{\partial}{\partial t}\left(D_y\right) + \frac{\sigma_z}{\varepsilon} D_y;$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = k_x \frac{\partial}{\partial t}\left(D_z\right) + \frac{\sigma_x}{\varepsilon} D_z;$$

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- Similarly in the second Maxwell's curl equation,
- Subsititute Auxiliary field vectors

$$B_x = \mu \frac{s_z}{s_x} H_x ; B_y = \mu \frac{s_x}{s_y} H_y ; B_z = \mu \frac{s_y}{s_z} H_z$$

- Similarly we can write $\nabla \times \vec{E} = -j\omega\mu_0 \overleftrightarrow{\Lambda}\vec{H}$ as

$$\nabla \times \vec{H} = j\omega\varepsilon_0 \overleftrightarrow{\Lambda}\vec{E};$$

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$$\begin{bmatrix} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \end{bmatrix} = -\frac{\partial}{\partial t}\begin{bmatrix} k_y & 0 & 0 \\ 0 & k_z & 0 \\ 0 & 0 & k_x \end{bmatrix}\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} - \frac{1}{\varepsilon}\begin{bmatrix} \sigma_y & 0 & 0 \\ 0 & \sigma_z & 0 \\ 0 & 0 & \sigma_x \end{bmatrix}\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix}$$

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- In component wise form,

$$\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}=-k_y\frac{\partial B_x}{\partial t}-\frac{\sigma_y}{\varepsilon}B_x;$$

$$\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}=-k_z\frac{\partial B_y}{\partial t}-\frac{\sigma_z}{\varepsilon}B_y;$$

$$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=-k_x\frac{\partial B_z}{\partial t}-\frac{\sigma_x}{\varepsilon}B_z$$

$$\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}=k_y\frac{\partial}{\partial t}\left(D_x\right)+\frac{\sigma_y}{\varepsilon}D_x;$$

$$\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}=k_z\frac{\partial}{\partial t}\left(D_y\right)+\frac{\sigma_z}{\varepsilon}D_y;$$

$$\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}=k_x\frac{\partial}{\partial t}\left(D_z\right)+\frac{\sigma_x}{\varepsilon}D_z;$$

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$$D_x = \varepsilon \frac{s_z}{s_x} E_x ; D_y = \varepsilon \frac{s_x}{s_y} E_y ; D_z = \varepsilon \frac{s_y}{s_z} E_z$$

- Auxiliary field update equation

$$\left(k_x + \frac{\sigma_x}{j\omega\varepsilon} \right) D_x = \varepsilon \left(k_z + \frac{\sigma_z}{j\omega\varepsilon} \right) E_x$$

$$\Rightarrow \frac{\partial}{\partial t}\left(k_x D_x \right) + \frac{\sigma_x}{\varepsilon} D_x = \varepsilon \left[\frac{\partial}{\partial t}\left(k_z E_x \right) + \frac{\sigma_z}{\varepsilon} E_x \right]$$

$$\frac{\partial}{\partial t}\left(k_y D_y \right) + \frac{\sigma_y}{\varepsilon} D_y = \varepsilon \left[\frac{\partial}{\partial t}\left(k_x E_y \right) + \frac{\sigma_x}{\varepsilon} E_y \right]$$

$$\frac{\partial}{\partial t}\left(k_z D_z \right) + \frac{\sigma_z}{\varepsilon} D_z = \varepsilon \left[\frac{\partial}{\partial t}\left(k_y E_z \right) + \frac{\sigma_y}{\varepsilon} E_z \right]$$

FDTD: An Introduction

- Auxiliary field update equation

$$\frac{\partial}{\partial t}\left(k_x B_x\right)+\frac{\sigma_x}{\varepsilon} B_x=\mu\left[\frac{\partial}{\partial t}\left(k_z H_x\right)+\frac{\sigma_z}{\varepsilon} H_x\right]$$

$$\frac{\partial}{\partial t}\left(k_y B_y\right)+\frac{\sigma_y}{\varepsilon} B_y=\mu\left[\frac{\partial}{\partial t}\left(k_x H_y\right)+\frac{\sigma_x}{\varepsilon} H_y\right]$$

$$\frac{\partial}{\partial t}\left(k_z B_z\right)+\frac{\sigma_z}{\varepsilon} B_z=\mu\left[\frac{\partial}{\partial t}\left(k_y H_z\right)+\frac{\sigma_y}{\varepsilon} H_z\right]$$

FDTD: An Introduction

$$\frac{\partial D_x}{\partial t} = \frac{D_x\Big|_{i+1/2,j,k}^{n+1} - D_x\Big|_{i+1/2,j,k}^{n}}{\Delta t}$$

- Note that the Dx is to be calculated at (n+1/2) time instant
- But it is not available, can be taken as average of Dx at (n+1) and n time instants

$$D_x = D_x\Big|_{i+1/2,j,k}^{n+1/2} = \frac{D_x\Big|_{i+1/2,j,k}^{n+1} + D_x\Big|_{i+1/2,j,k}^{n}}{2}$$

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- After substituting the finite difference approximations

$$\frac{H_z\Big|_{i+1/2,j+1/2,k}^{n+1/2} - H_z\Big|_{i+1/2,j-1/2,k}^{n+1/2}}{\Delta y} - \frac{H_y\Big|_{i+1/2,j,k+1/2}^{n+1/2} - H_y\Big|_{i+1/2,j,k-1/2}^{n+1/2}}{\Delta z}$$

$$= k_y\left(\frac{D_x\Big|_{i+1/2,j,k}^{n+1} - D_x\Big|_{i+1/2,j,k}^{n}}{\Delta t}\right) + \frac{\sigma_y}{\varepsilon}\left(\frac{D_x\Big|_{i+1/2,j,k}^{n+1} + D_x\Big|_{i+1/2,j,k}^{n}}{2}\right)$$

FDTD: An Introduction

- Discretization in space and time

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = k_y \frac{\partial}{\partial t}\left(D_x\right) + \frac{\sigma_y}{\varepsilon} D_x;$$

$$\frac{\partial H_z}{\partial y} = \frac{H_z\Big|_{i+1/2,j+1/2,k}^{n+1/2} - H_z\Big|_{i+1/2,j-1/2,k}^{n+1/2}}{\Delta y}$$

$$\frac{\partial H_y}{\partial z} = \frac{H_y\Big|_{i+1/2,j,k+1/2}^{n+1/2} - H_y\Big|_{i+1/2,j,k-1/2}^{n+1/2}}{\Delta z}$$

3D FDTD: Electric fields are calculated at "integer" time-steps and magnetic field at "half-integer" time-steps

FDTD: An Introduction

- Combining common terms of Dx at (n+1) and n time instants

$$\frac{H_z\Big|_{i+1/2,j+1/2,k}^{n+1/2} - H_z\Big|_{i+1/2,j-1/2,k}^{n+1/2}}{\Delta y} - \frac{H_y\Big|_{i+1/2,j,k+1/2}^{n+1/2} - H_y\Big|_{i+1/2,j,k-1/2}^{n+1/2}}{\Delta z}$$

$$= \left(\frac{k_y}{\Delta t} + \frac{\sigma_y}{2\varepsilon}\right) D_x\Big|_{i+1/2,j,k}^{n+1} - D_x\Big|_{i+1/2,j,k}^{n} \left(\frac{k_y}{\Delta t} - \frac{\sigma_y}{2\varepsilon}\right)$$

$$= \left(\frac{2\varepsilon k_y + \Delta t \sigma_y}{2\varepsilon \Delta t}\right) D_x\Big|_{i+1/2,j,k}^{n+1} - D_x\Big|_{i+1/2,j,k}^{n} \left(\frac{2\varepsilon k_y - \Delta t \sigma_y}{2\varepsilon \Delta t}\right)$$

FDTD: An Introduction

- The update equation for Dx can be written as

$$\left(\frac{2\varepsilon k_y + \Delta t\sigma_y}{2\varepsilon\Delta t}\right) D_x\Big|_{i+1/2,j,k}^{n+1} = \left(\frac{2\varepsilon k_y - \Delta t\sigma_y}{2\varepsilon\Delta t}\right) D_x\Big|_{i+1/2,j,k}^{n} + \frac{H_z\Big|_{i+1/2,j+1/2,k}^{n+1/2} - H_z\Big|_{i+1/2,j-1/2,k}^{n+1/2}}{\Delta y} - \frac{H_y\Big|_{i+1/2,j,k+1/2}^{n+1/2} - H_y\Big|_{i+1/2,j,k-1/2}^{n+1/2}}{\Delta z}$$

FDTD: An Introduction

- Can be written compactly after defining two new constants

$$D_x\Big|_{i+1/2,j,k}^{n+1} = C_1^{D_x} D_x\Big|_{i+1/2,j,k}^{n}$$

$$+C_2^{D_x}\frac{H_z\Big|_{i+1/2,j+1/2,k}^{n+1/2} - H_z\Big|_{i+1/2,j-1/2,k}^{n+1/2}}{\Delta y} - C_2^{D_x}\frac{H_y\Big|_{i+1/2,j,k+1/2}^{n+1/2} - H_y\Big|_{i+1/2,j,k-1/2}^{n+1/2}}{\Delta z};$$

$$C_1^{D_x} = \left(\frac{2\varepsilon k_y - \Delta t\sigma_y}{2\varepsilon k_y + \Delta t\sigma_y}\right); C_2^{D_x} = \left(\frac{2\varepsilon\Delta t}{2\varepsilon k_y + \Delta t\sigma_y}\right)$$

FDTD: An Introduction

- Calculation for update equation for Ex

$$\frac{\partial}{\partial t}\left(k_x D_x\right)+\frac{\sigma_x}{\varepsilon}D_x=\varepsilon\left[\frac{\partial}{\partial t}\left(k_z E_x\right)+\frac{\sigma_z}{\varepsilon}E_x\right]$$

$$\frac{\partial E_x}{\partial t}=\frac{E_x\big|_{i+1/2,j,k}^{n+1}-E_x\big|_{i+1/2,j,k}^{n}}{\Delta t}$$

$$E_x=E_x\big|_{i+1/2,j,k}^{n+1/2}=\frac{E_x\big|_{i+1/2,j,k}^{n+1}+E_x\big|_{i+1/2,j,k}^{n}}{2}$$

FDTD: An Introduction

- Substituting finite difference approximations

$$k_x\left(\frac{D_x\big|_{i+1/2,j,k}^{n+1}-D_x\big|_{i+1/2,j,k}^{n}}{\Delta t}\right)+\frac{\sigma_x}{\varepsilon}\left(\frac{D_x\big|_{i+1/2,j,k}^{n+1}+D_x\big|_{i+1/2,j,k}^{n}}{2}\right)$$

$$=\varepsilon k_z\left(\frac{E_x\big|_{i+1/2,j,k}^{n+1}-E_x\big|_{i+1/2,j,k}^{n}}{\Delta t}\right)+\sigma_z\left(\frac{E_x\big|_{i+1/2,j,k}^{n+1}+E_x\big|_{i+1/2,j,k}^{n}}{2}\right)$$

$$=\left(\frac{\varepsilon k_z}{\Delta t}+\frac{\sigma_z}{2}\right)E_x\Big|_{i+1/2,j,k}^{n+1}-\left(\frac{\varepsilon k_z}{\Delta t}-\frac{\sigma_z}{2}\right)E_x\Big|_{i+1/2,j,k}^{n}$$

FDTD: An Introduction

- Combining common terms, we have,

$$\left(\frac{k_x}{\Delta t}+\frac{\sigma_x}{2\varepsilon}\right)D_x\Big|_{i+1/2,j,k}^{n+1}-\left(\frac{k_x}{\Delta t}-\frac{\sigma_x}{2\varepsilon}\right)D_x\Big|_{i+1/2,j,k}^{n}$$

$$=\left(\frac{\varepsilon k_z}{\Delta t}+\frac{\sigma_z}{2}\right)E_x\Big|_{i+1/2,j,k}^{n+1}-\left(\frac{\varepsilon k_z}{\Delta t}-\frac{\sigma_z}{2}\right)E_x\Big|_{i+1/2,j,k}^{n}$$

$$\Rightarrow\left(2\varepsilon k_x+\sigma_x\Delta t\right)D_x\Big|_{i+1/2,j,k}^{n+1}-\left(2\varepsilon k_x-\sigma_x\Delta t\right)D_x\Big|_{i+1/2,j,k}^{n}$$

$$=\left(2\varepsilon k_z+\sigma_z\Delta t\right)E_x\Big|_{i+1/2,j,k}^{n+1}-\left(2\varepsilon k_z-\sigma_z\Delta t\right)E_x\Big|_{i+1/2,j,k}^{n}$$

FDTD: An Introduction

- Written in a compact form by defining 3 new constants

$$\left(2\varepsilon k_z+\sigma_z\Delta t\right)E_x\Big|_{i+1/2,j,k}^{n+1}=\left(2\varepsilon k_z-\sigma_z\Delta t\right)E_x\Big|_{i+1/2,j,k}^{n}$$

$$+\left(2\varepsilon k_x+\sigma_x\Delta t\right)D_x\Big|_{i+1/2,j,k}^{n+1}-\left(2\varepsilon k_x-\sigma_x\Delta t\right)D_x\Big|_{i+1/2,j,k}^{n}$$

$$\Rightarrow E_x\Big|_{i+1/2,j,k}^{n+1}=C_1^{E_x}\,D_x\Big|_{i+1/2,j,k}^{n+1}-C_2^{E_x}\,D_x\Big|_{i+1/2,j,k}^{n}+C_3^{E_x}\,E_x\Big|_{i+1/2,j,k}^{n};$$

$$C_1^{E_x}=\frac{\left(2\varepsilon k_x+\sigma_x\Delta t\right)}{\left(2\varepsilon k_z+\sigma_z\Delta t\right)},C_2^{E_x}=\frac{\left(2\varepsilon k_x-\sigma_x\Delta t\right)}{\left(2\varepsilon k_z+\sigma_z\Delta t\right)},C_3^{E_x}=\frac{\left(2\varepsilon k_z-\sigma_z\Delta t\right)}{\left(2\varepsilon k_z+\sigma_z\Delta t\right)}$$

FDTD: An Introduction

- Reflection error
- Lot of reflection error at the boundary of PML and free space
- Can be reduced by scaling the PML parameters
- But still there is lot of error especially at the low frequencies
- Scale factor,

$$s_x = k_x + \frac{\sigma_x}{j\omega\varepsilon}$$

- where σx and αx are assumed to be positive real and κx is real and ≥ 1

FDTD: An Introduction

- What happens when the frequency goes to zero?
- How do we overcome this?
- Add α_x in the scale factor expression as follows,

$$s_x = k_x + \frac{\sigma_x}{\alpha_x + j\omega\varepsilon}$$

- Assume that the wave entering the PML has a complex propagation constant

$$\gamma_x = jk_x$$

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- When the wave enters the PML

$$e^{-(jk_x)s_x x} = e^{-(jk_x)\left(k_x + \frac{\sigma_x}{\alpha_x + j\omega\varepsilon}\right)x}$$

$$= e^{-\left(jk_x k_x + jk_x \frac{\sigma_x}{\alpha_x + j\omega\varepsilon}\right)x}$$

- It is seen that k_x will amplify the attenuation
- At high frequency,

$$\omega >> \frac{\alpha_x}{\varepsilon_0}$$

FDTD: An Introduction

- CFS-PML will attenuate the wave just like that of UPML
- For low frequency,

$$\omega << \frac{\alpha_x}{\varepsilon_0}$$

- the wave will not attenuate at all
- The break point frequency can be calculated as

$$f_b = \frac{\alpha_x}{2\pi\varepsilon_0}$$