FDTD: Advances

- 3-D ADI FDTD

$$\frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon_0\varepsilon_r}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) \qquad \frac{\partial H_x}{\partial t} = \frac{1}{\mu_0\mu_r}\left(\frac{\partial E_y}{\partial z} - \frac{\partial E_z}{\partial y}\right)$$

$$\frac{\partial E_y}{\partial t} = \frac{1}{\varepsilon_0\varepsilon_r}\left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) \qquad \frac{\partial H_y}{\partial t} = \frac{1}{\mu_0\mu_r}\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right)$$

$$\frac{\partial E_z}{\partial t} = \frac{1}{\varepsilon_0\varepsilon_r}\left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) \qquad \frac{\partial H_z}{\partial t} = \frac{1}{\mu_0\mu_r}\left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right)$$

- Two-step scheme in time
  - First half time step (First procedure) (n+1/2←n)
  - Second half time step (Second procedure) (n+1←n+1/2)

FDTD: Advances

- 1)First Procedure (n+1/2←n) $\dfrac{\partial E_x}{\partial t} = \dfrac{1}{\varepsilon_0 \varepsilon_r}\left(\dfrac{\partial H_z}{\partial y} - \dfrac{\partial H_y}{\partial z}\right)$
- $\dfrac{\partial H_z}{\partial y}$ first term on RHS replaced with implicit difference approximation

  $\dfrac{\partial H_y}{\partial z}$ 2nd term on RHS replaced with explicit difference approximation

$$\frac{E_x\Big|_{i+1/2,j,k}^{n+1/2} - E_x\Big|_{i+1/2,j,k}^{n}}{\frac{\Delta t}{2}}$$

$$= \frac{1}{\varepsilon}\left(\frac{H_z\Big|_{i+1/2,j+1/2,k}^{n+1/2} - H_z\Big|_{i+1/2,j-1/2,k}^{n+1/2}}{\Delta y} - \frac{H_y\Big|_{i+1/2,j,k+1/2}^{n} - H_y\Big|_{i+1/2,j,k-1/2}^{n}}{\Delta z}\right)$$

FDTD: Advances

- 2)Second procedure (n+1←n+1/2) $\frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon_0 \varepsilon_r}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)$
- $\frac{\partial H_z}{\partial y}$ replaced with explicit difference approximation
- $\frac{\partial H_y}{\partial z}$ replaced with implicit difference approximation

$$\frac{E_x\big|_{i+1/2,j,k}^{n+1} - E_x\big|_{i+1/2,j,k}^{n+1/2}}{\frac{\Delta t}{2}}$$

$$= \frac{1}{\varepsilon}\left(\frac{H_z\big|_{i+1/2,j+1/2,k}^{n+1/2} - H_z\big|_{i+1/2,j-1/2,k}^{n+1/2}}{\Delta y} - \frac{H_y\big|_{i+1/2,j,k+1/2}^{n+1} - H_y\big|_{i+1/2,j,k-1/2}^{n+1}}{\Delta z}\right)$$

FDTD: Advances

- First Procedure (n+1/2←n) {*continued*}

$$\frac{2\varepsilon}{\Delta t}\left(E_x\Big|_{i+1/2,j,k}^{n+1/2} - E_x\Big|_{i+1/2,j,k}^{n}\right)$$

$$=\left(\frac{H_z\Big|_{i+1/2,j+1/2,k}^{n+1/2} - H_z\Big|_{i+1/2,j-1/2,k}^{n+1/2}}{\Delta y} - \frac{H_y\Big|_{i+1/2,j,k+1/2}^{n} - H_y\Big|_{i+1/2,j,k-1/2}^{n}}{\Delta z}\right)$$

- Looking at the above equation for the first procedure
- We want to calculate E field in LHS at n+1/2 time instant
- But there are values of H field at n+1/2 time instant (implicit terms)
- How do we calculate and get rid of $H_z\Big|_{i+1/2,j+1/2,k}^{n+1/2}$ & $H_z\Big|_{i+1/2,j-1/2,k}^{n+1/2}$ in RHS?

FDTD: Advances

- Consider first half time step (n+1/2) of the 6th equation

$$H_z\Big|_{i+1/2,j+1/2,k}^{n+1/2} \qquad \frac{\partial H_z}{\partial t}=\frac{1}{\mu_0\mu_r}\left(\frac{\partial E_x}{\partial y}-\frac{\partial E_y}{\partial x}\right)$$

$$H_z\Big|_{i+1/2,j+1/2,k}^{n+1/2}$$

$$=H_z\Big|_{i+1/2,j+1/2,k}^{n}+\frac{\Delta t}{2\mu}\left(\frac{E_x\Big|_{i+1/2,j+1,k}^{n+1/2}-E_x\Big|_{i+1/2,j,k}^{n+1/2}}{\Delta y}-\frac{E_y\Big|_{i+1,j+1/2,k}^{n}-E_y\Big|_{i,j+1/2,k}^{n}}{\Delta x}\right)$$

- Note that j are located at j+1/2, j and j+1

FDTD: Advances

- Consider first half time step (n+1/2) of

$$H_z\Big|_{i+1/2,j-1/2,k}^{n+1/2} \qquad \frac{\partial H_z}{\partial t} = \frac{1}{\mu_0\mu_r}\left(\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x}\right)$$

$$H_z\Big|_{i+1/2,j-1/2,k}^{n+1/2} = H_z\Big|_{i+1/2,j-1/2,k}^{n} + \frac{\Delta t}{2\mu}\left(\frac{E_x\Big|_{i+1/2,j,k}^{n+1/2} - E_x\Big|_{i+1/2,j-1,k}^{n+1/2}}{\Delta y} - \frac{E_y\Big|_{i+1,j-1/2,k}^{n} - E_y\Big|_{i,j-1/2,k}^{n}}{\Delta x}\right)$$

- Note that j are located at j-1/2, j and j-1

FDTD: Advances

- Substitute the above expressions of $H_z\big|_{i+1/2,j+1/2,k}^{n+1/2}$ and
- $H_z\big|_{i+1/2,j-1/2,k}^{n+1/2}$ (implicit (n+1/2) terms of H fields in the RHS) in the first procedure

$$\frac{2\varepsilon}{\Delta t}\left(E_x\big|_{i+1/2,j,k}^{n+1/2} - E_x\big|_{i+1/2,j,k}^{n}\right)$$

$$= \left(\frac{H_z\big|_{i+1/2,j+1/2,k}^{n+1/2} - H_z\big|_{i+1/2,j-1/2,k}^{n+1/2}}{\Delta y} - \frac{H_y\big|_{i+1/2,j,k+1/2}^{n} - H_y\big|_{i+1/2,j,k-1/2}^{n}}{\Delta z} \right)$$

FDTD: Advances

Hence, we have,

$$\frac{2\varepsilon}{\Delta t}E_x\Big|_{i+1/2,j,k}^{n+1/2}=\frac{2\varepsilon}{\Delta t}E_x\Big|_{i+1/2,j,k}^{n}$$

$$+\frac{1}{\Delta y}\left(H_z\Big|_{i+1/2,j+1/2,k}^{n}+\frac{\Delta t}{2\mu}\left(\frac{E_x\Big|_{i+1/2,j+1,k}^{n+1/2}-E_x\Big|_{i+1/2,j,k}^{n+1/2}}{\Delta y}-\frac{E_y\Big|_{i+1,j+1/2,k}^{n}-E_y\Big|_{i,j+1/2,k}^{n}}{\Delta x}\right)\right)$$

$$-\frac{1}{\Delta y}\left(H_z\Big|_{i+1/2,j-1/2,k}^{n}+\frac{\Delta t}{2\mu}\left(\frac{E_x\Big|_{i+1/2,j,k}^{n+1/2}-E_x\Big|_{i+1/2,j-1,k}^{n+1/2}}{\Delta y}-\frac{E_y\Big|_{i+1,j-1/2,k}^{n}-E_y\Big|_{i,j-1/2,k}^{n}}{\Delta x}\right)\right)$$

$$-\frac{1}{\Delta z}\left(H_y\Big|_{i+1/2,j,k+1/2}^{n}-H_y\Big|_{i+1/2,j,k-1/2}^{n}\right)$$

FDTD: Advances

Rearranging terms (taking all terms of E-field with n+1/2 time instant to LHS) $\{RHS\ has\ only\ fields\ at\ n\ time\ instants\}$

$$-\frac{\Delta t}{2\mu(\Delta y)^2}E_x\Big|_{i+1/2,j-1,k}^{n+1/2}+\left(\frac{2\varepsilon}{\Delta t}+\frac{\Delta t}{\mu(\Delta y)^2}\right)E_x\Big|_{i+1/2,j,k}^{n+1/2}-\frac{\Delta t}{2\mu(\Delta y)^2}E_x\Big|_{i+1/2,j+1,k}^{n+1/2}$$

$$=\frac{2\varepsilon}{\Delta t}E_x\Big|_{i+1/2,j,k}^{n}+\left(\frac{H_z\Big|_{i+1/2,j+1/2,k}^{n}-H_z\Big|_{i+1/2,j-1/2,k}^{n}}{\Delta y}-\frac{H_y\Big|_{i+1/2,j,k+1/2}^{n}-H_y\Big|_{i+1/2,j,k-1/2}^{n}}{\Delta z}\right)$$

$$+\frac{\Delta t}{2\mu}\left(\left(\frac{E_y\Big|_{i+1,j-1/2,k}^{n}-E_y\Big|_{i,j-1/2,k}^{n}}{\Delta x\Delta y}\right)-\left(\frac{E_y\Big|_{i+1,j+1/2,k}^{n}-E_y\Big|_{i,j+1/2,k}^{n}}{\Delta x\Delta y}\right)\right)$$

FDTD: Advances

- Points to be noted:
- LHS there are three terms dependent on Ex at time instant n+1/2 with coefficients
- Note the space locations are (i+1/2,{j+1 or j or j-1},k)
- RHS second term do not have any coefficient, there are four terms
- Defining some new coefficients (α_1, β_1, γ_1) for three coefficients of three terms in LHS
  - and (p_1, q_1 and r_1) for three coefficients of three terms in RHS and simplify

FDTD: Advances

$$\alpha_1 E_x\Big|_{i+1/2,j-1,k}^{n+1/2} + \beta_1 E_x\Big|_{i+1/2,j,k}^{n+1/2} + \gamma_1 E_x\Big|_{i+1/2,j+1,k}^{n+1/2}$$

$$= p_1 E_x\Big|_{i+1/2,j,k}^{n} + \left(\frac{H_z\big|_{i+1/2,j+1/2,k}^{n} - H_z\big|_{i+1/2,j-1/2,k}^{n}}{\Delta y} - \frac{H_y\big|_{i+1/2,j,k+1/2}^{n} - H_y\big|_{i+1/2,j,k-1/2}^{n}}{\Delta z} \right)$$

$$+ \left(q_1 \left(\frac{E_y\big|_{i+1,j-1/2,k}^{n} - E_y\big|_{i,j-1/2,k}^{n}}{\Delta x \Delta y} \right) - r_1 \left(\frac{E_y\big|_{i+1,j+1/2,k}^{n} - E_y\big|_{i,j+1/2,k}^{n}}{\Delta x \Delta y} \right) \right) = T_1\Big|_{i,j,k}^{n}$$

where material dependent coefficients are calculated at i+1/2,{j+1/2 or j or j-1/2},k

$$\alpha_1 = -\frac{2\Delta t}{\mu(\Delta y)^2}\bigg|_{i+1/2,j-1/2,k} ;\ \beta_1 = \left(\frac{2\varepsilon}{\Delta t} + \frac{\Delta t}{\mu(\Delta y)^2} \right)\bigg|_{i+1/2,j,k} ;\ \gamma_1 = -\frac{2\Delta t}{\mu(\Delta y)^2}\bigg|_{i+1/2,j+1/2,k} ;$$

$$p_1 = \frac{2\varepsilon}{\Delta t}\bigg|_{i+1/2,j,k} ;\ q_1 = \frac{2\Delta t}{\mu}\bigg|_{i+1/2,j-1/2,k} ;\ r_1 = \frac{2\Delta t}{\mu}\bigg|_{i+1/2,j+1/2,k} ;$$

FDTD: Advances

- Final equation for first procedure

$$+\alpha_1 E_x\Big|_{i+1/2,j-1,k}^{n+1/2} + \beta_1 E_x\Big|_{i+1/2,j,k}^{n+1/2} + \gamma_1 E_x\Big|_{i+1/2,j+1,k}^{n+1/2} = T_1\Big|_{i,j,k}^{n}$$

- In matrix form (for j=1,2,3,…,m)

$$\begin{bmatrix} \beta_1(1) & \gamma_1(1) & 0 & \cdots & 0 \\ \alpha_1(2) & \beta_1(2) & \gamma_1(2) & \cdots & 0 \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ 0 & \cdots & \alpha_1(m-1) & \beta_1(m-1) & \gamma_1(m-1) \\ 0 & \cdots & 0 & \alpha_1(m) & \beta_1(m) \end{bmatrix} \begin{bmatrix} E_x^{n+1/2}(1) \\ E_x^{n+1/2}(2) \\ \vdots \\ E_x^{n+1/2}(m-1) \\ E_x^{n+1/2}(m) \end{bmatrix} = \begin{bmatrix} T_1^n(1) \\ T_1^n(2) \\ \vdots \\ T_1^n(m-1) \\ T_1^n(m) \end{bmatrix}$$

FDTD: Advances

- In second procedure (n+1 ← n+1/2) $\frac{\partial E_x}{\partial t} = \frac{1}{\varepsilon_0 \varepsilon_r}\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)$
- $\frac{\partial H_z}{\partial y}$ replaced with explicit difference approximation
- $\frac{\partial H_y}{\partial z}$ replaced with implicit difference approximation

$$\frac{E_x\Big|_{i+1/2,j,k}^{n+1} - E_x\Big|_{i+1/2,j,k}^{n+1/2}}{\frac{\Delta t}{2}} = \frac{1}{\varepsilon}\left(\frac{H_z\Big|_{i+1/2,j+1/2,k}^{n+1/2} - H_z\Big|_{i+1/2,j-1/2,k}^{n+1/2}}{\Delta y} - \frac{H_y\Big|_{i+1/2,j,k+1/2}^{n+1} - H_y\Big|_{i+1/2,j,k-1/2}^{n+1}}{\Delta z}\right)$$

FDTD: Advances

- Similar equation for second procedure

$$-\alpha_2 E_x\Big|_{i+1/2,j,k-1}^{n+1} + \beta_2 E_x\Big|_{i+1/2,j,k}^{n+1} - \gamma_2 E_x\Big|_{i+1/2,j,k+1}^{n+1} = T_2\Big|_{i,j,k}^{n+1/2}$$

- In matrix form (k=1,2,3,…,m)

$$\begin{bmatrix} \beta_2(1) & \gamma_2(1) & 0 & \cdots & 0 \\ \alpha_2(2) & \beta_2(2) & \gamma_2(2) & \cdots & 0 \\ \vdots & \cdots & \ddots & \cdots & \vdots \\ 0 & \cdots & \alpha_2(m-1) & \beta_2(m-1) & \gamma_2(m-1) \\ 0 & \cdots & 0 & \alpha_2(m) & \beta_2(m) \end{bmatrix} \begin{bmatrix} E_x^{n+1}(1) \\ E_x^{n+1}(2) \\ \vdots \\ E_x^{n+1}(m-1) \\ E_x^{n+1}(m) \end{bmatrix} = \begin{bmatrix} T_2^{n+1/2}(1) \\ T_2^{n+1/2}(2) \\ \vdots \\ T_2^{n+1/2}(m-1) \\ T_2^{n+1/2}(m) \end{bmatrix}$$

FDTD: Advances

$$T_2\Big|_{i,j,k}^{n+1/2}$$

$$= p_2 E_x\Big|_{i+1/2,j,k}^{n+1/2} + \left(\frac{H_z\Big|_{i+1/2,j+1/2,k}^{n+1/2} - H_z\Big|_{i+1/2,j-1/2,k}^{n+1/2}}{\Delta y} - \frac{H_y\Big|_{i+1/2,j,k+1/2}^{n+1/2} - H_y\Big|_{i+1/2,j,k-1/2}^{n+1/2}}{\Delta z} \right)$$

$$+ \left(q_2 \left(\frac{E_z\Big|_{i+1,j,k-1/2}^{n+1/2} - E_y\Big|_{i,j,k-1/2}^{n+1/2}}{\Delta x \Delta z} \right) - r_2 \left(\frac{E_z\Big|_{i+1,j,k+1/2}^{n+1/2} - E_z\Big|_{i+1,j,k+1/2}^{n+1/2}}{\Delta x \Delta z} \right) \right)$$

where material dependent coefficients are calculated at i+1/2,j, {k+1/2 or k or k-1/2}

$$\alpha_2 = -\frac{2\Delta t}{\mu(\Delta z)^2}\Bigg|_{i+1/2,j,k-1/2}; \beta_2 = \left(\frac{2\varepsilon}{\Delta t} + \frac{\Delta t}{\mu(\Delta y)^2} \right)\Bigg|_{i+1/2,j,k}; \gamma_2 = -\frac{2\Delta t}{\mu(\Delta z)^2}\Bigg|_{i+1/2,j,k+1/2};$$

$$p_2 = \frac{2\varepsilon}{\Delta t}\Bigg|_{i+1/2,j,k}; q_2 = \frac{2\Delta t}{\mu}\Bigg|_{i+1/2,j,k-1/2}; r_2 = \frac{2\Delta t}{\mu}\Bigg|_{i+1/2,j,k+1/2};$$

FDTD: Advances

- 1-D CN FDTD

$$\frac{\partial E_x}{\partial t} = -\frac{1}{\varepsilon_0 \varepsilon_r}\left(\frac{\partial H_y}{\partial z}\right) \qquad \frac{\partial H_y}{\partial t} = -\frac{1}{\mu_0 \mu_r}\left(\frac{\partial E_x}{\partial z}\right)$$

- From time discretization (RHS has an implicit term)

$$\frac{E_x^{n+1} - E_x^n}{\Delta t} = -\frac{1}{2\varepsilon}\left(\frac{\partial H_y^{n+1}}{\partial z} + \frac{\partial H_y^n}{\partial z}\right)$$

$$\frac{H_y^{n+1} - H_y^n}{\Delta t} = -\frac{1}{2\mu}\left(\frac{\partial E_x^{n+1}}{\partial z} + \frac{\partial E_x^n}{\partial z}\right)$$

FDTD: Advances

- Reordering

$$E_x^{n+1} = E_x^n - \frac{\Delta t}{2\varepsilon}\left(\frac{\partial H_y^{n+1}}{\partial z} + \frac{\partial H_y^n}{\partial z}\right) \qquad (1)$$

$$H_y^{n+1} = H_y^n - \frac{\Delta t}{2\mu}\left(\frac{\partial E_x^{n+1}}{\partial z} + \frac{\partial E_x^n}{\partial z}\right) \qquad (2)$$

- Putting (2) in (1)

$$E_x^{n+1} = E_x^n - \frac{\Delta t}{2\varepsilon}\left(\frac{\partial\left(H_y^n - \frac{\Delta t}{2\mu}\left(\frac{\partial E_x^{n+1}}{\partial z} + \frac{\partial E_x^n}{\partial z}\right)\right)}{\partial z} + \frac{\partial H_y^n}{\partial z}\right)$$

FDTD: Advances

$$E_x^{n+1} = E_x^n - \frac{\Delta t}{2\varepsilon}\left(\frac{\partial\left(H_y^n - \frac{\Delta t}{2\mu}\left(\frac{\partial E_x^{n+1}}{\partial z} + \frac{\partial E_x^n}{\partial z} \right)\right)}{\partial z} + \frac{\partial H_y^n}{\partial z} \right)$$

- Taking all terms with superscript (n+1) to the LHS, we have

$$E_x^{n+1} - \frac{\Delta t^2}{4\mu\varepsilon}\frac{\partial^2 E_x^{n+1}}{\partial z^2} = E_x^n + \frac{\Delta t^2}{4\mu\varepsilon}\frac{\partial^2 E_x^n}{\partial z^2} - \frac{\Delta t}{\varepsilon}\frac{\partial H_y^n}{\partial z}$$

FDTD: Advances

- Space discretization

$$E_x^{n+1} - \frac{\Delta t^2}{4\mu\varepsilon\Delta z^2}\left(E_x^{n+1}(i+1) - 2E_x^{n+1}(i) + E_x^{n+1}(i-1)\right)$$

$$= E_x^n + \frac{\Delta t^2}{4\mu\varepsilon\Delta z^2}\left(E_x^n(i+1) - 2E_x^n(i) + E_x^n(i-1)\right) - \frac{\Delta t}{\varepsilon\Delta z}\left(H_y^n(i) - H_y^n(i-1)\right)$$

- Putting $\frac{\Delta t^2}{4\mu\varepsilon\Delta z^2} = \frac{p}{2}; \frac{\Delta t}{\varepsilon\Delta z} = r$

$$E_x^{n+1} - \frac{p}{2}\left(E_x^{n+1}(i+1) - 2E_x^{n+1}(i) + E_x^{n+1}(i-1)\right)$$

$$= E_x^n + \frac{p}{2}\left(E_x^n(i+1) - 2E_x^n(i) + E_x^n(i-1)\right) - r\left(H_y^n(i) - H_y^n(i-1)\right)$$

FDTD: Advances

- Rearranging (taking all n+1 term to LHS)

$$-\frac{p}{2}E_x^{n+1}(i-1)+(1+p)E_x^{n+1}(i)-\frac{p}{2}E_x^{n+1}(i+1)$$

$$=\frac{p}{2}E_x^{n}(i-1)+(1-p)E_x^{n}(i)+\frac{p}{2}E_x^{n}(i+1)-r\left(H_y^{n}(i)-H_y^{n}(i-1)\right)$$

- In matrix form

$$(\mathbf{I}+p\mathbf{G})\mathbf{E}_x^{n+1}=(\mathbf{I}-p\mathbf{G})\mathbf{E}_x^{n}-r\mathbf{J}\mathbf{H}_y^{n}$$

FDTD: Advances

$$-\frac{p}{2}E_x^{n+1}(i-1)+(1+p)E_x^{n+1}(i)-\frac{p}{2}E_x^{n+1}(i+1)$$

$$=\frac{p}{2}E_x^{n}(i-1)+(1-p)E_x^{n}(i)+\frac{p}{2}E_x^{n}(i+1)-r\left(H_y^n(i)-H_y^n(i-1)\right)$$

$$(\mathbf{I}+p\mathbf{G})\mathbf{E}_x^{n+1}=(\mathbf{I}-p\mathbf{G})\mathbf{E}_x^{n}-r\mathbf{J}\mathbf{H}_y^n$$

- where for i=1,2,…,m

$$\mathbf{E}_x^{n+1}=\begin{bmatrix}E_x^{n+1}(1)\\E_x^{n+1}(2)\\E_x^{n+1}(3)\\E_x^{n+1}(4)\\\vdots\\E_x^{n+1}(m)\end{bmatrix}$$

$$\mathbf{G}=\begin{bmatrix}1 & -\frac{1}{2} & 0 & 0 & 0 & 0\\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 & 0\\ 0 & -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0\\ 0 & 0 & \ddots & \ddots & \ddots & 0\\ 0 & 0 & 0 & -\frac{1}{2} & 1 & -\frac{1}{2}\\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 1\end{bmatrix}_{m\times m}$$

FDTD: Advances

$$-\frac{p}{2}E_x^{n+1}(i-1)+(1+p)E_x^{n+1}(i)-\frac{p}{2}E_x^{n+1}(i+1)$$

$$=\frac{p}{2}E_x^{n}(i-1)+(1-p)E_x^{n}(i)+\frac{p}{2}E_x^{n}(i+1)-r\left(H_y^n(i)-H_y^n(i-1)\right)$$

$$(\mathbf{I}+p\mathbf{G})\mathbf{E}_x^{n+1}=(\mathbf{I}-p\mathbf{G})\mathbf{E}_x^{n}-r\mathbf{J}\mathbf{H}_y^n$$

$$\mathbf{I}+p\mathbf{G}=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&\ddots&\ddots&\ddots&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}+\begin{bmatrix}p&-\frac{p}{2}&0&0&0&0\\-\frac{p}{2}&p&-\frac{p}{2}&0&0&0\\0&-\frac{p}{2}&p&-\frac{1}{2}&0&0\\0&0&\ddots&\ddots&\ddots&0\\0&0&0&-\frac{p}{2}&p&-\frac{p}{2}\\0&0&0&0&-\frac{p}{2}&p\end{bmatrix}_{m\times m}$$

FDTD: Advances

$$-\frac{p}{2}E_x^{n+1}(i-1)+(1+p)E_x^{n+1}(i)-\frac{p}{2}E_x^{n+1}(i+1)$$

$$=\frac{p}{2}E_x^{n}(i-1)+(1-p)E_x^{n}(i)+\frac{p}{2}E_x^{n}(i+1)-r\left(H_y^n(i)-H_y^n(i-1)\right)$$

$$(\mathbf{I}+p\mathbf{G})\mathbf{E}_x^{n+1}=(\mathbf{I}-p\mathbf{G})\mathbf{E}_x^{n}-r\mathbf{J}\mathbf{H}_y^{n}$$

$$\mathbf{H}_y^n=\begin{bmatrix}H_y^n(1)\\H_y^n(2)\\H_y^n(3)\\H_y^n(4)\\\vdots\\H_y^n(m)\end{bmatrix}\qquad \mathbf{J}=\begin{bmatrix}1&0&0&0&0&0\\-1&1&0&0&0&0\\0&-1&1&0&0&0\\0&0&\ddots&\ddots&0&0\\0&0&0&-1&1&0\\0&0&0&0&-1&1\end{bmatrix}_{m\times m}$$

FDTD: Advances

- In matrix form

$$\left(\mathbf{I}+p\mathbf{G}\right)\mathbf{H}_y^{n+1}=\left(\mathbf{I}-p\mathbf{G}\right)\mathbf{E}_y^n-r'\mathbf{J}\mathbf{E}_y^n$$

$$\Rightarrow \mathbf{M}_1\mathbf{H}_y^{n+1}=\mathbf{M}_2\mathbf{E}_y^n+\mathbf{M}_3 E_y^n$$

$$\Rightarrow \mathbf{H}_y^{n+1}=\left(\mathbf{M}_1\right)^{-1}\mathbf{M}_2\mathbf{E}_y^n+\left(\mathbf{M}_1\right)^{-1}\mathbf{M}_3\mathbf{E}_y^n$$