10.2 Basic Steps in Method of Moments

- For [Z] is non-singular,
- Solve the unknown matrix [I] of amplitudes of basis function as

$$[I]=[Z]^{-1}[V]=[Y][V]$$

- *Galerkin's method*

$$b_n = w_n$$

- *Point matching or Collocation*
  - The testing function is a delta function

10.2 Basic Steps in Method of Moments

- *Methods for calculating inverse of a matrix*
- Seldom find the inverse of matrix directly $[Z]^{-1}$, because,
  - if we have ill-conditioned matrices,
  - it can give highly erroneous results
- MATLAB command 'pinv' finds pseudo inverse of a matrix
  - using the singular value decomposition
- For a matrix equation of the form AX=B,
- if small changes in B leads to large changes in the solution X,
  - then we call A is ill-conditioned

10.2 Basic Steps in Method of Moments

- The *condition number of a matrix* is the
  - ratio of the largest singular value of a matrix to the smallest singular value
- Larger is this condition value
  - closer is the matrix to singularity
- It is always
  - greater than or equal to 1
- If it is close to one,
  - the matrix is *well conditioned*
  - which means its inverse can be computed with good accuracy

10.2 Basic Steps in Method of Moments

- If the condition number is large,
  - then the matrix is said to be *ill-conditioned*
- Practically,
  - such a matrix is almost singular, and
- the computation of its inverse, or
- solution of a linear system of equations is
  - prone to large numerical errors
- A matrix that is not invertible
  - has the condition number equal to infinity

10.2 Basic Steps in Method of Moments

- Sometimes pseudo inverse is also used for finding
  - approximate solutions to ill-conditioned matrices
- Preferable to use LU decomposition
  - to solve linear matrix equations
- LU factorization unlike Gaussian elimination,
  - do not make any modifications in the matrix B
- in solving the matrix equation

10.2 Basic Steps in Method of Moments

- Try to solving a matrix equation $[A][X]=[B]$
  - using LU factorization
- First express the matrix

$$[A]=[L][U]$$

$$[L]=\begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{N1} & l_{N2} & l_{N3} & \dots & l_{N1} \end{bmatrix} \quad and \quad [U]=\begin{bmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1N} \\ 0 & u_{22} & u_{23} & \cdots & u_{2N} \\ 0 & 0 & u_{33} & \cdots & u_{3N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & u_{NN} \end{bmatrix}$$

10.2 Basic Steps in Method of Moments

$$\therefore [L][U][X]=[B] \Rightarrow [L][Y]=[B]$$

- through the forward substitution

$$y_1 = \frac{b_1}{l_{11}}; y_i = \frac{1}{l_{ii}}\left[b_i - \sum_{k=1}^{i-1} l_{ik} y_k \right], i > 1$$

- through the backward substitution $[U][X]=[Y]$

$$x_N = \frac{y_N}{u_{NN}}; x_i = \frac{1}{u_{ii}}\left[y_i - \sum_{k=i+1}^{N} u_{ik} x_k \right], i < N$$

10.2 Basic Steps in Method of Moments

- This is more efficient than Gaussian elimination
  - since the RHS remain unchanged during the whole process
- The main issue here is to
  - find the lower and upper triangular matrices.
- MATLAB command for LU factorization of a matrix A is
  - [L U] = lu(A)

10.2 Basic Steps in Method of Moments

Example 10.1

- Consider a 1-D differential equation

$$-\frac{d^2 f(x)}{dx^2} = 3 + 2x^2$$

- subject to the boundary condition f(0)=f(1)=0
- Solve this differential equation using Galerkin's MoM

Solution:

- Note that for this case,

$$u = f(x)$$

10.2 Basic Steps in Method of Moments

$$k = 3 + 2x^2$$

$$L = -\frac{d^2}{dx^2}$$

- According to the nature of the known function $k = 3 + 2x^2$,
- it is natural to choose the basis function as $b_n(x) = x^n$
- However,
- the boundary condition f(1)=0
  - can't be satisfied with such a basis function

10.2 Basic Steps in Method of Moments

- A suitable basis function for this differential equation
  - taking into account of this boundary condition is

$$b_n(x) = x - x^{n+1}; n = 1, 2, \ldots, N$$

- Assume N=2 (the total number of subsections on the interval [0,1])
- Approximation of the unknown function

$$f(x) \cong I_1 b_1(x) + I_2 b_2(x) = I_1 \left(x - x^2\right) + I_2 \left(x - x^3\right)$$

10.2 Basic Steps in Method of Moments

- For Galerkin's MoM, the weighting functions are

$$w_m(x) = x - x^{m+1}; m = 1, 2, \ldots, M$$

- Choosing a square [Z] matrix where M=N=2

$$Z_{11} = \langle w_1, L(b_1) \rangle = \int_0^1 w_1(x) L(b_1(x))\, dx = \int_0^1 (x - x^2)(2)\, dx = \frac{1}{3}$$

$$L = -\frac{d^2}{dx^2}$$

$$Z_{12} = \langle w_1, L(b_2) \rangle = \int_0^1 w_1(x) L(b_2(x))\, dx = \int_0^1 (x - x^2)(6x)\, dx = \frac{1}{2}$$

10.2 Basic Steps in Method of Moments

$$Z_{21} = \left\langle w_2, L\left(b_1\right)\right\rangle = \int_0^1 w_2(x) L\left(b_1(x)\right) dx = \int_0^1 (x - x^3)\left(2\right) dx = \frac{1}{2}$$

$$Z_{22} = \left\langle w_2, L\left(b_2\right)\right\rangle = \int_0^1 w_2(x) L\left(b_2(x)\right) dx = \int_0^1 (x - x^3)\left(6x\right) dx = \frac{4}{5}$$

$$V_1 = \left\langle k, w_1 \right\rangle = \int_0^1 k(x) w_1(x) dx = \int_0^1 \left(3 + 2x^2\right)(x - x^2) dx = \frac{3}{5}$$

$k = 3 + 2x^2$

$$V_2 = \left\langle k, w_2 \right\rangle = \int_0^1 k(x) w_2(x) dx = \int_0^1 \left(3 + 2x^2\right)(x - x^3) dx = \frac{11}{12}$$

10.2 Basic Steps in Method of Moments

- *Therefore,*

$$[Z][I]=[V]\Rightarrow\begin{bmatrix}\frac{1}{3} & \frac{1}{2}\\ \frac{1}{2} & \frac{4}{5}\end{bmatrix}\begin{bmatrix}I_1\\ I_2\end{bmatrix}=\begin{bmatrix}\frac{3}{5}\\ \frac{11}{12}\end{bmatrix}$$

$$\Rightarrow[I]=\begin{bmatrix}I_1\\ I_2\end{bmatrix}=\begin{bmatrix}\frac{13}{10}\\ \frac{1}{3}\end{bmatrix}$$

- The unknown function f(x)

$$f(x)\cong I_1\left(x-x^2\right)+I_2\left(x-x^3\right)=\frac{13}{10}\left(x-x^2\right)+\frac{1}{3}\left(x-x^3\right)$$

10.2 Basic Steps in Method of Moments

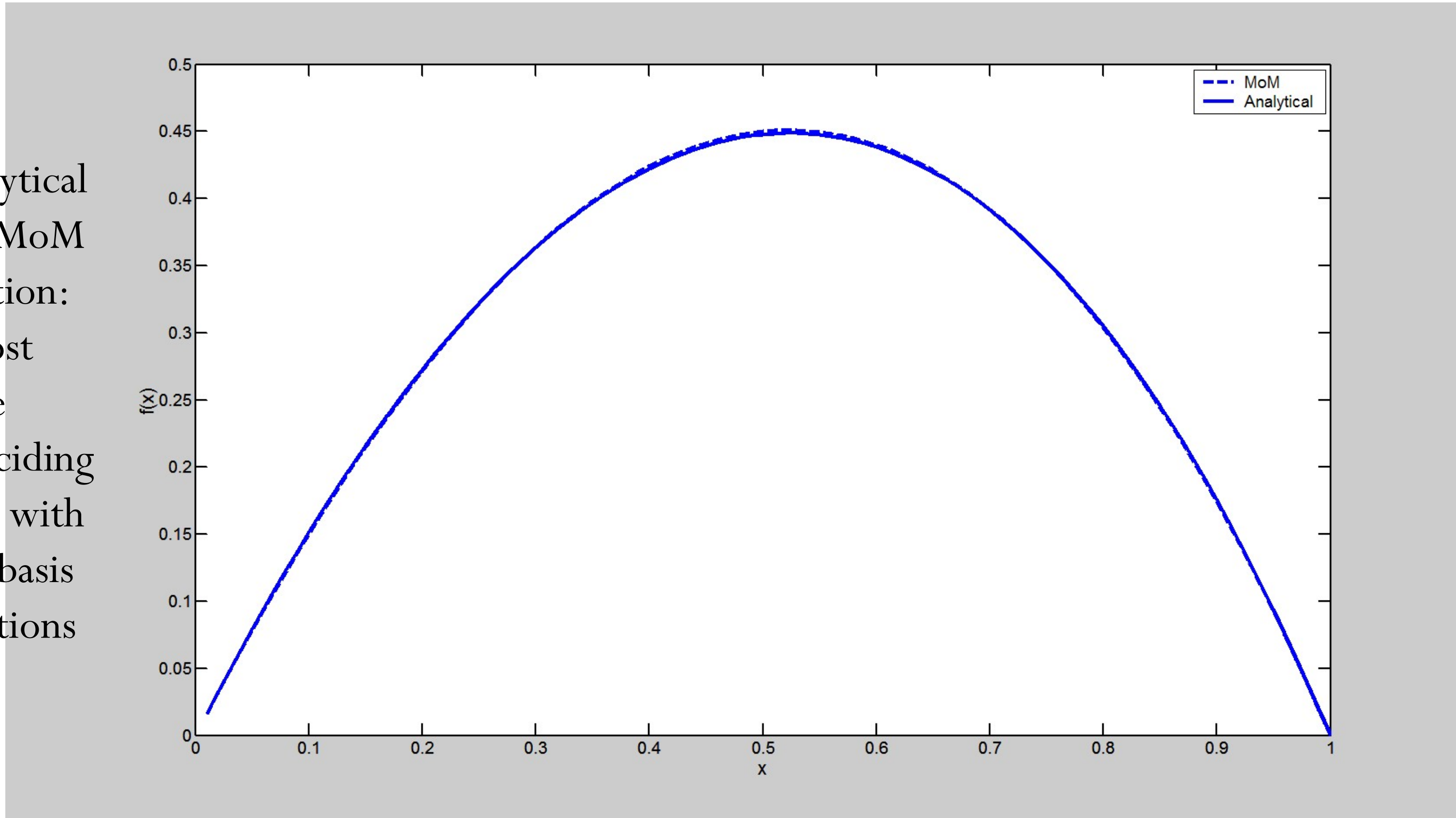
- The above function satisfies the given boundary conditions
  - $f(0)=f(1)=0$
- The analytical solution for this differential equation is

$$f(x)=\frac{5}{3}x-\frac{3}{2}x^2-\frac{1}{6}x^4$$

- Check whether the above solution using MoM is
  - different from the analytical solution obtained by direct integration (see Fig. 10.1)

10.2 Basic Steps in Method of Moments

Analytical and MoM solution: almost same coinciding even with two basis functions



- Fig. 10.1 Comparison of exact solution (analytical) and approximate solution (MoM) of Example 10.1

Programming Exercise 1 (Homework)

- Write a MATLAB program to solve Exercise 10.1
- Convergence analysis:
  - Perform convergence analysis by taking N=2,3,4
- Accuracy testing:
  - Check the accuracy of the MoM program by plotting the approximate solution obtained (convergent one) and comparing with the actual solution
- Programming exercise schedule:
  - All Programming exercise will be given on or before Friday
  - Submit it on or before next Wednesday

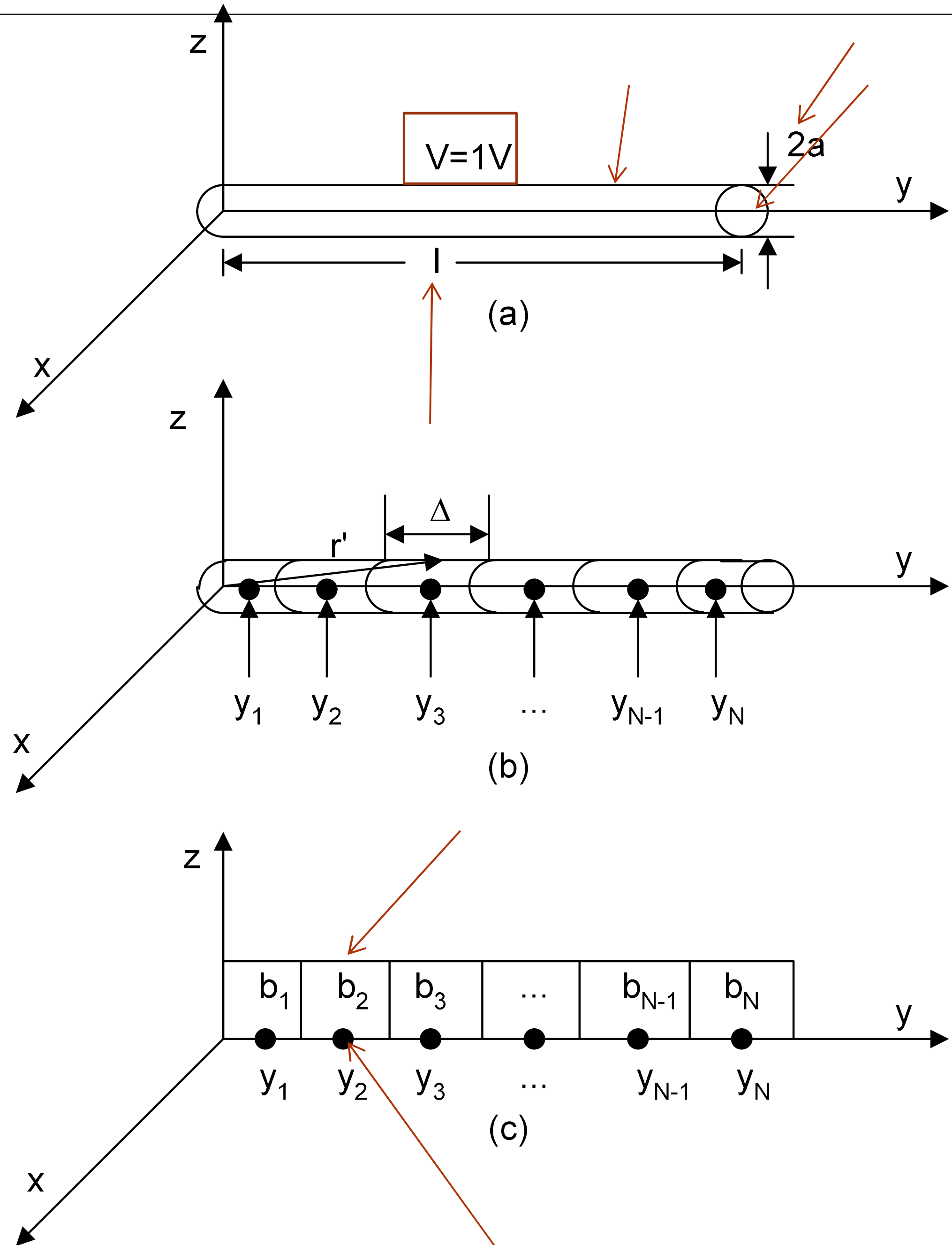
10.3 Introductory examples from electrostatics

- In electrostatics, the problem of finding the potential
  - due to a given charge distribution is often considered
- In practical scenario, it is very difficult to
  - specify a charge distribution
- We usually connect a conductor to a voltage source
  - and thus the voltage on the conductor is specified
- We will consider MoM
  - to solve for the electric charge distribution
- when an electric potential is specified
- Examples 2 and 3 discuss about calculation of inverse using LU decomposition and SVD

10.3 Introductory examples from electrostatics

- *1-D Electrostatic case: Charge density of a straight wire*
- Consider a straight wire of length l and radius a (assume a<<l),
  - placed along the y-axis as shown in Fig. 10.2 (a)
- The wire is applied to a constant electric potential of 1V
- Choosing observation along the wire axis (x=z=0) i.e., along the y-axis
  - and representing the charge density on the surface of the wire

$$1=\frac{1}{4\pi\varepsilon_0}\int_0^l \frac{\lambda(y')dy'}{R(y,y')}$$


z
V=1V
2a
y
l
x
(a)
z
Δ
r'
y
y1
y2
y3
...
yN-1
yN
x
(b)
z
b1
b2
b3
...
bN-1
bN
y
y1
y2
y3
...
yN-1
yN
x
(c)

10.3 Introductory examples from electrostatics

Fig. 10.2

- (a) Straight wire of length l and radius a applied with a constant potential of 1V
- (b) Its segmentation: $y_1, y_2, \ldots, y_N$ are observation points and r' shows a source point
- (c) Division of the charged strip into N sections

10.3 Introductory examples from electrostatics

- where

$$R(y, y') = R(\vec{r}, \vec{r}')\Big|_{x=z=0} = \sqrt{(y-y')^2 + (x')^2 + (z')^2} = \sqrt{(y-y')^2 + (a)^2}$$

- It is necessary to solve the integral equation
- to find the unknown function $\lambda(y')$
- The solution may be obtained numerically by
  - reducing the integral equation into a series of linear algebraic equations
  - that may be solved by conventional matrix techniques

10.3 Introductory examples from electrostatics

- (a) Approximate the unknown charge density $\lambda(y')$
  - by an expansion of N known basis functions with unknown coefficients

$$\lambda(y') = \sum_{n=1}^{N} I_n b_n(y')$$

$$1 = \frac{1}{4\pi\varepsilon_0} \int_0^l \frac{\lambda(y')dy'}{R(y,y')}$$

- Integral equation after substituting this is

$$4\pi\varepsilon_0 = \int_0^l \frac{\sum_{n=1}^{N} I_n b_n(y')dy'}{R(y,y')} = \sum_{n=1}^{N} I_n \int_0^l \frac{b_n(y')dy'}{R(y,y')}$$

10.3 Introductory examples from electrostatics

- Now we have divided the wire into N uniform segments each of length Δ as shown in Fig. 10.2 (b)
- We will choose our basis functions as pulse functions

$$b_n\left(y'\right)=\begin{cases}1 & for \quad (n-1)\Delta \le y' \le n\Delta \\ 0 & otherwise\end{cases}$$

b) Applying the testing or weighting functions

- Let us apply the testing functions as delta functions $\left[\partial\left(y-y_m\right)\right]$ for point matching

10.3 Introductory examples from electrostatics

- Integration of any function with this delta function
  - will give us the function value at $y = y_m$
- Replacing observation variable y by a fixed point such as y_m,
  - results in an integrand that is solely a function of y′
- so the integral may be evaluated.
- It leads to an equation
  - with N unknowns

$$4\pi\varepsilon_0 = \int_0^l \frac{\sum_{n=1}^{N} I_n b_n(y')dy'}{R(y,y')} = \sum_{n=1}^{N} I_n \int_0^l \frac{b_n(y')dy'}{R(y,y')}$$

$$4\pi\varepsilon_0 = I_1 \int_0^{\Delta} \frac{b_1(y')dy'}{R(y_m,y')} + I_2 \int_{\Delta}^{2\Delta} \frac{b_2(y')dy'}{R(y_m,y')} + \ldots + I_n \int_{(n-1)\Delta}^{n\Delta} \frac{b_n(y')dy'}{R(y_m,y')} + \ldots + I_N \int_{(N-1)\Delta}^{l} \frac{b_N(y')dy'}{R(y_m,y')}$$

10.3 Introductory examples from electrostatics

- Solution for these N unknown constants,
  - N linearly independent equations are required
- N equations may be produced
  - by choosing an observation point y_m on the wire
  - where **m=1,2,3…, N** and
  - at the center of each Δ length element
- as shown in Fig. 10.2 (c)
- Result in an equation of the form of the previous equation
  - corresponding to each observation point

10.3 Introductory examples from electrostatics

- For N such observation points we have

$$4\pi\varepsilon_0 = I_1 \int_0^{\Delta} \frac{b_1(y')dy'}{R(y_1, y')} + I_2 \int_{\Delta}^{2\Delta} \frac{b_2(y')dy'}{R(y_1, y')} + \ldots + I_n \int_{(n-1)\Delta}^{n\Delta} \frac{b_n(y')dy'}{R(y_1, y')} + \ldots + I_N \int_{(N-1)\Delta}^{l} \frac{b_N(y')dy'}{R(y_1, y')}$$

$$4\pi\varepsilon_0 = I_1 \int_0^{\Delta} \frac{b_1(y')dy'}{R(y_2, y')} + I_2 \int_{\Delta}^{2\Delta} \frac{b_2(y')dy'}{R(y_2, y')} + \ldots + I_n \int_{(n-1)\Delta}^{n\Delta} \frac{b_n(y')dy'}{R(y_2, y')} + \ldots + I_N \int_{(N-1)\Delta}^{l} \frac{b_N(y')dy'}{R(y_2, y')}$$

$$4\pi\varepsilon_0 = I_1 \int_0^{\Delta} \frac{b_1(y')dy'}{R(y_N, y')} + I_2 \int_{\Delta}^{2\Delta} \frac{b_2(y')dy'}{R(y_N, y')} + \ldots + I_n \int_{(n-1)\Delta}^{n\Delta} \frac{b_n(y')dy'}{R(y_N, y')} + \ldots + I_N \int_{(N-1)\Delta}^{l} \frac{b_N(y')dy'}{R(y_N, y')}$$