FDTD: Advances

DIGRESSION:

- **Frequency-dispersive medium**
- When the speed of light in a material is a function of frequency, the material said to be dispersive
- $v_p = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{\mu(\omega)\varepsilon(\omega)}} = \frac{1}{\sqrt{\mu(\omega)\varepsilon(\omega)}}$
- Most of the material parameters are frequency dependent
- For example, it is impossible to have a lossless dielectric with constant permittivity except free space
- $v_g = \frac{d\omega}{d\beta} = \left(\frac{d\beta}{d\omega}\right)^{-1}$ {If β is a function of ω, v_g may vary with ω}

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- **Linear and non-linear frequency-dispersive medium**
- For the components of the electric polarization P in the frequency domain one can write
- $P_i = \varepsilon_0\left(\chi_i^{1j} E_j + \chi_i^{2jk} E_j E_k + \chi_i^{3jkl} E_j E_k E_l + \cdots\right)$
- Ignoring higher order terms in the series on the right hand side of the above equation,
  - yields a polarization P that depends linearly on E

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- Frequency dependent first order susceptibilities
  - χ_i^{1j} that relate E to P give rise to linear dispersion
- The term linear refers to the linearity of P in E
  - and not to frequency dependence of the susceptibility χ_i^{1j}
- We usually consider three types of medium
  - Metal or conducting medium
  - Dielectric medium
  - Magnetic medium

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- Drude model
  - good model for conductors,
  - charges are assumed to move under the influence of the electric field
  - but they experience a damping force as well
- (In metals, free electrons, unbound to nucleus, low restoring force, no natural frequency (3rd term neglected in LHS of Lorentz model))

$$\frac{d^2}{dt^2}P_i + \Gamma_D \frac{d}{dt}P_i = \varepsilon_0 \chi_D E_i$$

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- Lorentz model for dielectrics,
  - charges are assumed to move under the influence of the electric field
  - but they experience a damping force as well

$$\frac{d^2}{dt^2}P_i + \Gamma_L \frac{d}{dt}P_i + \omega_0^2 P_i = \varepsilon_0 \chi_L E_i$$

- At a macroscopic level, all resonance mechanisms can be characterized using the Lorentz model
- This allows any number of resonances to be accounted for through a simple summation

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- Applications for dispersive-medium (ADE-FDTD),
  - Waveguides (dispersive since for TE_{mn} modes inside rectangular waveguide $\beta = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}, k = \frac{\omega}{c}$),
  - antenna structures,
  - integrated circuits,
  - bioelectromagnetic applications
  - Metamaterials (CN-FDTD employing Lorentz dispersive model of DNG metamaterial, https://ieeexplore.ieee.org/document/8262940)

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$$\chi_{e,Debye}(\omega) = \frac{P_i(\omega)}{\varepsilon_0 E_i(\omega)} = \frac{\chi_d}{j\Gamma_d \omega + \omega_0^2}$$

$$\chi_{e,Lorentz}(\omega) = \frac{P_i(\omega)}{\varepsilon_0 E_i(\omega)} = \frac{\chi_L}{-\omega^2 + j\Gamma_L \omega + \omega_0^2}$$

- FDTD method has been widely used to model dispersive media
  - because it allows the treatment of broadband response in a single simulation run
- For this purpose, one needs to
  - accurately and efficiently incorporate real material dispersions
- The common practice (Debye-Lorentz model) is to
  - fit a given permittivity function as the sum of multiple Debye poles or Lorentz pole pairs
- A very general equation for modeling complicated dielectrics and metals is the Drude-Lorentz model

$$\chi_{e,Drude}(\omega) = \frac{P_i(\omega)}{\varepsilon_0 E_i(\omega)} = \frac{\chi_L}{-\omega^2 + j\Gamma_D \omega}$$

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Stability analysis of FDTD:

- von Neumann method for analysing stability of finite difference equation (FDE)
  - by J. von Neumann for fluid dynamics during world war II
- **First step:**
- Write the spatial distribution of the voltage or electric field as a complex Fourier series (as proposed by J. von Neumann)
- $V(x,t) = \sum_{m=-\infty}^{\infty} V_m(x,t) = \sum_{m=-\infty}^{\infty} C_m(t)e^{jk_m x}$
- A general spatial Fourier component is given by
- $V_m(x,t) = C_m(t)e^{jk_m x}$

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Stability analysis of FDTD:

- For stability analysis of FDE we will use its discretized version
- $V_i^n = C(t^n)e^{jki\Delta x}$
- where we have dropped m from C and k because we are working with single component
- Also $V_{i\pm1}^n = C(t^n)e^{jk(i\pm1)\Delta x} = V_i^n e^{\pm jk\Delta x}$
- *Substitute discretized Fourier component in the FDE*

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- **Second step:**
- *Reduce the FDE into the form* $V_i^{n+1} = qV_i^n$ *so that the amplification factor q is expressed in terms of* $sin(k\Delta x), cos(k\Delta x)$ and Δt
- **Third step:**
- *Analyze q to determine the stability of FDE*

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- **Example 1:**
- *Forward-time centered space method (1st order forward difference approximation for time derivative & 2nd order centred difference approximation for space derivative)*
- $\frac{\partial V}{\partial t} + v_p \frac{\partial V}{\partial x} = 0$ (voltage on a transmission line)
- $\Rightarrow \frac{\partial V}{\partial t} = -v_p \frac{\partial V}{\partial x}$
- *FDE*
- $\frac{V_i^{n+1} - V_i^n}{\Delta t} = -v_p \frac{V_{i+1}^n - V_{i-1}^n}{2\Delta x}$
- $\Rightarrow V_i^{n+1} = V_i^n - \frac{v_p \Delta t}{2\Delta x}\left[V_{i+1}^n - V_{i-1}^n\right]$

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- 1st step:
- Put $V_{i\pm1}^{n} = V_{i}^{n} e^{\pm jk\Delta x}$ in FDE
- $V_{i}^{n+1} = V_{i}^{n} - \frac{v_p \Delta t}{2\Delta x}\left[V_{i}^{n} e^{jk\Delta x} - V_{i}^{n} e^{-jk\Delta x}\right]$
- 2nd step:
- $V_{i}^{n+1} = \left[1 - j\frac{v_p \Delta t}{\Delta x}\sin(k\Delta x)\right] V_{i}^{n} = qV_{i}^{n}$
- 3rd step:
- $|q| > 1$ unconditionally unstable

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- **Example 2:**
- **Leap frog manner: conventional FDTD (***2nd order centred difference approximation for time derivative & 2nd order centred difference approximation for space derivative***)**
- $\frac{\partial V}{\partial t} = -v_p \frac{\partial V}{\partial x}$
- *FDE*
- $\frac{V_i^{n+1} - V_i^{n-1}}{2\Delta t} = -v_p \frac{V_{i+1}^n - V_{i-1}^n}{2\Delta x}$
- $\Rightarrow V_i^{n+1} = V_i^n - \frac{v_p \Delta t}{\Delta x}[V_{i+1}^n - V_{i-1}^n]$

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- 1st step:
- Put $V_{i\pm1}^{n} = V_i^n e^{\pm jk\Delta x}$ in FDE and also noting that $V_i^n = qV_i^{n-1}$
- $V_i^{n+1} = \frac{V_i^n}{q} - \frac{v_p\Delta t}{2\Delta x}\left[V_i^n e^{jk\Delta x} - V_i^n e^{-jk\Delta x}\right]$
- 2nd step:
- $V_i^{n+1} = \left[\frac{1}{q} - j\frac{v_p\Delta t}{\Delta x}\sin(k\Delta x)\right]V_i^n = qV_i^n$
- 3rd step:
- $q=\left[\frac{1}{q} - j\frac{v_p\Delta t}{\Delta x}\sin(k\Delta x)\right]$

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- $q = -jA \pm \sqrt{1 - A^2}; A = \frac{v_p \Delta t}{\Delta x} sin(k\Delta x)$
- Note that $|q| = 1$ provided $1 - A^2$ is positive
- Hence for stability $1 - A^2 \geq 0$ or $|A| \leq 1$
- which implies that $\left|\frac{v_p \Delta t}{\Delta x} sin(k\Delta x)\right| \leq 1$
- Or, $\left|\frac{v_p \Delta t}{\Delta x}\right| \leq 1$
- Or, $\Delta t \leq \frac{\Delta x}{|v_p|}$ which is CFL

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- **Example 3:**
- *Backward-time centered space method (1st order forward difference approximation for time derivative & 2nd order centred difference approximation for space derivative)*
- $\frac{\partial V}{\partial t} + v_p \frac{\partial V}{\partial x} = 0$ (voltage on a transmission line)
- $\Rightarrow \frac{\partial V}{\partial t} = -v_p \frac{\partial V}{\partial x}$
- *FDE*
- $\frac{V_i^{n+1} - V_i^n}{\Delta t} = -v_p \frac{V_{i+1}^{n+1} - V_{i-1}^{n+1}}{2\Delta x}$ *(implicit FDTD)*
- $\Rightarrow V_i^{n+1} + \frac{v_p \Delta t}{2\Delta x}\left[V_{i+1}^{n+1} - V_{i-1}^{n+1}\right] = V_i^n$

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- 1^{st} step:
- Put $\mathrm{V}_{\mathrm{i}\pm 1}^{\mathrm{n}} = \mathrm{V}_{\mathrm{i}}^{\mathrm{n}} \mathrm{e}^{\pm \mathrm{jk}\Delta \mathrm{x}}$ in FDE and also noting that $\mathrm{V}_{\mathrm{i}}^{\mathrm{n}+1} = q\mathrm{V}_{\mathrm{i}}^{\mathrm{n}}$
- $\Rightarrow \frac{v_p \Delta t}{2\Delta x} V_{i+1}^{n+1} + V_i^{n+1} - \frac{v_p \Delta t}{2\Delta x} V_{i-1}^{n+1} = V_i^n$
- $\Rightarrow \frac{v_p \Delta t}{2\Delta x} qV_{i+1}^{n} + qV_i^{n} - \frac{v_p \Delta t}{2\Delta x} qV_{i-1}^{n} = V_i^n$
- $\Rightarrow \frac{v_p \Delta t}{2\Delta x} qV_i^n \mathrm{e}^{\mathrm{jk}\Delta \mathrm{x}} + qV_i^{n} - \frac{v_p \Delta t}{2\Delta x} V_i^n \mathrm{e}^{-\mathrm{jk}\Delta \mathrm{x}} = V_i^n$
- 2^{nd} step:
- $\Rightarrow q\left(j\frac{v_p \Delta t}{\Delta x} sin(\mathrm{k}\Delta \mathrm{x}) + 1\right) V_i^n = V_i^n$

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- 3rd step:
- $q = \frac{1}{\left(j\frac{v_p \Delta t}{\Delta x} sin(\mathrm{k}\Delta \mathrm{x}) + 1\right)}$
- Since $\left| j\frac{v_p \Delta t}{\Delta x} sin(\mathrm{k}\Delta \mathrm{x}) + 1 \right| > 1$ and hence q<1
- Hence unconditionally stable
- **Example 4:**
- **1-D CN-FDTD**
- $\frac{\partial V}{\partial t} = -v_p \frac{\partial V}{\partial x}$

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- $\Rightarrow \left.\frac{\partial V}{\partial t}\right|_i^{n+\frac{1}{2}} = -\frac{v_p}{2}\left(\left.\frac{\partial V}{\partial x}\right|_i^{n+1} + \left.\frac{\partial V}{\partial x}\right|_i^{n}\right)$ *(implicit FDTD)*
- $\frac{V_i^{n+1}-V_i^n}{\Delta t} = -\frac{v_p}{2}\left(\frac{V_{i+1}^{n+1}-V_{i-1}^{n+1}}{2\Delta x} + \frac{V_{i+1}^{n}-V_{i-1}^{n}}{2\Delta x}\right)$
- $\Rightarrow KV_{i+1}^{n+1} + V_i^{n+1} - KV_{i-1}^{n+1} = -KV_{i+1}^{n} + V_i^{n} + KV_{i-1}^{n}$
- where $K = \frac{v_p \Delta t}{4\Delta x}$
- First step:
- $\mathrm{V}_{\mathrm{i}\pm 1}^{\mathrm{n}} = \mathrm{V}_{\mathrm{i}}^{\mathrm{n}} \mathrm{e}^{\pm \mathrm{jk}\Delta \mathrm{x}}$
- $KV_i^{n+1}\mathrm{e}^{\mathrm{jk}\Delta\mathrm{x}} + V_i^{n+1} - KV_i^{n+1}\mathrm{e}^{-\mathrm{jk}\Delta\mathrm{x}} = -KV_i^{n}\mathrm{e}^{\mathrm{jk}\Delta\mathrm{x}} + V_i^{n} + KV_i^{n}\mathrm{e}^{-\mathrm{jk}\Delta\mathrm{x}}$

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- Second step:
- $\left(1+2jksin(\mathrm{k}\Delta \mathrm{x})\right)V_i^{n+1} = \left(1-2jksin(\mathrm{k}\Delta \mathrm{x})\right)V_i^{n}$
- Or, $\frac{V_i^{n+1}}{V_i^{n}} = \frac{\left(1-2jksin(\mathrm{k}\Delta \mathrm{x})\right)}{\left(1+2jksin(\mathrm{k}\Delta \mathrm{x})\right)}$
- Or, $|q| = 1$ for any values of k, $\Delta \mathrm{x}$ and $\Delta \mathrm{t}$
- Unconditionally stable
- Questions raised:
- Question: How ADE FDTD is different from NORMAL FDTD and ADI FDTD?
  - Ans: It is designed for frequency-dispersive medium

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- Question: ADE FDTD is faster than ADI FDTD? Is it also provides unconditional stability?
  - Ans: Depends on your implementation
- Question: ADE FDTD only valid for meta materials?
  - Ans: No for all dispersive materials as well as normal materials
- Question: In DRUDE model coupling frequency is called as plasma frequency, is there any reason behind it?
  - Answer: Plasma frequency is the resonance frequency for metals
  - Below plasma frequency electric permittivity is imaginary and metal behaves like good conductors

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- At plasma frequency, both real and imaginary part becomes significant, metals are very lossy
- Above plasma frequency metals are almost transparent and weakly absorbing
- Plasma frequency is of the order of few thousand THz

- Question: How ADI FDTD is faster than normal FDTD?
  - Ans: Because you can take time step size beyond CFL