FDTD: Advances

- How to get numerators of D(2x), D(2y)?
  - Ans: D(2x) is shown here you can find D(2y) similarly
  - $\frac{\partial^2}{\partial x^2} H_z^{n+1}\left(i+\frac{1}{2}, j+1/2\right)$
  - $= \frac{\partial}{\partial x}\left\{\frac{H_z^{n+1}(i+1, j+1/2) - H_z^{n+1}(i, j+1/2)}{\Delta x}\right\}$
  - $= \left\{\frac{H_z^{n+1}(i+3/2, j+1/2) - H_z^{n+1}(i+1/2, j+1/2)}{\Delta x^2}\right\} - \left\{\frac{H_z^{n+1}(i+1/2, j+1/2) - H_z^{n+1}(i-1/2, j+1/2)}{\Delta x^2}\right\}$
  - $= \left\{\frac{H_z^{n+1}(i+3/2, j+1/2) - 2H_z^{n+1}\left(i+\frac{1}{2}, j+\frac{1}{2}\right) + H_z^{n+1}(i-1/2, j+1/2)}{\Delta x^2}\right\}$

$$(1+j\omega\tau_0)\mathbf{D}=(\varepsilon_0\varepsilon_\infty\times \quad j\omega\tau_0+\varepsilon_0\varepsilon_{dc})\mathbf{E}$$

FDTD: Advances

- Frequency domain equation for Debye model

$$\mathbf{D}+j\omega\tau_0\mathbf{D}=\varepsilon_0\varepsilon_{dc}\mathbf{E}+j\omega\tau_0\varepsilon_0\varepsilon_\infty\mathbf{E}$$

- Take IFT

$$\bar{D}+\tau_0\frac{\partial\bar{D}}{\partial t}=\varepsilon_0\varepsilon_{dc}\bar{E}+\tau_0\varepsilon_0\varepsilon_\infty\frac{\partial\bar{E}}{\partial t}$$

- This differential equation can be discretized as usual to find an update equation for electric field

FDTD: An Introduction

$$\bar{E}^{n+1} = \left(\frac{\Delta t + 2\tau_0}{2\tau_0 \varepsilon_0 \varepsilon_\infty + \varepsilon_0 \varepsilon_{dc} \Delta t} \right) \bar{D}^{n+1}$$

$$+ \left(\frac{\Delta t - 2\tau_0}{2\tau_0 \varepsilon_0 \varepsilon_\infty + \varepsilon_0 \varepsilon_{dc} \Delta t} \right) \bar{D}^{n} + \left(\frac{2\tau_0 \varepsilon_0 \varepsilon_\infty - \varepsilon_0 \varepsilon_{dc} \Delta t}{2\tau_0 \varepsilon_0 \varepsilon_\infty + \varepsilon_0 \varepsilon_{dc} \Delta t} \right) \bar{E}^{n}$$

Update equation for magnetic field (no dispersion in magnetic permeability)

$$\frac{\partial \bar{H}}{\partial t} = -\frac{1}{\mu} \nabla \times \bar{E}$$

$$\Rightarrow \bar{H}^{n+1/2} = \bar{H}^{n-1/2} - \frac{\Delta t}{\mu} \left[\nabla \times \bar{E} \right]^{n}$$

FDTD: An Introduction

- Update equation for electric flux density (use the value of $\bar{H}^{n+1/2}$ from the previous update equation)

$$\frac{\partial \bar{D}}{\partial t} = \nabla \times \bar{H}$$

$$\Rightarrow \bar{D}^{n+1} = \bar{D}^{n} - \Delta t \left[\nabla \times \bar{H} \right]^{n+1/2}$$

$$\bar{E}^{n+1} = \left(\frac{\Delta t + 2\tau_0}{2\tau_0 \varepsilon_0 \varepsilon_\infty + \varepsilon_0 \varepsilon_{dc} \Delta t} \right) \bar{D}^{n+1} + \left(\frac{\Delta t - 2\tau_0}{2\tau_0 \varepsilon_0 \varepsilon_\infty + \varepsilon_0 \varepsilon_{dc} \Delta t} \right) \bar{D}^{n} + \left(\frac{2\tau_0 \varepsilon_0 \varepsilon_\infty - \varepsilon_0 \varepsilon_{dc} \Delta t}{2\tau_0 \varepsilon_0 \varepsilon_\infty + \varepsilon_0 \varepsilon_{dc} \Delta t} \right) \bar{E}^{n}$$

- We can use value of $\bar{D}^{n+1}$, $\bar{D}^{n}$ and $\bar{E}^{n}$ to find $\bar{E}^{n+1}$
- Formulation for multiple Debye poles
- For materials having susceptibilty with multiple (e.g. a total of M) poles, we can express the permittivity in the frequency domain as:

$$\varepsilon_c(\omega) = \varepsilon_0 \left[\varepsilon_\infty + \sum_{m=1}^{M} \frac{\Delta \varepsilon_m}{1 + j\omega \tau_{0m}} \right]; \Delta \varepsilon_m = \varepsilon_{dc}^{m} - \varepsilon_\infty^{m}$$

FDTD: An Introduction

- $\Delta\varepsilon_m$ Change in the real part of the relative permittivity in the vicinity of the mth Debye pole, specified by the relaxation time τ_{0m}
- The polarization current $\bar{J}_p(t)=\frac{\partial \bar{P}}{\partial t}$ for the mth pole would be

$$\varepsilon_{c,Debye}(\omega)=\varepsilon_0\left[\varepsilon_\infty+\frac{(\varepsilon_{m,dc}-\varepsilon_{m,\infty})}{(1+j\omega\tau_{0m})}\right]=\varepsilon_0\varepsilon_\infty+\frac{\varepsilon_0\Delta\varepsilon_m}{(1+j\omega\tau_{0m})}$$

$$\Rightarrow \chi_{em,Debye}(\omega)=\frac{\varepsilon_0\Delta\varepsilon_m}{(1+j\omega\tau_{0m})}$$

$$\Rightarrow j\omega\mathbf{J}_{pm}(\omega)=(\chi_{em,Debye}(\omega))\mathbf{E}(\omega)$$

$$\mathbf{J}_{pm}(\omega)=j\omega\varepsilon_0\frac{\Delta\varepsilon_m}{1+j\omega\tau_{0m}}\mathbf{E}(\omega)$$

FDTD: An Introduction

- Maxwell equation

$$\Delta\times\bar{H}=\varepsilon_0\varepsilon_\infty\frac{\partial\bar{E}}{\partial t}+\sum_{m=1}^{M}\bar{J}_{pm}$$

- Update equation for Polarization current

$$\mathbf{J}_{pm}(\omega)+j\omega\tau_{0m}\mathbf{J}_{pm}(\omega)=j\omega\varepsilon_0\Delta\varepsilon_m\mathbf{E}(\omega)$$

$$\mathbf{J}_{pm}(\omega)=j\omega\varepsilon_0\frac{\Delta\varepsilon_m}{1+j\omega\tau_{0m}}\mathbf{E}(\omega)$$

- IFT $\bar{J}_{pm}+\tau_{0m}\frac{d\bar{J}_{pm}}{dt}=\varepsilon_0\Delta\varepsilon_m\frac{d\bar{E}}{dt}$

$$\bar{J}_{pm}\Big|^{n+1}=\left(\frac{2\tau_{0m}-\Delta t}{2\tau_{0m}+\Delta t}\right)\bar{J}_{pm}\Big|^{n}+\left(\frac{2\varepsilon_0\Delta\varepsilon_m\Delta t}{2\tau_{0m}+\Delta t}\right)\left[\frac{\bar{E}\Big|^{n+1}-\bar{E}\Big|^{n}}{\Delta t}\right]$$

FDTD: An Introduction

- We can obtain

$$\bar{J}_{pm}\Big|^{n+1/2} = \frac{\bar{J}_{pm}\Big|^{n+1} + \bar{J}_{pm}\Big|^{n}}{2} = \left(\frac{2\tau_{0m}}{2\tau_{0m} + \Delta t}\right)\bar{J}_{pm}\Big|^{n} + \left(\frac{\varepsilon_0 \Delta\varepsilon_m \Delta t}{2\tau_{0m} + \Delta t}\right)\left[\frac{\bar{E}\Big|^{n+1} - \bar{E}\Big|^{n}}{\Delta t}\right]$$

- Therefore

$$\left[\nabla \times \bar{H}\right]^{n+1/2} = \varepsilon_0 \varepsilon_\infty \left[\frac{\bar{E}^{n+1} - \bar{E}^{n}}{\Delta t}\right] + \sum_{m=1}^{M} P_{pm}^{n+1/2}$$

- Putting the update equation of polarization current and rearranging

FDTD: An Introduction

$$\bar{J}_{pm}\Big|^{n+1} = \left(\frac{2\tau_{0m} - \Delta t}{2\tau_{0m} + \Delta t}\right)\bar{J}_{pm}\Big|^{n} + \left(\frac{2\varepsilon_0 \Delta\varepsilon_m \Delta t}{2\tau_{0m} + \Delta t}\right)\left[\frac{\bar{E}\big|^{n+1} - \bar{E}\big|^{n}}{\Delta t}\right]$$

$$\bar{E}^{n+1} = \bar{E}^{n} + \left(\frac{2\Delta t}{2\varepsilon_0\varepsilon_\infty + \sum_{m=1}^{M}\beta_m}\right)\left[\left[\nabla\times\bar{H}\right]^{n+1/2} - \frac{1}{2}\sum_{m=1}^{M}\left(1+k_m\right)\bar{J}_{pm}^{n}\right]$$

where

$$k_m = \left(\frac{2\tau_{0m} - \Delta t}{2\tau_{0m} + \Delta t}\right) \qquad \beta_m = \left(\frac{2\varepsilon_0 \Delta\varepsilon_m \Delta t}{2\tau_{0m} + \Delta t}\right)$$

FDTD: An Introduction

$$\bar{E}^{n+1} = \bar{E}^{n} + \left(\frac{2\Delta t}{2\varepsilon_0 \varepsilon_\infty + \sum_{m=1}^{M} \beta_m} \right) \left[\left[\nabla \times \bar{H} \right]^{n+1/2} - \frac{1}{2} \sum_{m=1}^{M} (1+k_m) \bar{J}_{pm}^{\,n} \right]$$

$$\bar{E}^n, \bar{P}^n, \bar{H}^{n+1/2} \rightarrow \bar{E}^{n+1} \rightarrow \bar{P}^{n+1}$$

$$\bar{J}_{pm}\Big|^{n+1} = \left(\frac{2\tau_{0m} - \Delta t}{2\tau_{0m} + \Delta t} \right) \bar{J}_{pm}\Big|^{n} + \left(\frac{2\varepsilon_0 \Delta\varepsilon_m \Delta t}{2\tau_{0m} + \Delta t} \right) \left[\frac{\bar{E}\big|^{n+1} - \bar{E}\big|^{n}}{\Delta t} \right]$$

$$\bar{J}_p(t) = \frac{\partial \bar{P}}{\partial t}$$

$$\bar{H}^{n+3/2}$$

$$\frac{\partial \bar{H}}{\partial t} = -\frac{1}{\mu} \nabla \times \bar{E}$$

$$\Rightarrow \bar{H}^{n+1/2} = \bar{H}^{n-1/2} - \frac{\Delta t}{\mu} \left[\nabla \times \bar{E} \right]^{n}$$

After which use Faraday's law to find and proceed with updating

$$\left[\nabla \times \bar{E} \right]^{n+1} = -\mu \left[\frac{\bar{H}^{n+3/2} - \bar{H}^{n+1/2}}{\Delta t} \right]$$

FDTD: An Introduction

- Lorentz materials

$$\varepsilon_{Lorentz}(\omega) = \varepsilon_0\left(\varepsilon_\infty + \chi_{e,Lorentz}(\omega)\right)$$

- where

$$\chi_{e,Lorentz}(\omega) = \frac{P_i(\omega)}{\varepsilon_0 E_i(\omega)} = \frac{\chi_L}{-\omega^2 + j\Gamma_L\omega + \omega_0^2}$$

- In terms of $\Delta\varepsilon_m = \varepsilon_{dc}^m - \varepsilon_\infty^m$

$$\varepsilon_{Lorentz}(\omega) = \varepsilon_0\left(\varepsilon_\infty + \frac{\Delta\varepsilon\omega_0^2}{-\omega^2 + 2j\omega\delta + \omega_0^2}\right)$$

FDTD: Advances

- For Lorentz material with M pole-pairs,

$$\varepsilon_{Lorentz}(\omega) = \varepsilon_0 \left(\varepsilon_\infty + \sum_{m=1}^{M} \frac{\Delta\varepsilon_m \omega_{0m}^2}{\omega_{0m}^2 + 2j\omega\delta_m - \omega^2} \right)$$

- Therefore

$$\mathbf{J}_{pm}(\omega) = j\omega\mathbf{P} = \varepsilon_0 \Delta\varepsilon_m \omega_{0m}^2 \frac{j\omega}{\omega_{0m}^2 + 2j\omega\delta_m - \omega^2}\mathbf{E}$$

FDTD: Advances

- Rearranging

$$\omega_{0m}^2 \mathbf{J}_{pm} + 2j\omega\delta_m \mathbf{J}_{pm} - \omega^2 \mathbf{J}_{pm} = \varepsilon_0 \Delta\varepsilon_m \omega_{0m}^2 j\omega \mathbf{E}$$

- Taking IFT

$$\omega_{0m}^2 \bar{J}_{pm} + 2\delta_m \frac{\partial \bar{J}_{pm}}{\partial t} + \frac{\partial^2 \bar{J}_{pm}}{\partial t^2} = \varepsilon_0 \Delta\varepsilon_m \omega_{0m}^2 \frac{\partial \bar{E}}{\partial t}$$

FDTD: Advances

- Discretize using second-order centered differences for first and second derivaties

$$\omega_{0m}^2 \bar{J}_{pm} + 2\delta_m \frac{\bar{J}_{pm}^{n+1} - \bar{J}_{pm}^{n-1}}{2\Delta t} + \frac{\bar{J}_{pm}^{n+1} - 2\bar{J}_{pm}^{n} + \bar{J}_{pm}^{n-1}}{\Delta t^2}$$

$$= \varepsilon_0 \Delta\varepsilon_m \omega_{0m}^2 \frac{\bar{E}_m^{n+1} - \bar{E}_m^{n-1}}{2\Delta t}$$

FDTD: Advances

- Hence

$$\overline{J}_{pm}^{\,n+1/2} = \frac{\overline{J}_{pm}^{\,n+1} + \overline{J}_{pm}^{\,n}}{2} = \frac{1}{2}\left[\left(1 + A_{1m}\right)\overline{J}_{pm}^{\,n} + A_{2m}\overline{J}_{pm}^{\,n-1} + A_{3m}\left(\frac{\overline{E}_{m}^{\,n+1} - \overline{E}_{m}^{\,n-1}}{2\Delta t}\right)\right]$$

- Now we can use this equation in

$$A_{1m} = \frac{2-\omega_{0m}^2}{1+\delta_m \Delta t}; A_{2m} = \frac{\delta_m \Delta t - 1}{1+\delta_m \Delta t} A_{3m} = \frac{\varepsilon_0 \Delta\varepsilon_m \omega_{0m}^2 \Delta t^2}{1+\delta_m \Delta t}$$

FDTD: Advances

- Update equation for polarization current

$$\overline{J}_{pm}^{n+1} = A_{1m}\overline{J}_{pm}^{n} + A_{2m}\overline{J}_{pm}^{n-1} + A_{3m}\left(\frac{\overline{E}_m^{n+1} - \overline{E}_m^{n-1}}{2\Delta t}\right)$$

- Now we can use this equation in the following equation

$$\Delta \times \overline{H} = \varepsilon_0 \varepsilon_\infty \frac{\partial \overline{E}}{\partial t} + \sum_{m=1}^{M} \overline{J}_{pm}$$

FDTD: Advances

- The update equation for electric field as

$$\bar{E}^{n+1} = C_1 \bar{E}^n + C_2 \bar{E}^{n-1} + C_3 \left[\left(\Delta \times \bar{H} \right)^{n+1/2} - \frac{1}{2} \sum_{m=1}^{M} \left\{ \left(1 + A_{1m} \right) \bar{J}_{pm}^{n} + A_{2m} \bar{J}_{pm}^{n-1} \right\} \right]$$

- where the constants are

$$C_1 = \frac{2\varepsilon_0 \varepsilon_m}{2\varepsilon_0 \varepsilon_\infty + \frac{1}{2}\sum A_{3m}}; C_2 = \frac{\frac{1}{2}\sum A_{3m}}{2\varepsilon_0 \varepsilon_\infty + \frac{1}{2}\sum A_{3m}} C_3 = \frac{2\Delta t}{2\varepsilon_0 \varepsilon_\infty + \frac{1}{2}\sum A_{3m}}$$

FDTD: Advances

- Finally

$$\bar{E}^{n}, \bar{E}^{n-1}, \bar{P}^{n}, \bar{P}^{n-1}, \bar{H}^{n+1/2} \rightarrow \bar{E}^{n+1} \rightarrow \bar{P}^{n+1}$$

- Drude materials

$$\chi_{e,Drude}(\omega) = \frac{\chi_L}{-\omega^2 + j\Gamma_D \omega} = -\frac{\omega_r^2}{\omega^2 - j\frac{1}{\tau_r}\omega}$$

- For multiple pole

$$\varepsilon_{Drude}(\omega) = \varepsilon_0 \varepsilon_\infty - \varepsilon_0 \sum_{m=1}^{M} \frac{\omega_{rm}^2}{\omega^2 - \frac{j\omega}{\tau_{rm}}}$$

FDTD: Advances

$$\varepsilon_{Drude}(\omega) = \varepsilon_0 \varepsilon_\infty - \varepsilon_0 \sum_{m=1}^{M} \frac{\omega_{rm}^2}{\omega^2 - \frac{j\omega}{\tau_{rm}}}$$

- Polarization current

$$\mathbf{J}_{pm}(\omega) = j\omega \mathbf{P} = -j\omega\varepsilon_0 \frac{\omega_{rm}^2}{\omega^2 - \frac{j\omega}{\tau_{rm}}} \mathbf{E}$$

$$\omega^2 \mathbf{J}_{pm} - \frac{j\omega}{\tau_{rm}} \mathbf{J}_{pm} = -j\omega\varepsilon_0 \omega_{rm}^2 \mathbf{E}$$

$$-j\omega \mathbf{J}_{pm} - \frac{1}{\tau_{rm}} \mathbf{J}_{pm} = -\varepsilon_0 \omega_{rm}^2 \mathbf{E}$$

$$\frac{\partial \bar{J}_{pm}}{\partial t} + \frac{1}{\tau_{rm}} J_{pm} = \varepsilon_0 \omega_{rm}^2 \bar{E}$$

FDTD: Advances

- Discretize

$$\frac{\bar{J}_{pm}^{n+1} - \bar{J}_{pm}^{n}}{\Delta t} + \frac{1}{\tau_{rm}} \frac{\bar{J}_{pm}^{n+1} + \bar{J}_{pm}^{n}}{2} = \varepsilon_0 \omega_{rm}^2 \frac{\bar{E}^{n+1} + \bar{E}^{n}}{2}$$

- which yields $\bar{J}_{pm}^{n+1} = \alpha_m \bar{J}_{pm}^{n} + \beta_m \left(\frac{\bar{E}^{n+1} + \bar{E}^{n}}{2} \right);$

$$\alpha_m = \left(\frac{2 - \frac{\Delta t}{\tau_{rm}}}{2 + \frac{\Delta t}{\tau_{rm}}} \right), \beta_m = \left(\frac{\varepsilon_0 \omega_{rm}^2 \Delta t}{2 + \frac{\Delta t}{\tau_{rm}}} \right)$$

FDTD: Advances

- Now

$$\bar{J}_{pm}^{n+1/2}=\frac{\bar{J}_{pm}^{n+1}+\bar{J}_{pm}^{n}}{2}=\frac{1}{2}\left[\left(1+\alpha_m\right)\bar{J}_{pm}^{n}+\beta_m\left(\frac{\bar{E}^{n+1}+\bar{E}^{n}}{2}\right)\right]$$

- Finally

$$\bar{E}^{n+1}=\left(\frac{2\varepsilon_0\varepsilon_\infty-\Delta t\sum\beta_m}{2\varepsilon_0\varepsilon_\infty+\Delta t\sum\beta_m}\right)\bar{E}^{n}+\left(\frac{2\Delta t}{2\varepsilon_0\varepsilon_\infty+\sum\beta_m}\right)\left[\left(\Delta\times\bar{H}\right)^{n+1/2}-\frac{1}{2}\sum_{m=1}^{M}\left\{\left(1+\alpha_m\right)\bar{J}_{pm}^{n}\right\}\right]$$

FDTD: An Introduction

References

1. T. Rylander, P. Ingelstrom and A. Bondeson, *Computational Electromagnetics*, Springer, 2013
2. J. M. Jin, *Theory and Computation of Electromagnetic Fields*, IEEE Press, 2010
3. U. S. Inan and R. A. Marshall, *Numerical Electromagnetics The FDTD Method*, Cambridge University Press, 2011
4. S. D. Gedney, *Introduction to the Finite-Difference Time-Domain Method for Electromagnetics*, Morgan & Claypool, 2011