Finite Element Method

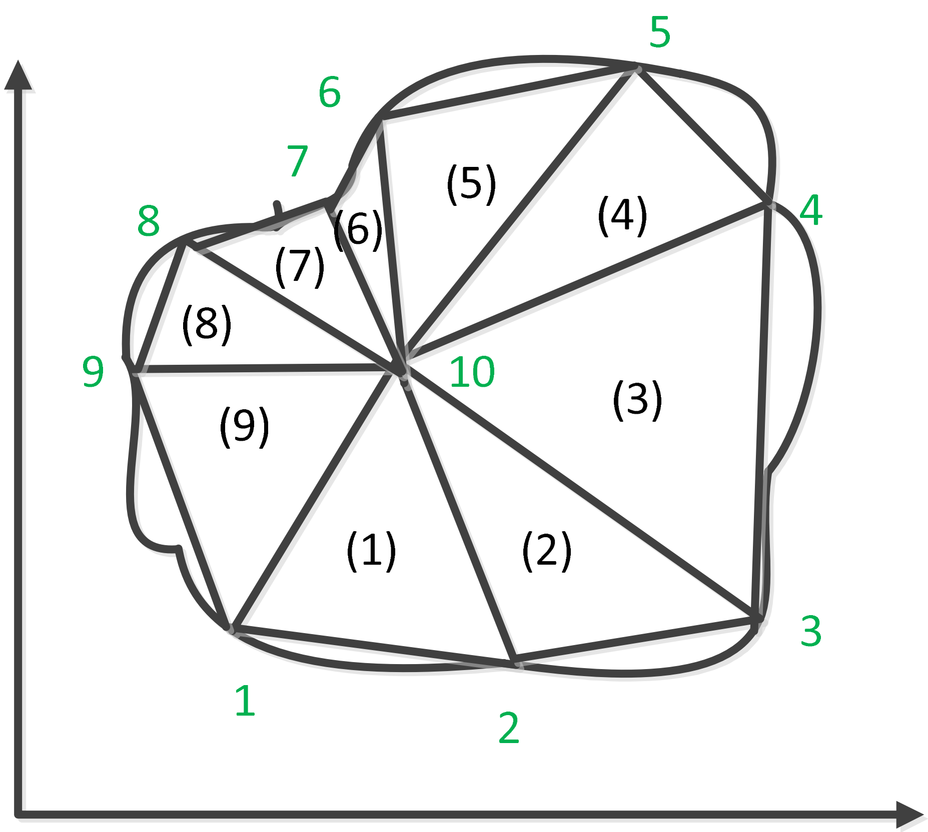
Prof. Rakhesh Singh Kshetrimayum

Introduction

- FEM involves basically four steps
- *Finite element discretization:*
  - Discretizing the solution region into a finite number of subregions or elements
- *Element governing equation:*
  - Deriving governing equations for a typical element
- *Assembling all elements:*
  - assembling of all elements in the solution region
- *Solving the resulting equation:*
  - Solving the system of equations obtained

Introduction

- *Finite element discretization:* i node number and (j) element number



Introduction

- We find an approximation for the potential Ve within an element e
  - then interrelate the potential distribution in various elements
    - Such that the potential is continuous across the inter-element boundaries
- The approximate solution for the whole region is

$$V(x,y) \cong \sum_{e=1}^{N} V_e(x,y)$$

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- where N is the number of triangular elements into which the solution region is divided
- The most common form of approximation of Ve within an element is polynomial approximation, namely
  - Triangular element

$$V_e(x,y) = a + bx + cy$$

  - Quadrilateral element

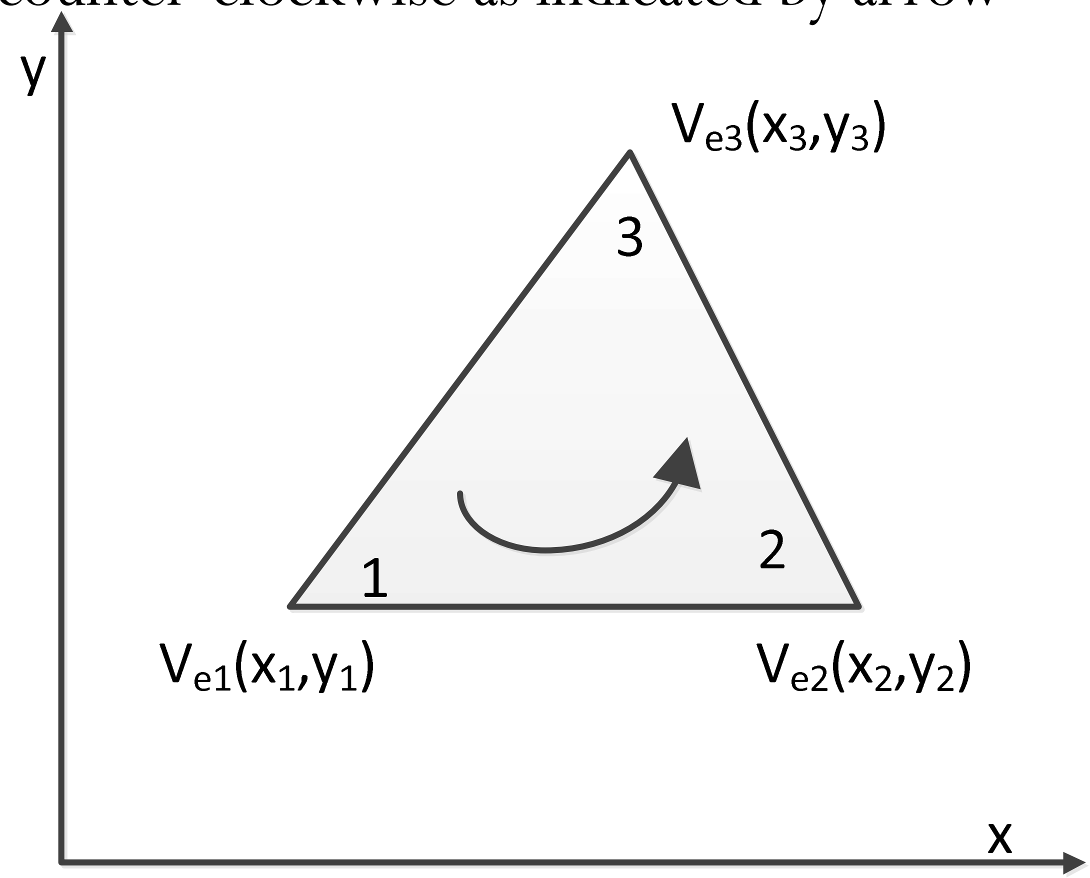
$$V_e(x,y) = a + bx + cy + dxy$$

Introduction

- The constants a, b, c and d are to be determined
- The potential Ve in general is nonzero within the element e but zero outside e
- *Element governing equation:*
- Consider a triangular element shown in Fig.
- The potential V_{e1}, V_{e2} and V_{e3} at nodes 1,2 and 3 respectively are obtained as

Introduction

- Typical triangular element: local numbering 1-2-3 must proceed counter-clockwise as indicated by arrow

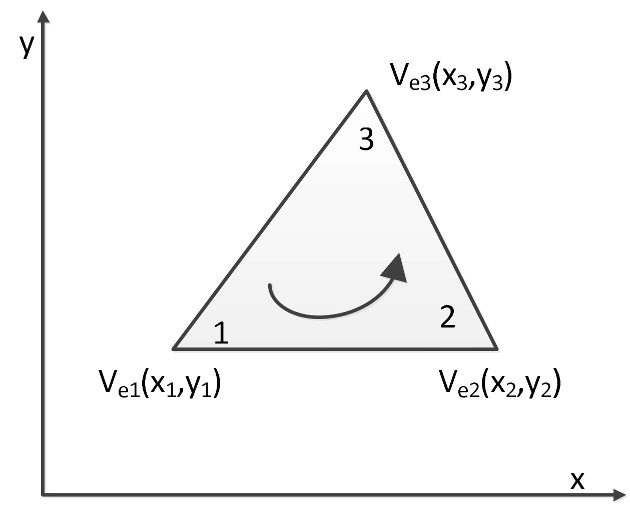


Introduction

$$\begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\because V_e(x,y) = a + bx + cy$$

$$\Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix}$$



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$$\Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} x_2 y_3 - x_3 y_2 & x_3 y_1 - x_1 y_3 & x_1 y_2 - x_2 y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix}$$

- where A is the area of element e, i.e.,

$$A = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{1}{2} \left\{ \left(x_1 y_2 - x_2 y_1 \right) + \left(x_3 y_1 - x_1 y_3 \right) + \left(x_2 y_3 - x_3 y_2 \right) \right\}$$

$$= \frac{1}{2} \left\{ \left(x_2 - x_1 \right)\left(y_3 - y_1 \right) - \left(x_3 - x_1 \right)\left(y_2 - y_1 \right) \right\}$$

Introduction

- The value A is positive if the nodes are numbered counter-clockwise (starting from any node) as shown by arrow in the Fig.
- We may express the above equation also as

$$V_e = \sum_{i=1}^{3} \alpha_i(x,y) V_{ei}$$

- where

$$\alpha_1 = \frac{1}{2A}\left[\left(x_2 y_3 - x_3 y_2\right) + \left(y_2 - y_3\right)x + \left(x_3 - x_2\right)y\right]$$

$$\alpha_2 = \frac{1}{2A}\left[\left(x_3 y_1 - x_1 y_3\right) + \left(y_3 - y_1\right)x + \left(x_1 - x_3\right)y\right]$$

$$\alpha_3 = \frac{1}{2A}\left[\left(x_1 y_2 - x_2 y_1\right) + \left(y_1 - y_2\right)x + \left(x_2 - x_1\right)y\right]$$

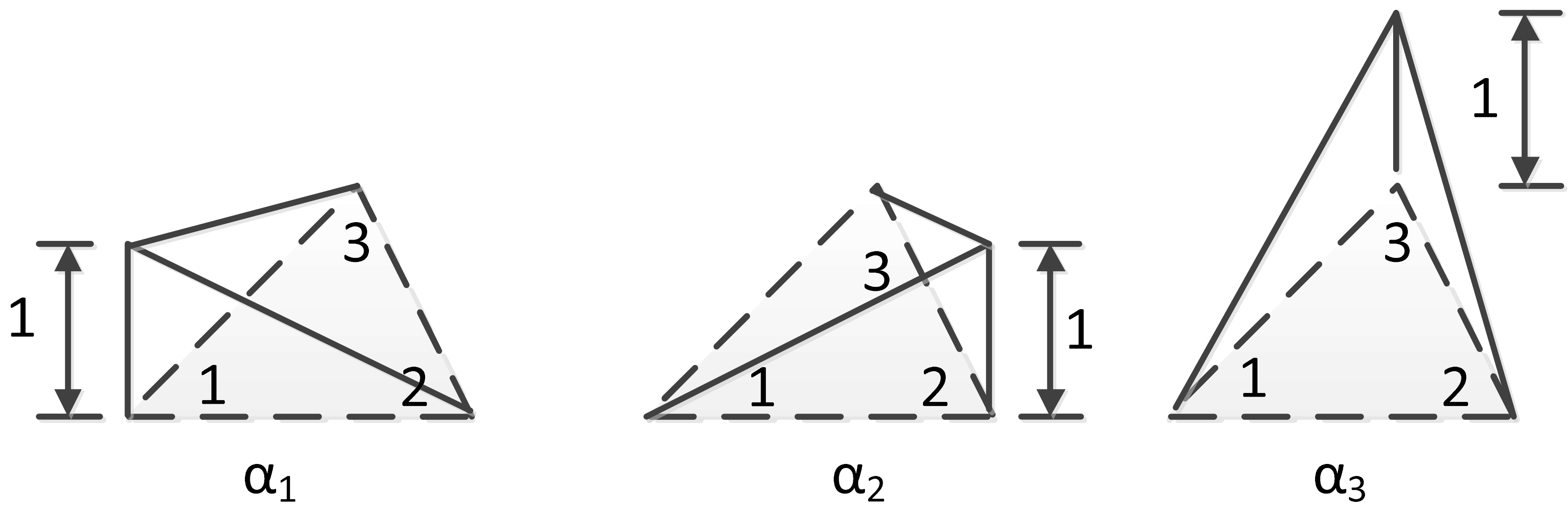
Introduction

- The potential at any point (x,y) within the element provided that the potentials at the vertices are known
  - This is unlike FDTD when the potential is known at the grid points only
- Here the α_i are the interpolation functions
  - They are also called as *element shape functions*
  - And they have the following properties:

$$\alpha_i = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Introduction

- Shape functions for α_1, α_2 and α_3 for a triangular element



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- Also $\sum_{i=1}^{3} \alpha_i(x, y) = 1$

- The shape functions α_1, α_2 and α_3 are illustrated in Fig.
- The functional corresponding to Laplace's equation $\nabla^2 V = 0$ is given by

$$W_e = \frac{1}{2}\int \varepsilon \left|\vec{E}_e\right|^2 ds = \frac{1}{2}\int \varepsilon \left|\nabla V_e\right|^2 ds$$

Physically the functional We is the energy per unit length associated with the element e

Introduction

$$\nabla V_e = \sum_{i=1}^{3} V_{ei} \nabla \alpha_i \qquad \because V_e = \sum_{i=1}^{3} \alpha_i(x,y) V_{ei}$$

- Therefore, $$W_e = \frac{1}{2}\int \varepsilon \left|\vec{E}_e\right|^2 ds = \frac{1}{2}\int \varepsilon \left|\nabla V_e\right|^2 ds$$

$$\therefore W_e = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} \varepsilon V_{ei}\left[\int \nabla \alpha_i \bullet \nabla \alpha_j ds\right] V_{ej}$$

- If we define the term in brackets as

$$C_{ij}^{(e)} = \int \nabla \alpha_i \bullet \nabla \alpha_j ds$$

- Therefore $$W_e = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3} \varepsilon V_{ei} C_{ij}^{(e)} V_{ej}$$

Introduction

- In matrix form,

$$W_e = \frac{1}{2}\varepsilon\left[V_e\right]^t\left[C^{(e)}\right]\left[V_e\right]$$

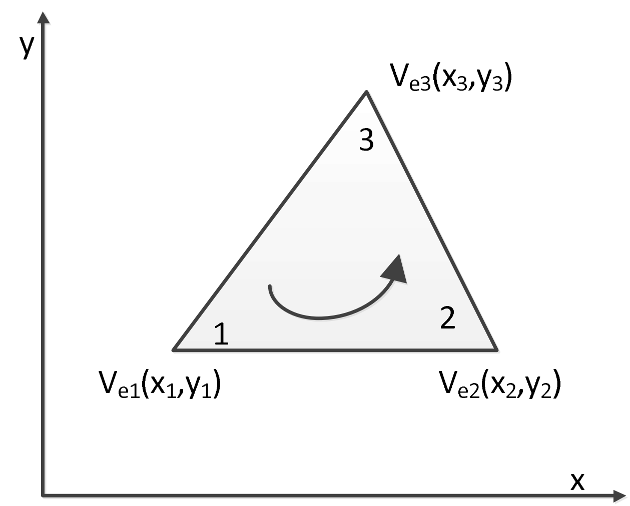
- where the subscript t denotes the transpose and

$$\left[V_e\right] = \begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix}$$

Introduction

Element coefficient matrix or stiffness matrix for element e

$$\left[C^{(e)}\right]=\begin{bmatrix} C_{11}^{(e)} & C_{12}^{(e)} & C_{13}^{(e)} \\ C_{21}^{(e)} & C_{22}^{(e)} & C_{23}^{(e)} \\ C_{31}^{(e)} & C_{32}^{(e)} & C_{33}^{(e)} \end{bmatrix}$$



Introduction

- The element $C_{ij}^{(e)}$ of the coefficient matrix may be regarded as the coupling between nodes i and j (for instance)

$$C_{12}^{(e)} = \int \nabla \alpha_1 \bullet \nabla \alpha_2 ds$$

$$= \frac{1}{4A^2}\left[\left(y_2 - y_3\right)\left(y_3 - y_1\right) + \left(x_3 - x_2\right)\left(x_1 - x_3\right)\right]\int ds$$

$$= \frac{1}{4A}\left[-\left(y_3 - y_2\right)\left(y_3 - y_1\right) - \left(x_3 - x_2\right)\left(x_3 - x_1\right)\right]$$

$$= C_{21}^{(e)}$$

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- Similarly,

$$C_{13}^{(e)} = C_{31}^{(e)}$$

$$= \frac{1}{4A}\left[-\left(y_2 - y_3\right)\left(y_2 - y_1\right) - \left(x_2 - x_3\right)\left(x_2 - x_1\right)\right]$$

$$C_{23}^{(e)} = C_{32}^{(e)}$$

$$= \frac{1}{4A}\left[-\left(y_1 - y_3\right)\left(y_1 - y_2\right) - \left(x_1 - x_3\right)\left(x_1 - x_2\right)\right]$$

Introduction

- How to remember? (off-diagonal elements)
- For $C_{12}^{(e)}$, find remaining vertex and it is 3
- $-(y_3-y_2)(y_3-y_1)$ plus $-(x_3-x_2)(x_3-x_1)$ multiplied by $\frac{1}{4A}$
- Similarly for
- For $C_{13}^{(e)}$, find remaining vertex and it is 2
- $-(y_2-y_3)(y_2-y_1)$ plus $-(x_2-x_3)(x_2-x_1)$ multiplied by $\frac{1}{4A}$
- For $C_{23}^{(e)}$, find remaining vertex and it is 1
- $-(y_1-y_2)(y_1-y_3)$ plus $-(x_1-x_3)(x_1-x_1)$ multiplied by $\frac{1}{4A}$

Introduction

How to remember? (diagonal elements)

- Find the remaining vertices
- For $C_{11}^{(e)}$, $(y_2 - y_3)^2$ plus $(x_2 - x_3)^2$ multiplied by $\frac{1}{4A}$

$$C_{11}^{(e)} = \frac{1}{4A}\left[(y_2 - y_3)^2 + (x_2 - x_3)^2\right]$$

- For $C_{22}^{(e)}$, $(y_3 - y_1)^2$ plus $(x_3 - x_1)^2$ multiplied by $\frac{1}{4A}$

$$C_{22}^{(e)} = \frac{1}{4A}\left[(y_3 - y_1)^2 + (x_3 - x_1)^2\right]$$

- For $C_{33}^{(e)}$, $(y_1 - y_2)^2$ plus $(x_1 - x_2)^2$ multiplied by $\frac{1}{4A}$

$$C_{33}^{(e)} = \frac{1}{4A}\left[(y_1 - y_2)^2 + (x_1 - x_2)^2\right]$$

Introduction

- *Assembling all elements:*
- The energy associated with the assemblage of elements is

$$W = \sum_{e=1}^{N} W_e = \frac{1}{2}\varepsilon \left[V\right]^t \left[C\right]\left[V\right]$$

- where

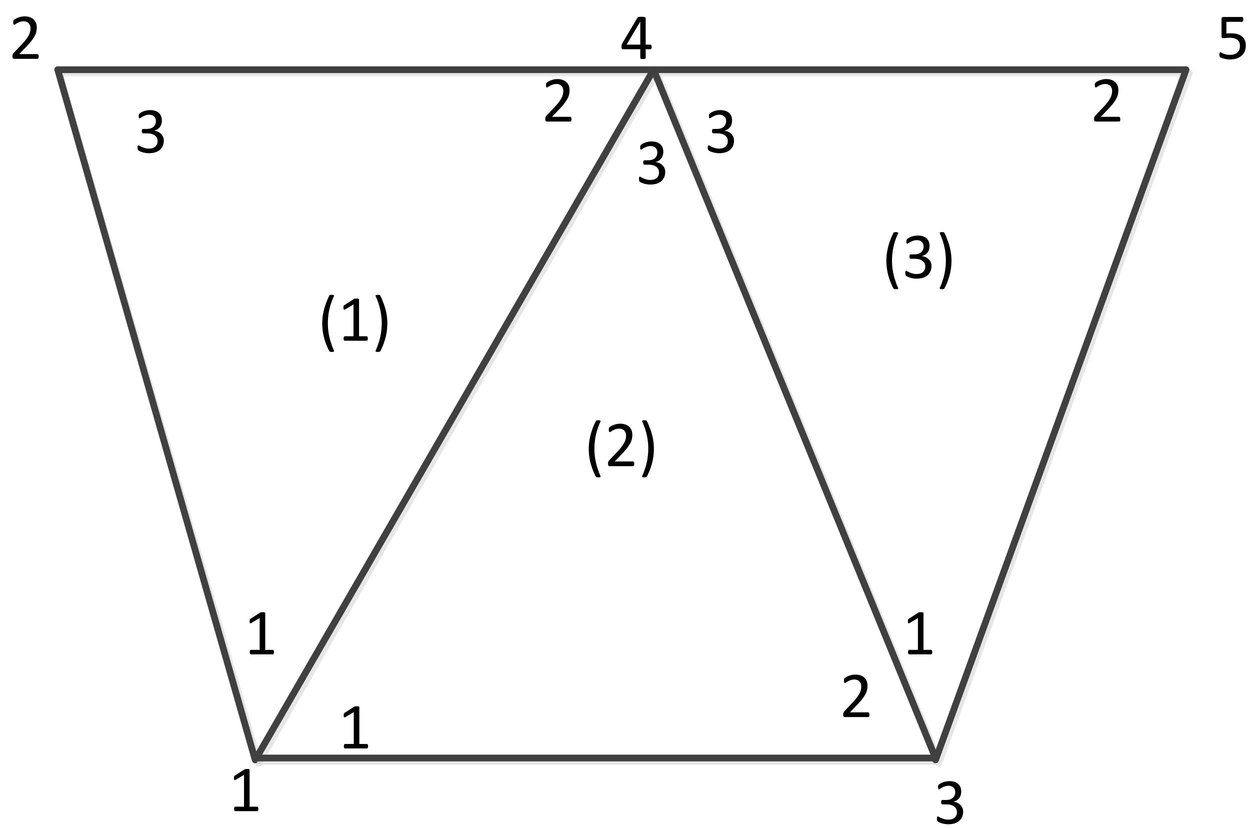
$$\left[V\right] = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \\ \\ \end{bmatrix}$$

Introduction

  - n is the number of nodes
  - N is the number of elements
- [C] is the overall or global coefficients matrix
  - which is the assemblage of individual element coefficient matrix
- The process by which individual element coefficient matrices are assembled
  - to obtain the global coefficient matrix is illustrated with an example

Introduction

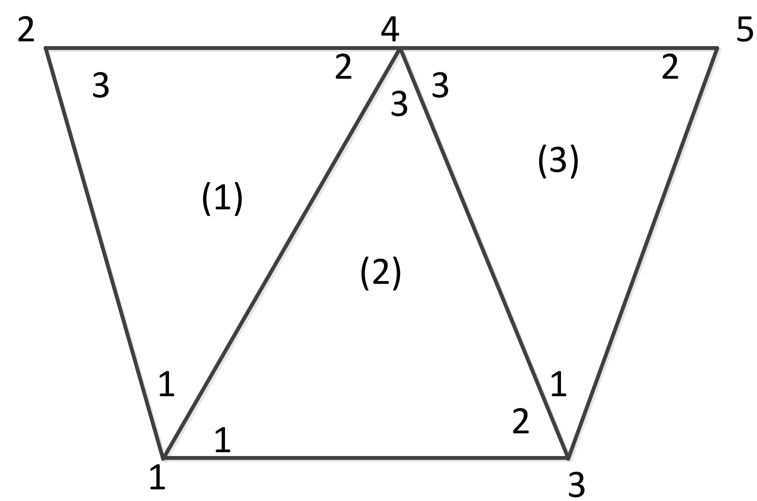
- Example: A solution region is discretized into 3 triangular elements



Introduction

- Global numbering (number exterior to figure) : 1,2,3,4,5
- Local numbering (always counter-clockwise): 1,2,3
- For five nodes, n=5 and
  - N=3 (three elements),
    - the global coefficient matrix [C] is

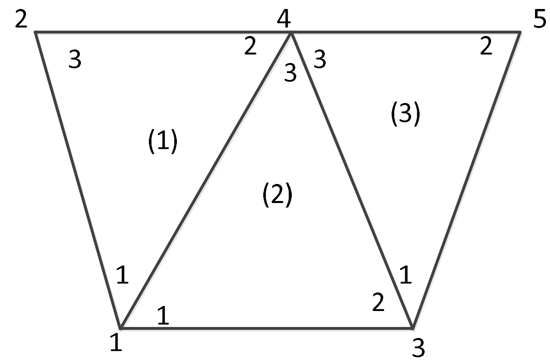
$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} \end{bmatrix}$$



Introduction

- Properties of global coefficient matrix [C] :
  - It is symmetric, i.e., Cij=Cji
  - Cij=0 if no coupling exists, making the matrix sparse
  - It is singular
- *How to find the elements of the global coefficient matrix [C]?*
- For example,
- a) Element (1) and (2) have global node 1 in common, hence

$$C_{11} = C_{11}^{(1)} + C_{11}^{(2)}$$



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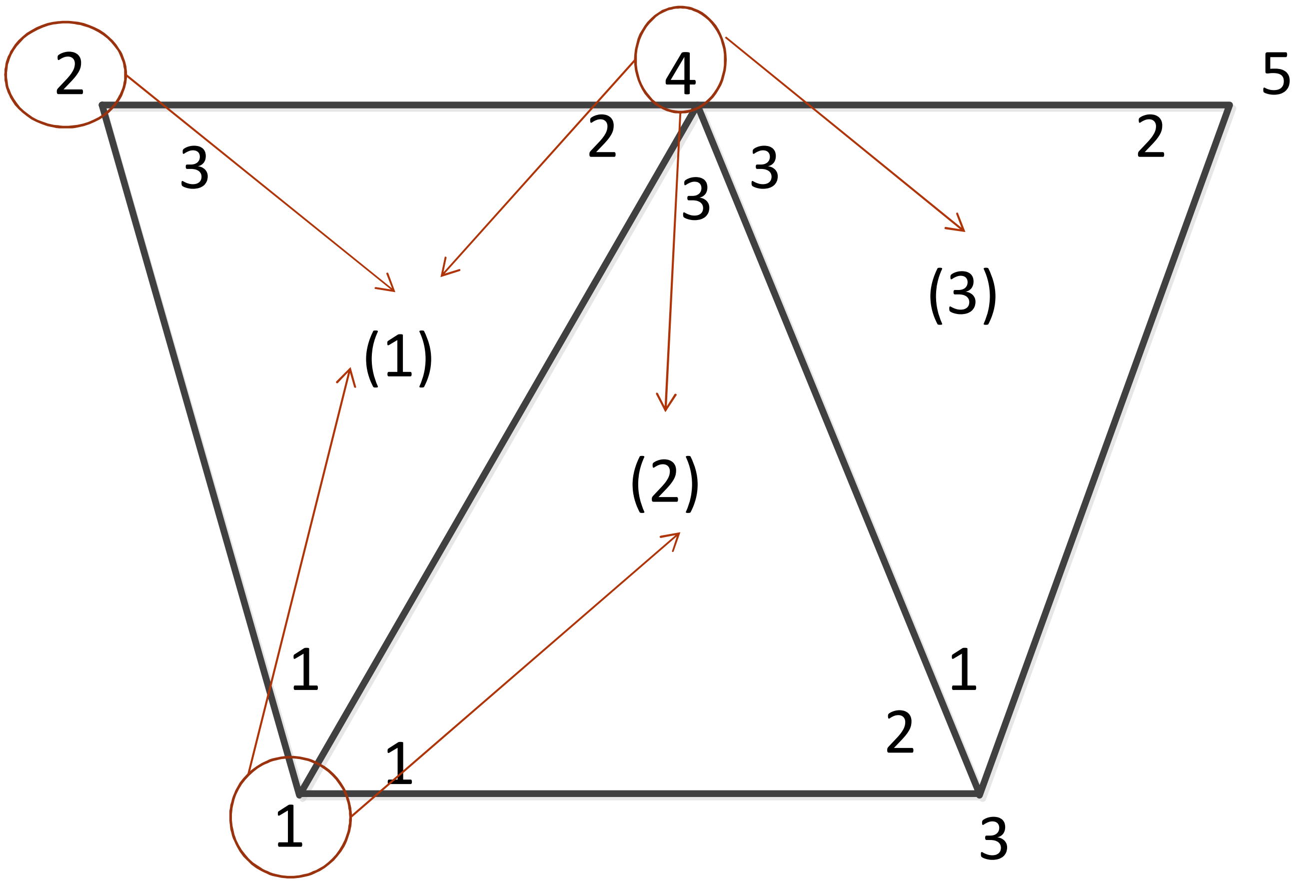

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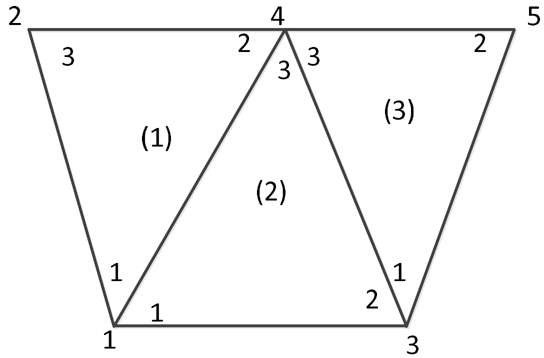
- b) Node 2 belongs to element 1 only, hence



$$C_{22} = C_{33}^{(1)}$$

- c) Node 4 belongs to elements 1,2 and 3 hence

$$C_{44} = C_{22}^{(1)} + C_{33}^{(2)} + C_{33}^{(3)}$$

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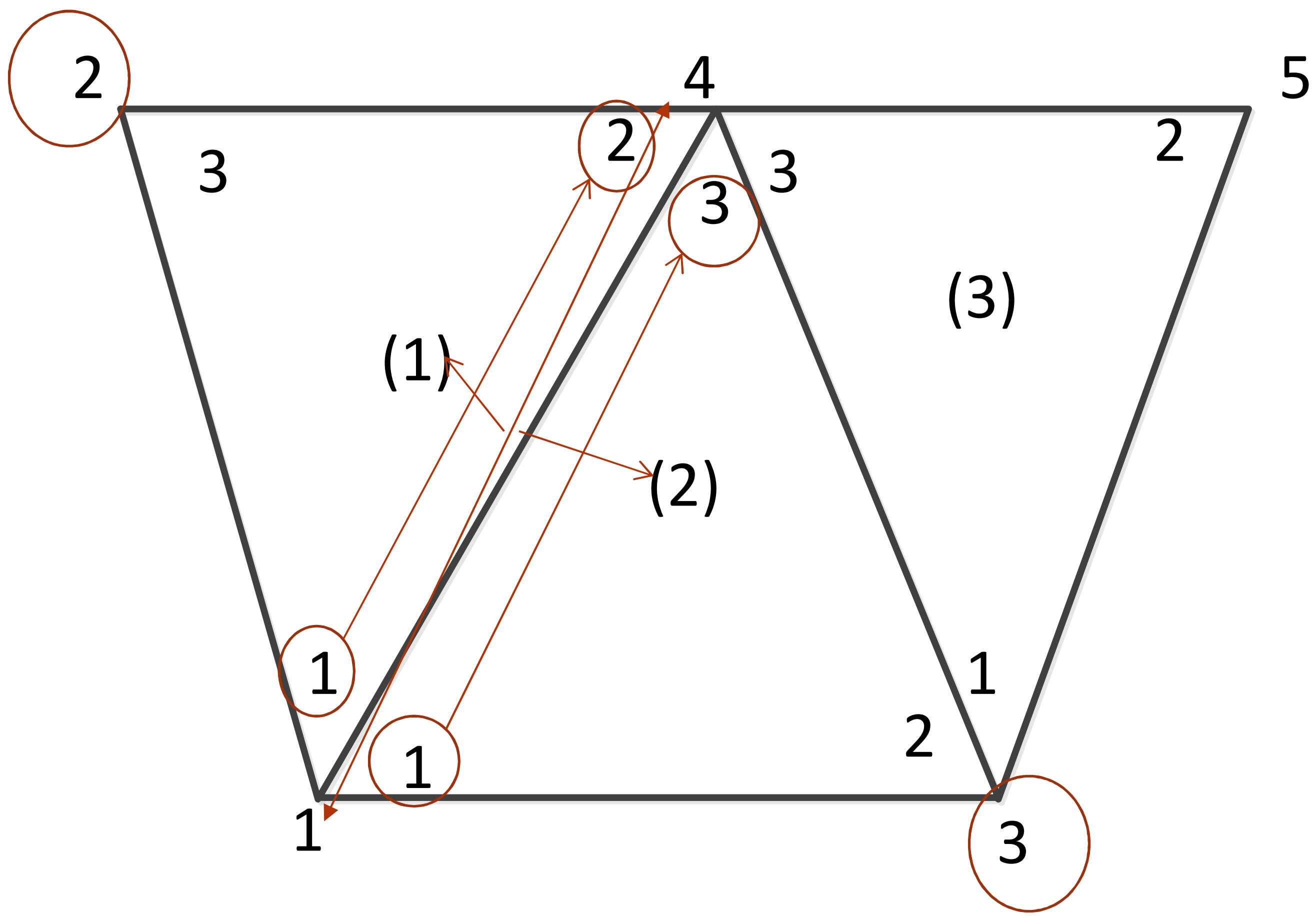

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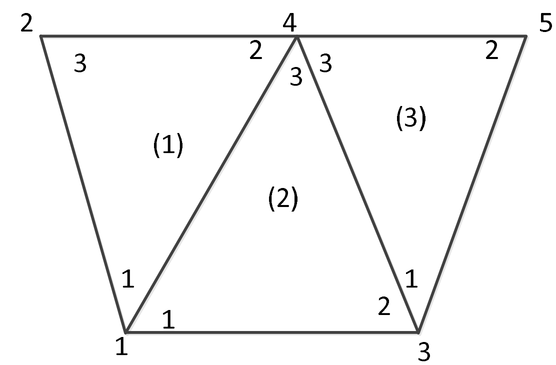

- d) Node 1 and 4 belong simultaneously to element 1 and 2 hence

$$C_{14} = C_{41} = C_{12}^{(1)} + C_{13}^{(2)}$$

- e) Since there is no coupling between nodes 2 and 3, hence

$$C_{23} = C_{32} = 0$$

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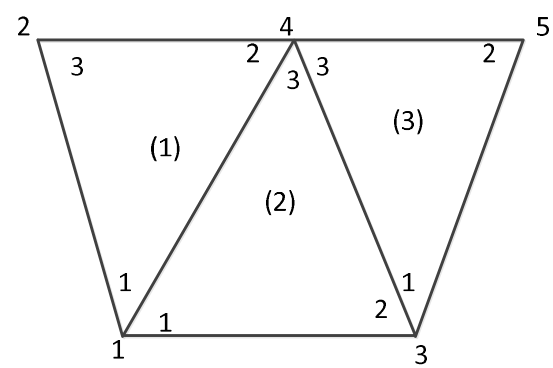
- Therefore, the global coefficient matrix is given by

$$[C]=\begin{bmatrix} C_{11}^{(1)}+C_{11}^{(2)} & C_{13}^{(1)} & C_{12}^{(2)} & C_{12}^{(1)}+C_{13}^{(2)} & 0 \\ C_{31}^{(1)} & C_{33}^{(1)} & 0 & C_{32}^{(1)} & 0 \\ C_{21}^{(2)} & 0 & C_{22}^{(2)}+C_{11}^{(23)} & C_{23}^{(2)}+C_{13}^{(3)} & C_{12}^{(3)} \\ C_{21}^{(1)}+C_{31}^{(2)} & C_{23}^{(1)} & C_{32}^{(2)}+C_{31}^{(3)} & C_{22}^{(1)}+C_{33}^{(2)}+C_{33}^{(3)} & C_{32}^{(3)} \\ 0 & 0 & C_{21}^{(3)} & C_{23}^{(3)} & C_{22}^{(3)} \end{bmatrix}$$



Introduction

- *Solving the resulting equation:*
- Note that Laplace's equation is satisfied when the total energy in the solution region is minimum
- Thus we require the partial derivatives of W w.r.t. each nodal value of potential be zero, i.e.,

$$\frac{\partial W}{\partial V_1} = \frac{\partial W}{\partial V_2} = \cdots = \frac{\partial W}{\partial V_n} = 0$$

$$\Rightarrow \frac{\partial W}{\partial V_k} = 0, k = 1, 2, \cdots, n$$

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- Consider the previous example of 5 nodes
- For example, $\frac{\partial W}{\partial V_1} = 0$ implies that

$$W = \frac{1}{2}\varepsilon \begin{bmatrix} V_1 & V_2 & V_3 & V_4 & V_5 \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{bmatrix}$$

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$$W=\frac{1}{2}\varepsilon\begin{bmatrix}V_1 & V_2 & V_3 & V_4 & V_5\end{bmatrix}\begin{bmatrix}C_{11}V_1+C_{12}V_2+C_{13}V_3+C_{14}V_4+C_{15}V_5\\C_{21}V_1+C_{22}V_2+C_{23}V_3+C_{24}V_4+C_{25}V_5\\C_{31}V_1+C_{32}V_2+C_{33}V_3+C_{34}V_4+C_{35}V_5\\C_{41}V_1+C_{42}V_2+C_{43}V_3+C_{44}V_4+C_{45}V_5\\C_{51}V_1+C_{52}V_2+C_{53}V_3+C_{54}V_4+C_{55}V_5\end{bmatrix}$$

- V_1 dependent terms are shown in green color

$$W=\frac{1}{2}\varepsilon\begin{bmatrix}V_1C_{11}V_1+V_1C_{12}V_2+V_1C_{13}V_3+V_1C_{14}V_4+V_1C_{15}V_5+\\V_2C_{21}V_1+V_2C_{22}V_2+V_2C_{23}V_3+V_2C_{24}V_4+V_2C_{25}V_5+\\V_3C_{31}V_1+V_3C_{32}V_2+V_3C_{33}V_3+V_3C_{34}V_4+V_3C_{35}V_5+\\V_4C_{41}V_1+V_4C_{42}V_2+V_4C_{43}V_3+V_4C_{44}V_4+V_4C_{45}V_5+\\V_5C_{51}V_1+V_5C_{52}V_2+V_5C_{53}V_3+V_5C_{54}V_4+V_5C_{55}V_5\end{bmatrix}$$

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- Therefore,

$$\frac{\partial W}{\partial V_1} = 2C_{11}V_1 + C_{12}V_2 + C_{13}V_3 + C_{14}V_4 + C_{15}V_5 + V_2C_{21} + V_3C_{31} + V_4C_{41} + V_5C_{51}$$

$$\Rightarrow C_{11}V_1 + C_{12}V_2 + C_{13}V_3 + C_{14}V_4 + C_{15}V_5 = 0$$

- Usually, $\frac{\partial W}{\partial V_k}$ leads to $\sum_{i=1}^{n} V_i C_{ik} = 0$
- where n is the number of nodes in the mesh
- Writing the above equation for all nodes k=1,2,…,n, we obtain a set of simultaneous equations from which the solution $[V]^t = \begin{bmatrix} V_1 & V_2 & \cdots & V_n \end{bmatrix}$ can be found

Introduction

(i) Iteration method

- Suppose node 1 is a free node
- A free node is where the potential is unknown
- Whereas, a fixed node is where the potential is prescribed
- Since $C_{11}V_1 + C_{12}V_2 + C_{13}V_3 + C_{14}V_4 + C_{15}V_5 = 0$
- We have

$$V_1 = -\frac{C_{12}V_2 + C_{13}V_3 + C_{14}V_4 + C_{15}V_5}{C_{11}} = -\frac{1}{C_{11}}\sum_{i=2}^{5} C_{1i}V_i$$

Introduction

- Thus in general for node k in a mesh with n nodes, we have,

$$V_k = -\frac{1}{C_{kk}} \sum_{i=1, i\neq k}^{n} C_{ki} V_i$$

- where the k node is a free node
- Since $C_{ki}=0$ if node k is not directly connected to node i, so nodes which are directly linked to node k contribute to V_k in the above equation
- The iteration process starts by assigning the potential of fixed nodes equal to zero or to the average potential