Introduction

$$V_{avg} = \frac{1}{2}\left(V_{\min} + V_{\max}\right)$$

- where V_{min} and V_{max} are the maximum and minimum values of V at the fixed nodes
- Doubts:
- Q. 1: How did we come up with the equation of alpha in slide 10?
- Answer:
- Shape functions are obtained from
- $V_e(x,y) = a + bx + cy$ where (a,b,c) are

Introduction

$$\Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} x_2 y_3 - x_3 y_2 & x_3 y_1 - x_1 y_3 & x_1 y_2 - x_2 y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \begin{bmatrix} V_{e1} \\ V_{e2} \\ V_{e3} \end{bmatrix}$$

- where A is the area of element e, i.e.,

$$A = \frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = \frac{1}{2}\left\{ \left(x_1 y_2 - x_2 y_1\right) + \left(x_3 y_1 - x_1 y_3\right) + \left(x_2 y_3 - x_3 y_2\right) \right\}$$

$$= \frac{1}{2}\left\{ \left(x_2 - x_1\right)\left(y_3 - y_1\right) - \left(x_3 - x_1\right)\left(y_2 - y_1\right) \right\}$$

Introduction

- We may express the below equation as

$$V_e(x,y) = a + bx + cy$$

$$V_e = \sum_{i=1}^{3} \alpha_i(x,y) V_{ei}$$

- where

$$\alpha_1 = \frac{1}{2A}\left[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y\right]$$

$$\alpha_2 = \frac{1}{2A}\left[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y\right]$$

$$\alpha_3 = \frac{1}{2A}\left[(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y\right]$$

Introduction

- Q. 2: Anticlockwise approach assigning the node numbers is not applicable for global nodes numbering?
- Ans: All the calculations for global coefficients are done from local elements coefficient matrix so no need
- Q3: Global coupling coefficient matrix (C) is always singular?
- Ans: Yes, for example, det (C)=0

$$[C]=\begin{bmatrix} 1.2357 & -0.7786 & 0 & -0.4571 \\ -0.7786 & 1.25 & -0.4571 & -0.0143 \\ 0 & -0.4571 & 0.8238 & -0.3667 \\ -0.4571 & -0.0143 & -0.3667 & 0.8381 \end{bmatrix}$$

Introduction

- Q4:
- In the last part, we are finding the partial derivative of W wrt Vk.
  - Does that mean we are finding the optimum value of V to have minimum energy?
  - If yes and if for a fixed node, if we are finding the value of Vk, does it not tamper with the original value of V, given for that particular node?
- Ans: Yes, we will find it only for free nodes

Introduction

$$V_{avg} = \frac{1}{2}\left(V_{\min} + V_{\max}\right)$$

- where $V_{\min}$ and $V_{\max}$ are the maximum and minimum values of V at the fixed nodes

(2) Band matrix method

- If all the free nodes (f) are numbered first and all prescribed nodes (p) last, we can write

$$W = \frac{1}{2}\varepsilon\begin{bmatrix} V_f & V_p \end{bmatrix}\begin{bmatrix} C_{ff} & C_{fp} \\ C_{pf} & C_{pp} \end{bmatrix}\begin{bmatrix} V_f \\ V_p \end{bmatrix} = \frac{1}{2}\varepsilon\begin{bmatrix} V_f C_{ff} + V_p C_{pf} \\ V_f C_{fp} + V_p C_{pp} \end{bmatrix}\begin{bmatrix} V_f \\ V_p \end{bmatrix}$$

Introduction

$$W = \frac{1}{2}\varepsilon \begin{bmatrix} V_f C_{ff} V_f + V_p C_{pf} V_f + \\ V_f C_{fp} V_p + V_p C_{pp} V_p \\ \\ \end{bmatrix}$$

- What are free and prescribed nodes?
  - Nodes which are free
  - Nodes with the prescribed or fixed potentials

Introduction

- Since V_p is constant, we differentiate only w.r.t. V_f, hence

$$\begin{bmatrix} C_{ff} & C_{fp} \end{bmatrix} \begin{bmatrix} V_f \\ V_p \end{bmatrix} = 0 \Rightarrow \left[C_{ff}\right]\left[V_f\right] = -\left[C_{fp}\right]\left[V_p\right]$$

- This equation can be written as

$$[A][V] = [B] \Rightarrow [V] = [A]^{-1}[B]$$

- where

$$[V] = \left[V_f\right]; [A] = \left[C_{ff}\right]; [B] = -\left[C_{fp}\right]\left[V_p\right]$$

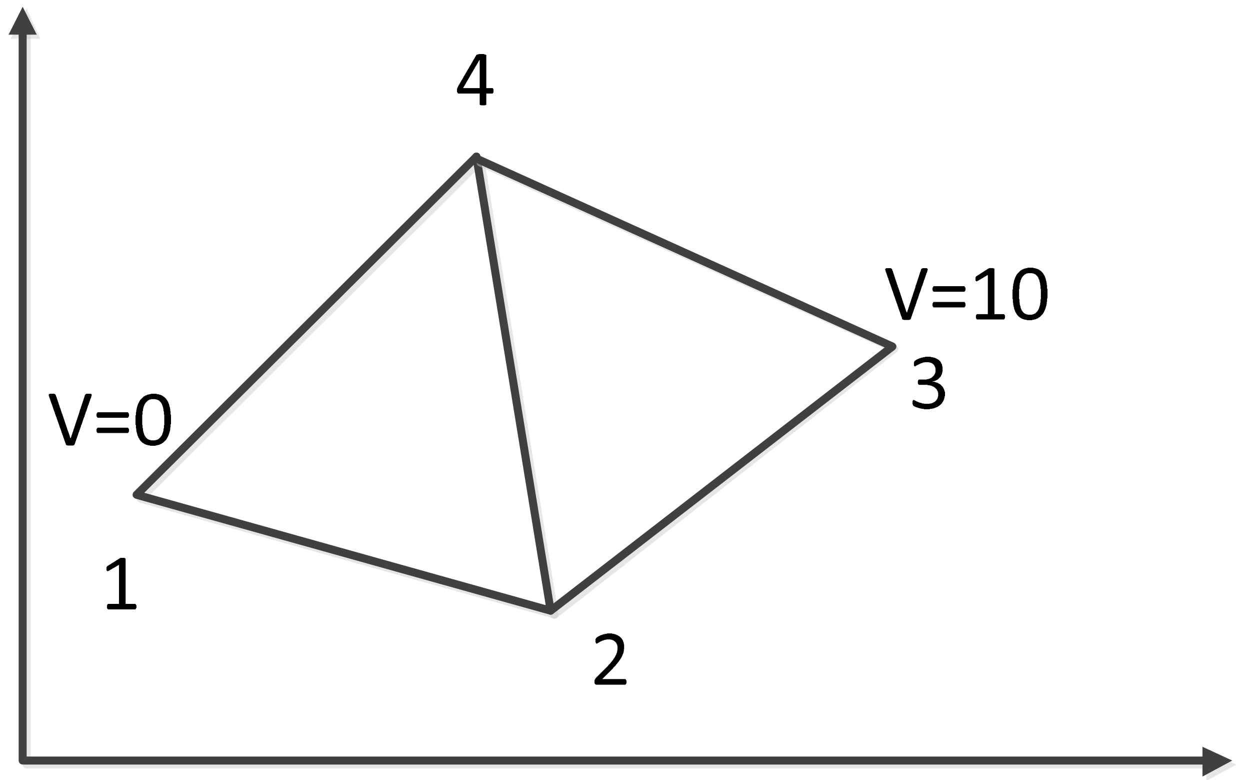
Introduction

- Since A is in general non-singular, the potential at the free nodes can be found out easily
- Example: Consider the two element mesh shown in Fig. Using FEM find the potentials within the mesh.

Node	x	y
1	0.8	1.8
2	1.4	1.4
3	2.1	2.1
4	1.2	2.7

Introduction

- Two element mesh



Introduction

- Define

$$P_1 = (y_2 - y_3), P_2 = (y_3 - y_1), P_3 = (y_1 - y_2)$$

$$Q_1 = -(x_2 - x_3),\ Q_2 = -(x_3 - x_1),\ Q_3 = -(x_1 - x_2)$$

- Then

$$C_{ij}^{(e)} = \frac{1}{4A}\left(P_i P_j + Q_i Q_j\right)$$

- where

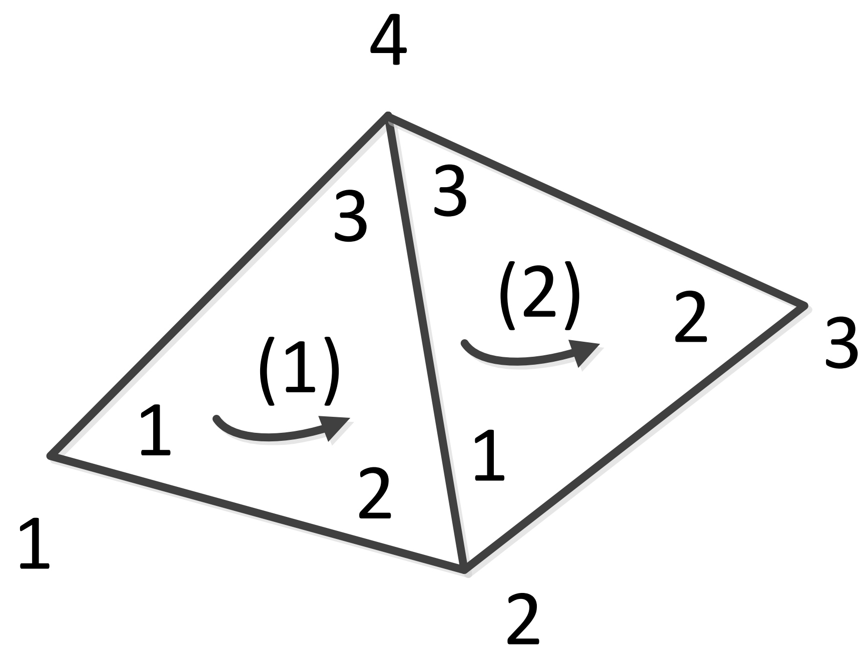
$$A = \frac{1}{2}\left(P_2 Q_3 - P_3 Q_2\right)$$

Introduction

Element	1	2
(P_1,P_2,P_3)	(-1.3,0.9,0.4)	(-0.6,1.3,-0.7)
(Q_1,Q_2,Q_3)	(-.2,-.4,0.6)	(-0.9,0.2,0.7)
A	0.35	0.525

Introduction

- Element numbering and local node numbering



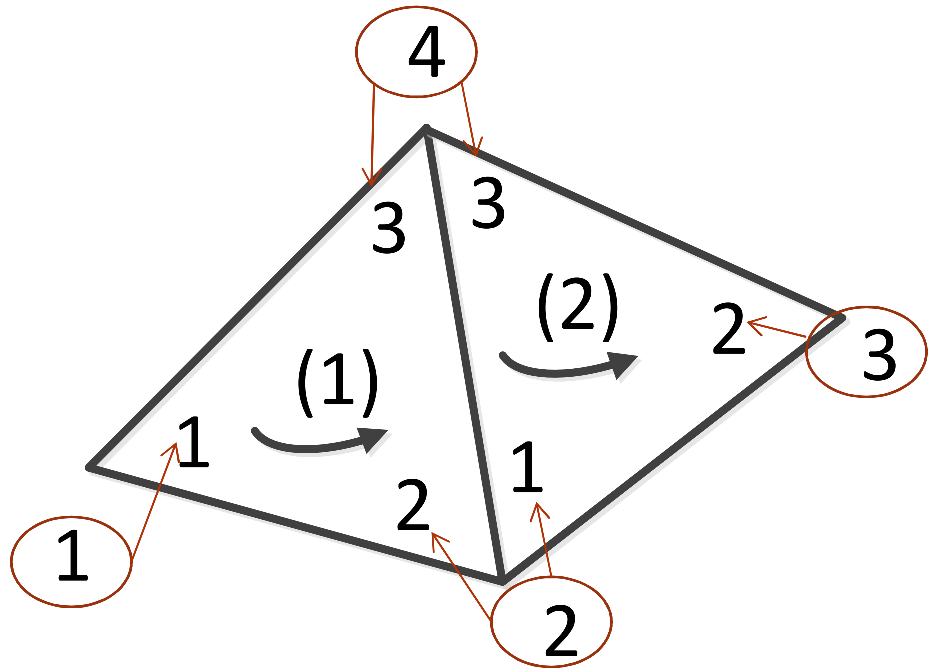
Introduction

$$\left[C^{(1)}\right]=\begin{bmatrix} 1.2357 & -0.7786 & -0.4571 \\ -0.7786 & 0.6929 & 0.0857 \\ -0.4571 & 0.0857 & 0.37114 \end{bmatrix}$$

$$\left[C^{(2)}\right]=\begin{bmatrix} 0.5571 & -0.4571 & -0.1 \\ -0.4571 & 0.8238 & -0.3667 \\ -0.1 & -0.3667 & 0.4667 \end{bmatrix}$$

Introduction

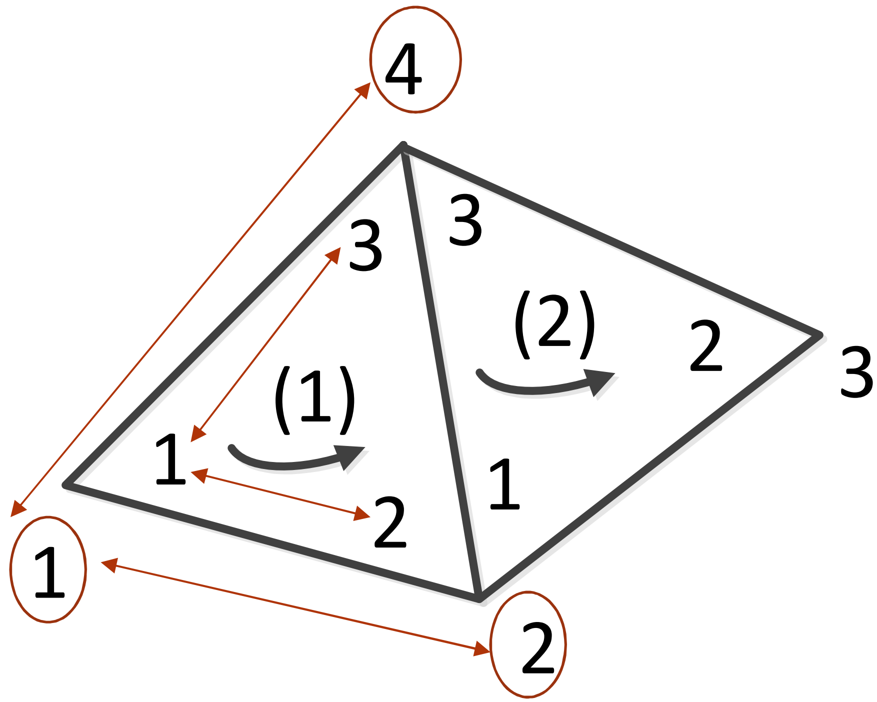
- Diagonal elements of Global coefficient matrix



Introduction

$$[C] = \begin{bmatrix} C_{11} = C_{11}^{(1)} & & & \\ & C_{22} = C_{22}^{(1)} + C_{11}^{(2)} & & \\ & & C_{33} = C_{22}^{(2)} & \\ & & & C_{44} = C_{33}^{(1)} + C_{33}^{(2)} \\ & & & \end{bmatrix}$$

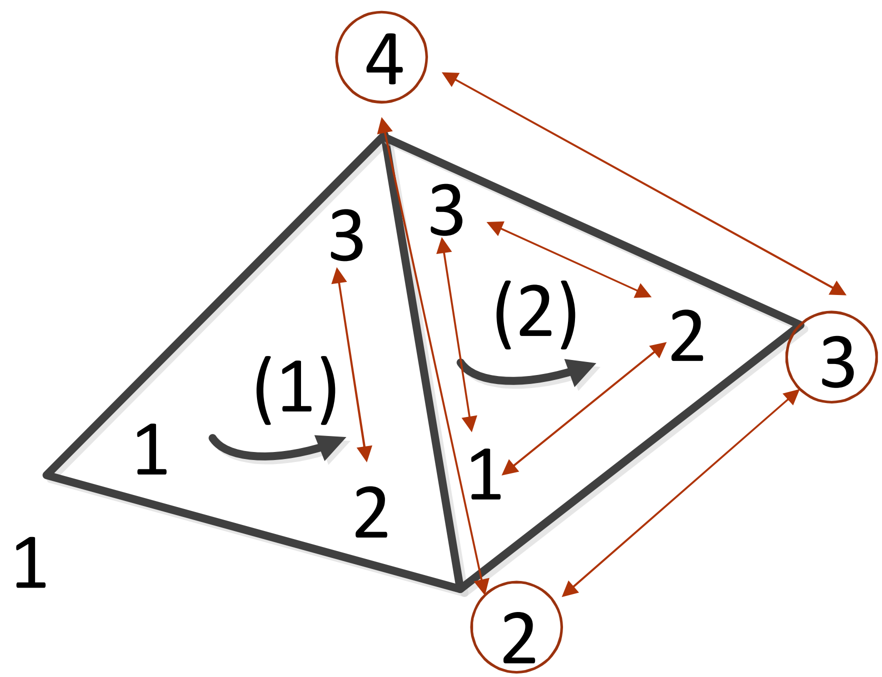
Introduction



Introduction

$$[C]=\begin{bmatrix} C_{11}=C_{11}^{(1)} & C_{12}=C_{12}^{(1)} & C_{13}=0 & C_{14}=C_{13}^{(1)} \\ C_{21}=C_{12} & C_{22}=C_{22}^{(1)}+C_{11}^{(2)} & & \\ C_{31}=C_{13} & & C_{33}=C_{22}^{(2)} & \\ C_{41}=C_{14} & & & C_{44}=C_{33}^{(1)}+C_{33}^{(2)} \\ & & & \end{bmatrix}$$

Introduction



Introduction

$$[C]=\begin{bmatrix} C_{11}=C_{11}^{(1)} & C_{12}=C_{12}^{(1)} & C_{13}=0 & C_{14}=C_{13}^{(1)} \\ C_{21}=C_{12} & C_{22}=C_{22}^{(1)}+C_{11}^{(2)} & C_{23}=C_{12}^{(2)} & C_{24}=C_{23}^{(1)}+C_{13}^{(2)} \\ C_{31}=C_{13} & C_{32}=C_{23} & C_{33}=C_{22}^{(2)} & C_{34}=C_{23}^{(2)} \\ C_{41}=C_{14} & C_{42}=C_{24} & C_{43}=C_{34} & C_{44}=C_{33}^{(1)}+C_{33}^{(2)} \end{bmatrix}$$

Introduction

$$[C] = \begin{bmatrix} 1.2357 & -0.7786 & 0 & -0.4571 \\ -0.7786 & 1.25 & -0.4571 & -0.0143 \\ 0 & -0.4571 & 0.8238 & -0.3667 \\ -0.4571 & -0.0143 & -0.3667 & 0.8381 \end{bmatrix}$$

- Note that $\sum_{i=1}^{4} C_{ij} = \sum_{j=1}^{4} C_{ij} = 0$
- It may be used to check if C is properly obtained
- For free nodes, we can found out as follows

Introduction

First iteration (note $V_1=0, V_3=10$, prescribed) $\because V_k = -\frac{1}{C_{kk}} \sum_{i=1, i\neq k}^{n} C_{ki} V_i$

$$V_2 = -\frac{1}{C_{22}}\left(C_{21}V_1 + C_{23}V_3 + C_{24}V_4\right) = -\frac{1}{1.25}\left(-4.571 - 0.0143V_4\right)$$

$$V_4 = -\frac{1}{C_{44}}\left(C_{41}V_1 + C_{42}V_2 + C_{43}V_3\right) = -\frac{1}{0.831}\left(-3.667 - 0.0143V_2\right)$$

Setting $V_2=V_4=0$ initially, we have, $V_2=3.6568, V_4=4.413$

Use this value for second iteration, we have, $V_2=3.707, V_4=4.476$

Introduction

- Band matrix method:

$$\begin{bmatrix} C_{22} & C_{24} \\ C_{42} & C_{44} \\ & \end{bmatrix}\begin{bmatrix} V_2 \\ V_4 \\ \end{bmatrix} = -\begin{bmatrix} C_{21} & C_{23} \\ C_{41} & C_{43} \\ & \end{bmatrix}\begin{bmatrix} V_1 \\ V_3 \\ \end{bmatrix} \quad \because \left[C_{ff}\right]\left[V_f\right] = -\left[C_{fp}\right]\left[V_p\right]$$

- Equivalently,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & C_{22} & 0 & C_{24} \\ 0 & 0 & 1 & 0 \\ 0 & C_{42} & 0 & C_{44} \\ & & & \end{bmatrix}\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -C_{21} & -C_{23} \\ 0 & 1 \\ -C_{41} & -C_{43} \\ & \end{bmatrix}\begin{bmatrix} V_1 \\ V_3 \\ \end{bmatrix}$$

Introduction

- Or,
- [C][V]=[B]
- where

$$[C]=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1.25 & 0 & -0.0143\\ 0 & 0 & 1 & 0\\ 0 & -0.0143 & 0 & 0.8381\\ & & & \end{bmatrix},[B]=\begin{bmatrix}0\\ 4.571\\ 10\\ 3.667\\ \end{bmatrix}$$

- Therefore,

$$[V]=[C]^{-1}[B]=\begin{bmatrix}0\\ 3.708\\ 10.0\\ 4.438\\ \end{bmatrix}$$

Introduction

- What is Functional?
- Consider a 1-D region of space in the range $x_1 \le x \le x_2$
- A function $\varphi(x)$ and its derivative $\varphi' = d\varphi/dx$ are defined in the region along with $F(\varphi, \varphi', x)$
- A general expression of the functions and independent variable
- Consider the volume integral of the expression

$$I = \int_{x_1}^{x_2} F\left(\phi, \phi', x\right) dx$$

Introduction

- The quantity I takes on a value for each choice of the function φ
- It is called function of φ
- We seek a particular form of φ that gives a maximum or minimum value of the functional
- Assume that $\varphi(x)$ has an arbitrary variation over the range but at the ends the values are clamped
- $\Phi(x_1)= \varphi_1$, $\Phi(x_2)= \varphi_2$
- To seek a minimum, we investigate variations of the functional form of $\Phi(x)$

Introduction

- Let $\Phi_0(x)$ represent the desired minimizing function and define an arbitrary function $\eta(x)$ that satisfies the boundary conditions
- $\eta(x_1)= \eta(x_2)=0$
- Let ε be a small number and add the quantity $\varepsilon\,\eta(x)$ to define the variation of $\Phi(x)$ about the optimum point
- The functional becomes

$$I(\varepsilon)=\int_{x_1}^{x_2} F\left(\phi,\phi',x\right)dx=\int_{x_1}^{x_2} F\left(\phi_0+\varepsilon\eta,\phi'+\varepsilon\eta',x\right)dx$$

Introduction

- The condition dI/d ε =0 means that $\Phi_0(x)$ corresponds to the minimum (or maximum) of the functional

$$\frac{\partial I(\varepsilon)}{\partial \varepsilon}=\int_{x_1}^{x_2}\left(\frac{\partial F}{\partial \phi}\eta+\frac{\partial F}{\partial \phi'}\eta'\right)dx=0$$

- Using integration by parts of the second term in the above integration
- Take

$$v=\frac{\partial F}{\partial \phi'};u=\eta \Rightarrow du=\eta' dx$$

Introduction

- Integration by parts

$$\int v\,du = vu - \int u\,dv$$

- Therefore

$$\frac{\partial I(\varepsilon)}{\partial \varepsilon} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial \phi}\eta - \eta \frac{d\left(\frac{\partial F}{\partial \phi^{'}}\right)}{dx} \right) dx + \frac{\partial F}{\partial \phi^{'}}\eta \Bigg|_{x_1}^{x_2} = 0$$

Introduction

- Because of the boundary condition: $\eta(x_1) = \eta(x_2) = 0$

$$\frac{\partial I(\varepsilon)}{\partial \varepsilon} = \int_{x_1}^{x_2} \left(\frac{\partial F}{\partial \phi}\eta - \eta \frac{d\left(\frac{\partial F}{\partial \phi'}\right)}{dx} \right) dx + \frac{\partial F}{\partial \phi'}\eta \Big|_{x_1}^{x_2} = 0$$

- Hence,

$$\int_{x_1}^{x_2} \left(\frac{\partial F}{\partial \phi} - \frac{d\left(\frac{\partial F}{\partial \phi'}\right)}{dx} \right) \eta dx = 0$$

Introduction

- The function $\Phi(x)$ that gives minimum of the functional I therefore satisfies the equation which is called Euler equation

$$\frac{\partial F}{\partial \phi} - \frac{d\left(\frac{\partial F}{\partial \phi'}\right)}{dx} = 0; \phi' = \frac{d\phi}{dx}$$

- When the function $\Phi(x,y,z)$ depends on three independent variables, the 3-D form of the Euler equation

$$\frac{\partial F}{\partial \phi} - \frac{\partial\left(\frac{\partial F}{\partial \phi_x}\right)}{\partial x} - \frac{\partial\left(\frac{\partial F}{\partial \phi_y}\right)}{\partial y} - \frac{\partial\left(\frac{\partial F}{\partial \phi_z}\right)}{\partial z} = 0$$

- where $$\phi_x = \frac{d\phi}{dx}, \phi_y = \frac{d\phi}{dy}, \phi_z = \frac{d\phi}{dz}$$