# FEM

- FEM
- Converts PDE into a set of linear algebraic equations
  - To obtain approximate solutions to boundary-value problems (BVPs)
- Two methods:
- Variational method (Rayleigh-Ritz method):
  - Starts with variational representation of the BVPs
- Weighted residual method
  - Similar to MoM

# FEM

- Variational method (Rayleigh-Ritz method):
  - In BVPs,
  - it is often possible to replace the problem of integrating a differential equation
  - by the equivalent problem of seeking a function that gives a minimum value of some integral (functional)
- Problems of this type are called *variational problems*
- Was first presented by Rayleigh in 1877 and extended by Ritz in 1909

# FEM

- Use Calculus of Variations in solving BVPs
- What is Calculus of Variations?
- Calculus of Variations:
  - It is an extension of ordinary calculus
  - it is concerned primarily with the theory of maxima and minima
- In FEM, we will try to find the extrema of an integral expression involving a function of function (functionals)

# FEM

- Consider the problem of finding a function Φ(x) {***consider Φ is dependent only on x***} such that the function
- $I(\Phi) = \int_a^b F(x, \Phi, \Phi')dx$
- Subject to the boundary condition $\Phi(a) = A, \Phi(b) = B$
- The integrand $F(x, \Phi, \Phi')$ is a given function of x, Φ and $\Phi' = \frac{d\Phi}{dx}$
- The $I(\Phi)$ is called a *functional or variational principle*
- The problem here is finding an extremizing function Φ(x) for which the functional $I(\Phi)$ has an extremum

# FEM

- Let us introduce an operator $\delta$ called the *variational symbol*
- The variation $\delta\Phi$of a function $\Phi$(x) is an infinitesimal change in $\Phi$ for a fixed value of the independent variable x, i.e., for $\delta x = 0$
- Note that total differential of $F(x, \Phi, \Phi')$ is
- $\delta F = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial \Phi} d\Phi + \frac{\partial F}{\partial \Phi'} d\Phi'$
- where $\delta x = 0$ since x does not change when $\Phi$ changes from $\Phi + \delta\Phi$
- $\delta F = \frac{\partial F}{\partial \Phi} d\Phi + \frac{\partial F}{\partial \Phi'} d\Phi'$

# FEM

- Note that $\delta$ operator is similar to differential operator
- A necessary condition for $I(\Phi)$ to have an extremum is
- $\delta I = 0$
- Let h(x) be an increment in $\Phi$(x)
- In order that the boundary condition $\Phi(a) = A, \Phi(b) = B$ is satisfied
- $h(a) = 0,\ h(b) = 0$
- The corresponding increment in I is
- $\Delta I = I(\Phi + h) - I(\Phi)$

# FEM

- $\Delta I = I(\Phi + h) - I(\Phi)$
- $= \int_a^b [F(x, \Phi + h, \Phi' + h') - F(x, \Phi, \Phi')]dx$
- On applying Taylor's series expansion
- $\Delta I = \int_a^b [F_\Phi(x, \Phi, \Phi')h - F_{\Phi'}(x, \Phi, \Phi')h']dx + higher\ order\ terms = \delta I + O(h^2)$
- where $\delta I = \int_a^b [F_\Phi(x, \Phi, \Phi')h - F_{\Phi'}(x, \Phi, \Phi')h']dx$

# FEM

- Integration by parts

$$\int v du = vu - \int u dv$$

- Take $v = F_{\Phi'}(x, \Phi, \Phi'), u = h, du = h' dx$
- for the second term in the integrand $\delta I = \int_a^b [F_{\Phi}(x, \Phi, \Phi')h - F_{\Phi'}(x, \Phi, \Phi')h']dx$
- Integration by parts
- $\delta I = \int_a^b \left[F_{\Phi}(x, \Phi, \Phi')h - \frac{d}{dx}\left(F_{\Phi'}(x, \Phi, \Phi')\right)h\right] dx + F_{\Phi'}(x, \Phi, \Phi')h|_{x=a}^{x=b}$

# FEM

- The last term is zero
- $\delta I = \int_a^b \left[ F_\Phi(x, \Phi, \Phi')h - \frac{d}{dx}\left(F_{\Phi'}(x, \Phi, \Phi')\right)h \right] dx$
- For $\delta I = 0$, we have, integrand equal to zero
- $F_\Phi(x, \Phi, \Phi') - \frac{d}{dx}\left(F_{\Phi'}(x, \Phi, \Phi')\right) = 0$
- $\frac{\partial F}{\partial \Phi} - \frac{d}{dx}(F_{\Phi'}) = 0$
- This is called Euler's (Euler-Lagrange) equation
- Thus necessary condition for $I(\Phi)$ to have an extremum for a given function Φ(x) is that Φ(x) satisfies Euler's equation

# Introduction

- The function Φ(x) that gives minimum of the functional I therefore satisfies the equation which is called Euler equation

$$\frac{\partial F}{\partial \phi} - \frac{d\left(\frac{\partial F}{\partial \phi'}\right)}{dx} = 0; \phi' = \frac{d\phi}{dx}$$

- When the function Φ(x,y,z) depends on three independent variables, the 3-D form of the Euler equation

$$\frac{\partial F}{\partial \phi} - \frac{\partial\left(\frac{\partial F}{\partial \phi_x}\right)}{\partial x} - \frac{\partial\left(\frac{\partial F}{\partial \phi_y}\right)}{\partial y} - \frac{\partial\left(\frac{\partial F}{\partial \phi_z}\right)}{\partial z} = 0$$

- where $$\phi_x = \frac{d\phi}{dx}, \phi_y = \frac{d\phi}{dy}, \phi_z = \frac{d\phi}{dz}$$

# FEM

- Example:
- Given the functional
- $I(\Phi) = \int_S \left[\frac{1}{2}\left(\Phi_x^2 + \Phi_y^2\right) - f(x,y)\Phi\right] dxdy$
- Obtain the Euler's equation.
- Solution:
- The integrand of the function or variational principle is
- $F\left(x, y, \Phi, \Phi_x, \Phi_y\right) = \frac{1}{2}\left(\Phi_x^2 + \Phi_y^2\right) - f(x,y)\Phi$
- It has two independent variables

# FEM

- Therefore,

$$\frac{\partial F}{\partial \phi}-\frac{\partial\left(\frac{\partial F}{\partial \phi_x}\right)}{\partial x}-\frac{\partial\left(\frac{\partial F}{\partial \phi_y}\right)}{\partial y}-\frac{\partial\left(\frac{\partial F}{\partial \phi_z}\right)}{\partial z}=0$$

- where
- $F\left(x, y, \Phi, \Phi_x, \Phi_y\right)=\frac{1}{2}\left(\Phi_x^2+\Phi_y^2\right)-f(x, y)\Phi$
- Hence,
- $-f(x, y)-\Phi_{xx}-\Phi_{yy}=0$
- $\Rightarrow \Phi_{xx}+\Phi_{yy}=-f(x, y) \Rightarrow \nabla^2\Phi=-f(x, y)$
- which is the Poisson's equation

# FEM

- **Construction of functionals from PDE**
- We will try to find functional or variational principle for a given differential equation
- It involves four steps
- (1) Multiply the $\mathcal{L}\Phi = f$ (Euler's equation) with the variational $\delta\Phi$ of the dependent variable $\Phi$ and integrate over the domain of the problem
- (2) Use integration by parts to transfer the derivatives to variation $\delta\Phi$

# FEM

- (3) Express the boundary integrals in terms of the specified boundary condition
- (4) Bring the variational operator δ outside the integrals
- For instance,
- We want to find the functional or variational principle of the Poisson's equation
- $\nabla^2\Phi = -f(x,y) \Rightarrow -\nabla^2\Phi - f(x,y) = 0$
- (1) $\delta I = \int\int[-\nabla^2\Phi - f(x,y)]\delta\Phi dxdy = -\int\int\nabla^2\Phi\delta\Phi dxdy - \int\int f(x,y)\delta\Phi dxdy$

# FEM

$$\int v du = vu - \int u dv$$

- (2) Integration by parts
- Hence the first term of

$$\delta I = -\int\int \nabla^2 \Phi \delta\Phi dxdy - \int\int f(x,y)\delta\Phi dxdy$$

- is simplified as
- $\int\int(-\nabla^2\Phi)\delta\Phi dxdy =$
- $-\int\int\left(\Phi_{xx} + \Phi_{yy}\right)\delta\Phi dxdy =$
- $-\int\int(\Phi_{xx})\delta\Phi dxdy - \int\int\left(\Phi_{yy}\right)\delta\Phi dxdy$

# FEM

- In order to find using integration by parts $\int\int(\Phi_{xx})\delta\Phi dxdy$
- Let us take
$$\int vdu = vu - \int udv$$
- $v = \delta\Phi, du = \frac{\partial}{\partial x}\left(\frac{\partial\Phi}{\partial x}\right)dx = \Phi_{xx}dx, u = \frac{\partial\Phi}{\partial x} = \Phi_x$
- and $dv = \frac{\partial}{\partial x}\delta\Phi dx$
- Hence integration by parts of the first integration

$$\int\left[\int(\Phi_{xx})\delta\Phi dx\right]dy = \int\left[\delta\Phi\Phi_x - \int\Phi_x\frac{\partial}{\partial x}\delta\Phi dx\right]dy$$

# FEM

- In order to find using integration by parts $\int\int(\Phi_{yy})\delta\Phi dxdy$
- Let us take $$\int vdu = vu - \int udv$$
- $v = \delta\Phi, du = \frac{\partial}{\partial y}\left(\frac{\partial\Phi}{\partial y}\right)dy = \Phi_{yy}dy, u = \frac{\partial\Phi}{\partial y} = \Phi_y$
- and $dv = \frac{\partial}{\partial y}\delta\Phi dy$
- Hence integration by parts of the first integration

$$\int\left[\int(\Phi_{yy})\delta\Phi dy\right]dx = \int\left[\delta\Phi\Phi_y - \int\Phi_y\frac{\partial}{\partial y}\delta\Phi dy\right]dx$$

# FEM

- Hence
- $\delta I = \int\int \left[\Phi_x \frac{\partial}{\partial x}\delta\Phi + \Phi_y \frac{\partial}{\partial y}\delta\Phi - f(x,y)\delta\Phi\right] dxdy - \int \delta\Phi\Phi_x\, dy - \int \delta\Phi\Phi_y\, dx$
- $= \frac{\delta}{2}\int\int \left[(\Phi_x)^2 + (\Phi_y)^2 - 2f\Phi\right] dxdy - \delta\int \Phi\Phi_x\, dy - \delta\int \Phi\Phi_y\, dx$
- If we assume homogeneous Drichlet or Neuman boundary conditions
- $\delta I = \delta\int\int \frac{1}{2}\left[(\Phi_x)^2 + (\Phi_y)^2 - 2f\Phi\right] dxdy$

# Introduction

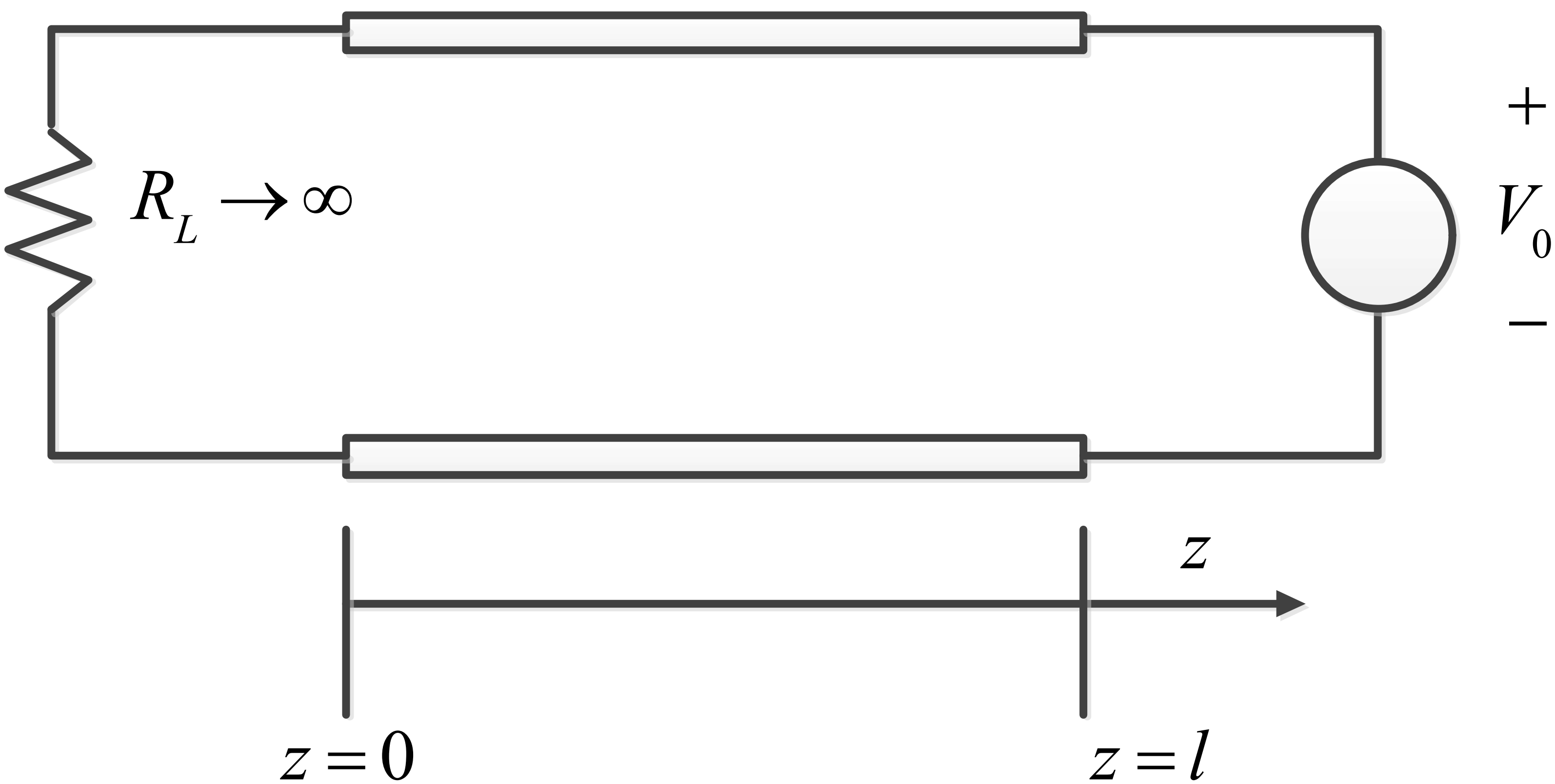
- Therefore variational principle or functional
- for Poisson's equation $\nabla^2\Phi = -f(x,y)$
- After taking all terms to RHS $-\nabla^2\Phi - f(x,y) = 0$ for
- $I = \int\int \frac{1}{2}\left[(\Phi_x)^2 + \left(\Phi_y\right)^2 - 2f\Phi\right] dxdy$
- For L$\Phi$=g is obtained by extremizing the functional

$$I(\Phi) = \frac{1}{2}\iint\left[\left|\nabla\Phi\right|^2 - k^2\Phi^2 + 2\Phi g\right]ds$$

- for Euler's equation $\nabla^2\Phi + k^2\Phi = g$ after taking all terms to RHS
- $-\nabla^2\Phi - k^2\Phi + g = 0$

# Introduction

- **FEM analysis of transmission lines (Variational approach)**



# Introduction

- Telegrapher's equation

$$\frac{\partial I(z,t)}{\partial z} = -C(z)\frac{\partial V(z,t)}{\partial t}$$

$$\frac{\partial V(z,t)}{\partial z} = -L(z)\frac{\partial I(z,t)}{\partial t}$$

- In frequency domain

$$\frac{\partial I(z)}{\partial z} = -j\omega C(z)V(z)$$

$$\frac{\partial V(z)}{\partial z} = -j\omega L(z)I(z)$$

# Introduction

- 1-D wave equation in TL

$$\frac{d}{dz}\left(\frac{1}{L}\frac{dV(z)}{dz}\right)=-j\omega\frac{dI(z)}{dz}=-\omega^2 CV$$

$$\Rightarrow \frac{d}{dz}\left(\frac{dV(z)}{dz}\right)+\omega^2 LCV=0$$

$$\Rightarrow \frac{d^2V(z)}{dz^2}+\omega^2 LCV=0$$

- Boundary condition $V(z=l)=V_0$

$$\because I(z=0)=0 \Rightarrow \left.\frac{dV}{dz}\right|_{z=0}=0$$

# Introduction

- Euler's equation $\nabla^2\Phi + k^2\Phi = g$
- Functional is

$$I(\Phi) = \frac{1}{2}\iint\left[\left|\nabla\Phi\right|^2 - k^2\Phi^2 + 2\Phi g\right]ds$$

- Euler's equation $\frac{1}{L}\frac{d^2V(z)}{dz^2} + \omega^2 CV = 0$
- Transfer all terms to RHS, $-\frac{1}{L}\frac{d^2V(z)}{dz^2} - \omega^2 CV = 0$
- Therefore, functional is

$$I = \frac{1}{2}\int_l\left[\frac{1}{L}\left(\frac{dV}{dz}\right)^2 - \omega^2 CV^2\right]dz$$

# Introduction

- Hence, the integrand of the functional is

$$F\left(V,\frac{dV}{dz},z\right)=\frac{1}{2}\left[\frac{1}{L}\left(\frac{dV}{dz}\right)^2-\omega^2CV^2\right]$$

- Therefore,

$$\frac{\partial F}{\partial V}=-\omega^2CV;\frac{\partial F}{\partial V_z}=\frac{1}{L}\frac{dV}{dz}=\frac{1}{L}V_z$$

# Introduction

- Euler equation

$$\because \frac{\partial F}{\partial \phi} - \frac{d\left(\frac{\partial F}{\partial \phi'}\right)}{dx} = 0$$

- Hence the PDE for this case is

$$\omega^2 CV + \frac{1}{L}\frac{d^2V}{dz^2} = 0$$

- It is the wave equation of the TL

# Introduction

- This functional can be expressed as

$$-\frac{1}{2}\omega^2\int_l \left[ LI^2 + CV^2 \right] dz \qquad \because -\frac{1}{j\omega L}\frac{\partial V}{\partial z} = I$$

- The integrand look like stored electric and magnetic energy per unit length, so energy-related functional

# Introduction

- The approximate voltage for element lying between $z_l$ and $z_r$:

$$V^e = \alpha_l(z)V_l + \alpha_r(z)V_r$$

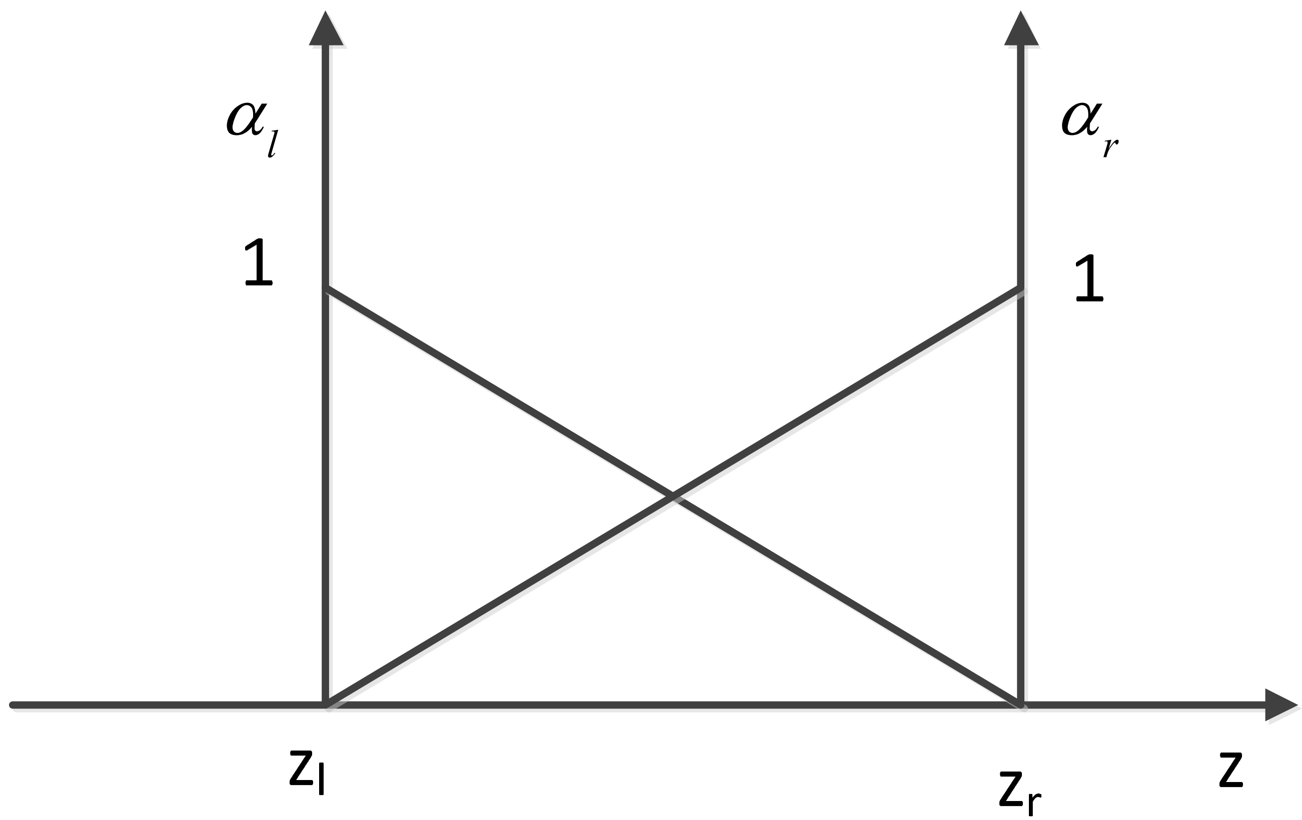
- where the interpolation functions are given by

$$\alpha_l(z) = \frac{z_r - z}{z_r - z_l}; \alpha_r(z) = \frac{z - z_l}{z_r - z_l}$$

- Assume that a TL of length l is discretized into N elements, each of length $h_e$=l/N

# Introduction

- Interpolation functions



# Introduction

- Substituting the approximate voltage in the functional

$$F^e\left(V^e\right)=\frac{1}{2}\int_{z_l}^{z_r}\left[\frac{1}{L}\left(\frac{dV^e}{dz}\right)^2-\omega^2C\left(V^e\right)^2\right]dz$$

$$F^e\left(V^e\right)=\frac{1}{2}\int_{z_l}^{z_r}\left[\frac{1}{L}\left(\frac{d\left(\alpha_l\left(z\right)V_l+\alpha_r\left(z\right)V_r\right)}{dz}\right)^2-\omega^2C\left(\alpha_l\left(z\right)V_l+\alpha_r\left(z\right)V_r\right)^2\right]dz$$