Introduction

- Substituting the approximate voltage in the functional

$$F^e\left(V^e\right)=\frac{1}{2}\int_{z_l}^{z_r}\left[\frac{1}{L}\left(\frac{dV^e}{dz}\right)^2-\omega^2C\left(V^e\right)^2\right]dz$$

$$V^e=\alpha_l(z)V_l+\alpha_r(z)V_r$$

$$F^e\left(V^e\right)=\frac{1}{2}\int_{z_l}^{z_r}\left[\frac{1}{L}\left(\frac{d\left(\alpha_l(z)V_l+\alpha_r(z)V_r\right)}{dz}\right)^2-\omega^2C\left(\alpha_l(z)V_l+\alpha_r(z)V_r\right)^2\right]dz$$

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- This can be written in matrix form as

$$F^e\left(V^e\right)=\begin{bmatrix} V_l & V_r \end{bmatrix}\left[\frac{1}{L_e}A-\omega^2 C_e B\right]\begin{bmatrix} V_l \\ V_r \end{bmatrix}$$

$$F^e\left(V^e\right)=\frac{1}{2}\int_{z_l}^{z_r}\left[\frac{1}{L}\left(\frac{d\left(\alpha_l\left(z\right)V_l+\alpha_r\left(z\right)V_r\right)}{dz}\right)^2-\omega^2 C\left(\alpha_l\left(z\right)V_l+\alpha_r\left(z\right)V_r\right)^2\right]dz$$

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$$F^e\left(V^e\right)=\frac{1}{2}\int_{z_l}^{z_r}\left[\frac{1}{L}\left(\frac{d\left(\alpha_l(z)V_l+\alpha_r(z)V_r\right)}{dz}\right)^2-\omega^2C\left(\alpha_l(z)V_l+\alpha_r(z)V_r\right)^2\right]dz$$

- where

$$A_{ij}=\int_{z_l}^{z_r}\frac{d\alpha_i}{dz}\frac{d\alpha_j}{dz}dz;B_{ij}=\int_{z_l}^{z_r}\alpha_i\alpha_j dz;$$

- with indices i and j taking both values of l and r
- Also note that $\alpha_l+\alpha_r=1$
- Therefore $V^e=\alpha_l(z)V_l+\alpha_r(z)V_r$

$$=\left(1-\alpha_r(z)\right)V_l+\alpha_r(z)V_r$$

- Since

$$\alpha_r(z)=\frac{z-z_r}{h_e};\frac{d\alpha_r(z)}{dz}=\frac{1}{h_e};dz=h_e d\alpha$$

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- Hence,

$$A_{lr}=h_e\int_0^1 \frac{d(1-\alpha)}{d\alpha}\frac{d\alpha}{dz}\frac{d\alpha}{d\alpha}\frac{d\alpha}{dz}d\alpha$$

$$\Rightarrow A_{lr}=h_e\int_0^1 (-1)\frac{1}{h_e}(1)\frac{1}{h_e}d\alpha=\frac{-1}{h_e}$$

$$A_{rr}=h_e\int_0^1 \frac{d\alpha}{d\alpha}\frac{d\alpha}{dz}\frac{d\alpha}{d\alpha}\frac{d\alpha}{dz}d\alpha$$

$$\Rightarrow A_{rr}=h_e\int_0^1 (1)\frac{1}{h_e}(1)\frac{1}{h_e}d\alpha=\frac{1}{h_e}$$

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- Similarly,

$$B_{lr} = h_e \int_0^1 (1-\alpha)\alpha d\alpha$$
$$= h_e \int_0^1 (\alpha - \alpha^2) d\alpha = h_e \left(\frac{\alpha^2}{2} - \frac{\alpha^3}{3} \right)\Bigg|_0^1 = \frac{h_e}{6}$$

$$B_{rr} = h_e \int_0^1 (1\alpha)\alpha d\alpha$$
$$= h_e \int_0^1 (\alpha^2) d\alpha = h_e \left(\frac{\alpha^3}{3} \right)\Bigg|_0^1 = \frac{h_e}{3}$$

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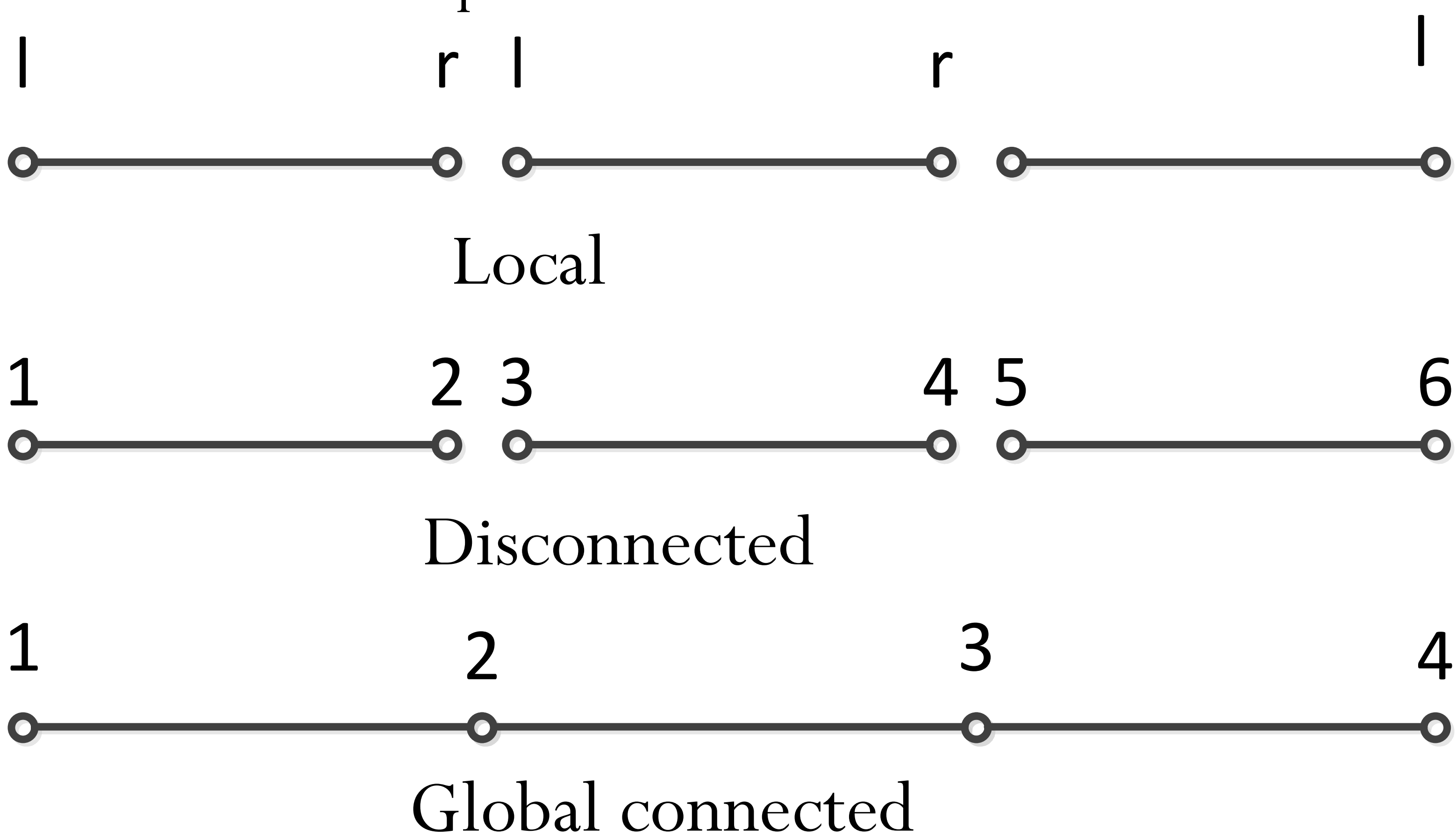
- Hence,

$$[A]=\frac{1}{h_e}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

$$[B]=\frac{h_e}{6}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$$

Introduction

- 1-D FEM example

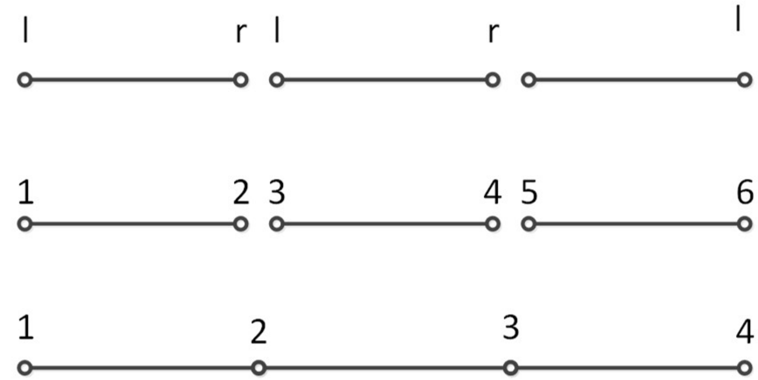


Introduction

- For the above example
- where

$$[V]_{discon} = [M]_{con}[V]_{con}$$

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{bmatrix}_{discon} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{con} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}_{con}$$



Introduction

$$F^e\left(V^e\right)=\begin{bmatrix} V_l & V_r \end{bmatrix}\left[\frac{1}{L_e}A-\omega^2 C_e B\right]\begin{bmatrix} V_l \\ V_r \end{bmatrix}$$

- Therefore

$$F^e\left(V^e\right)=\left[V\right]_{discon}^T\begin{bmatrix} \frac{1}{L_1}A-\omega^2 C_1 B & 0 & 0 & 0 \\ 0 & \frac{1}{L_2}A-\omega^2 C_2 B & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \frac{1}{L_N}A-\omega^2 C_N B \end{bmatrix}\left[V\right]_{discon}$$

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- In matrix form,

$$F^e\left(V^e\right)=\left[V\right]_{discon}^T\left[C\right]_{discon}\left[V\right]_{discon}$$

$$\Rightarrow F^e\left(V^e\right)=\left[V\right]_{con}^T\left[M\right]_{con}^T\left[C\right]_{discon}\left[M\right]_{con}\left[V\right]_{con}$$

$$\Rightarrow F^e\left(V^e\right)=\left[V\right]_{con}^T\left[C\right]_{con}\left[V\right]_{con}$$

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- where

$$\left[C\right]_{con}=\frac{1}{L}\left[M\right]_{con}^{T}\left[A\right]\left[M\right]_{con}-\omega^{2}C\left[M\right]_{con}^{T}\left[B\right]\left[M\right]_{con}$$

- since

$$F^{e}\left(V^{e}\right)=\begin{bmatrix}V_{l} & V_{r}\end{bmatrix}\left[\frac{1}{L_{e}}A-\omega^{2}C_{e}B\right]\begin{bmatrix}V_{l}\\V_{r}\end{bmatrix}=\left[V\right]_{discon}^{T}\left[C\right]_{discon}\left[V\right]_{discon}$$

- and

$$F^{e}\left(V^{e}\right)=\left[V\right]_{con}^{T}\left[C\right]_{con}\left[V\right]_{con}$$

Introduction

- Assuming

$$[C]_{con} = \frac{1}{L}[M]_{con}^{T}[A][M]_{con} - \omega^2 C[M]_{con}^{T}[B][M]_{con}$$

$$= [C]_{con1} - [C]_{con2}$$

- where

$$[C]_{con1} = \frac{1}{L}[M]_{con}^{T}[A][M]_{con}$$

$$[C]_{con2} = \omega^2 C[M]_{con}^{T}[B][M]_{con}$$

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$$[C]_{con1} = \frac{1}{L}[M]_{con}^{T}[A][M]_{con} \qquad [A] = \frac{1}{h_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

- It can be shown that

$$[C]_{con1} = \frac{1}{Lh_e}\begin{bmatrix} 1 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & & \ddots & & & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{bmatrix}$$

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$$[C]_{con2} = \omega^2 C [M]_{con}^T [B][M]_{con} \quad [B] = \frac{h_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ & \end{bmatrix}$$

- It can be shown that

$$[C]_{con2} = \frac{\omega^2 hC}{6}\begin{bmatrix} 2 & 1 & & & & & \\ 1 & 4 & 1 & & & & \\ & 1 & 4 & 1 & & & \\ & & & \ddots & & & \\ & & & 1 & 4 & 1 \\ & & & & 1 & 2 \\ & & & & & \end{bmatrix}$$

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- Hence, $F^e\left(V^e\right)=\left[V\right]_{con}^T\left[C\right]_{con}\left[V\right]_{con}$
- which is of the form

$$F(V)=\begin{bmatrix} V_1 & \cdots & V_N \end{bmatrix}\begin{pmatrix} C_{11} & C_{12} & & & & \\ C_{21} & C_{22} & C_{23} & & & \\ & C_{32} & C_{33} & C_{34} & & \\ & & & \ddots & & \\ & & & C_{N-1,N-2} & C_{N-1,N-1} & C_{N-1,N} \\ & & & & C_{N,N-1} & C_{NN} \end{pmatrix}\begin{bmatrix} V_1 \\ \vdots \\ V_N \end{bmatrix}$$

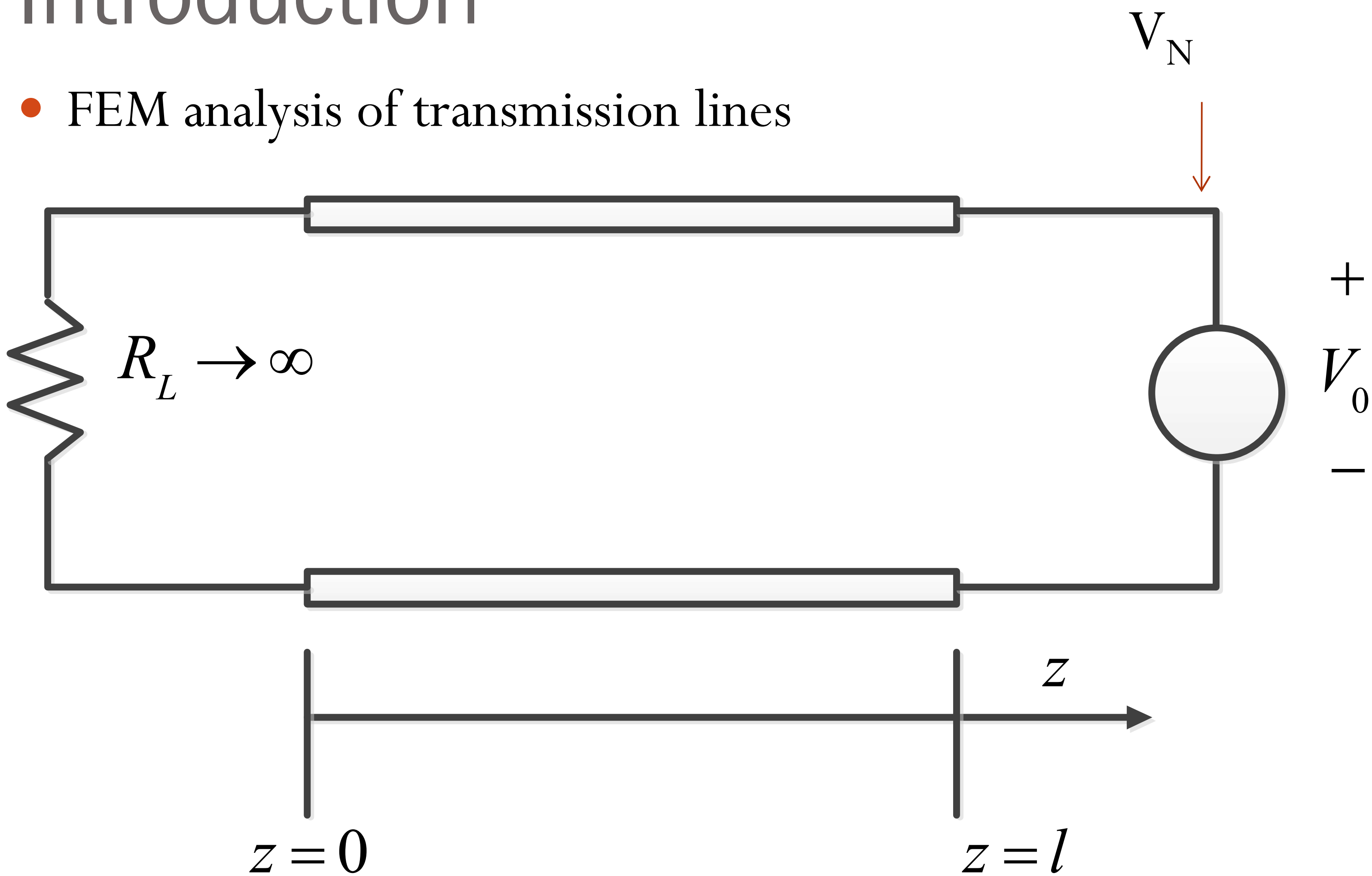
Introduction

- Therefore,

$$
\begin{aligned}
F(V) &= C_{11}V_1^2 + C_{12}V_1V_2 \\
&+ C_{21}V_2V_1 + C_{22}V_2^2 + C_{23}V_2V_3 \\
&+ C_{32}V_3V_2 + C_{33}V_3^2 + C_{34}V_3V_4 \\
&\vdots \\
&\cdots + C_{N-1,N}V_{N-1}V_N \\
&+ C_{N,N-1}V_NV_{N-1} + C_{N,N}V_N^2
\end{aligned}
$$

Introduction

- FEM analysis of transmission lines



Introduction

- V_N is a prescribed node
- Differentiating w.r.t. $V_1, V_2, \ldots, V_{N-1}$, we have,

$$\left.\frac{\partial F}{\partial\{V_1, V_2, \cdots, V_{N-1}\}}\right|^T = \begin{pmatrix} C_{11} & C_{12} & & & \\ C_{21} & C_{22} & C_{23} & & \\ & C_{32} & C_{33} & C_{34} & \\ & & & \ddots & \\ & & & C_{N-1,N-2} & C_{N-1,N-1} \\ & & & & \end{pmatrix} \begin{bmatrix} V_1 \\ \vdots \\ V_{N-1} \end{bmatrix}$$

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- And equating it to zero, we have,

$$\left.\frac{\partial F}{\partial\left\{V_1,V_2,\cdots,V_{N-1}\right\}}\right|^T=\begin{bmatrix}0\\ \vdots\\ 0\\ -C_{N-1,N}V_N\\ \\ \end{bmatrix}$$

Introduction

- **Solution of Poisson's equation**

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

- Deriving element governing equation
- We approximate the potential distribution and source terms by linear combination of local interpolation polynomials α_i

$$V_e = \sum_{i=1}^{3} V_{ei}\alpha_i(x,y); \boxed{\rho_{ve} = \sum_{i=1}^{3} \rho_{ei}\alpha_i(x,y)}$$

- Values of ρ_{ei} are known since $\rho_v(x,y)$ is prescribed and V_{ei} are to be determined

Introduction

- Euler's equation $\nabla^2\Phi + k^2\Phi = g$
- Functional is

$$I(\Phi) = \frac{1}{2}\iint\left[\left|\nabla\Phi\right|^2 - k^2\Phi^2 + 2\Phi g\right]ds$$

- Euler's equation $\nabla^2 V = -\frac{\rho_v}{\varepsilon}$
- Transfer all terms to RHS $-\varepsilon\nabla^2 V - \rho_v = 0$
- Therefore, functional is

$$F(V_e) = \frac{1}{2}\int\left[\varepsilon\left|\nabla V_e\right|^2 - 2\rho_{V_e} V_e\right]ds$$