Introduction

- Euler's equation $\nabla^2\Phi + k^2\Phi = g$
- Functional is

$$I(\Phi) = \frac{1}{2}\iint\left[\left|\nabla\Phi\right|^2 - k^2\Phi^2 + 2\Phi g\right]ds$$

- Euler's equation $$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$$
- Transfer all terms to RHS $-\varepsilon\nabla^2 V - \rho_v = 0$
- Therefore, functional is

$$F(V_e) = \frac{1}{2}\int\left[\varepsilon\left|\nabla V_e\right|^2 - 2\rho_{V_e} V_e\right]ds$$

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- Energy functional for this case is

$$F\left(V_e\right)=\frac{1}{2}\int\left[\varepsilon\left|\nabla V_e\right|^2-2\rho_{V_e}V_e\right]ds$$

- which can be expressed as

$$F\left(V_e\right)=\frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\varepsilon V_{ei}\int\left[\nabla\alpha_i\bullet\nabla\alpha_j\right]dsV_{ej}$$
$$-\sum_{i=1}^{3}\sum_{j=1}^{3}V_{ei}\int\left[\alpha_i\alpha_j\right]ds\rho_{ej}$$

$$V_e=\sum_{i=1}^{3}V_{ei}\alpha_i\left(x,y\right);\rho_{ve}=\sum_{i=1}^{3}\rho_{ei}\alpha_i\left(x,y\right)$$

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$$F(V_e) = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\varepsilon V_{ei}\int\left[\nabla\alpha_i \bullet \nabla\alpha_j\right] ds V_{ej}$$
$$-\sum_{i=1}^{3}\sum_{j=1}^{3} V_{ei}\int\left[\alpha_i\alpha_j\right] ds \rho_{ej}$$

- This can be written as

$$F(V_e) = \frac{1}{2}\varepsilon\left[V_e\right]^T\left[C^{(e)}\right]\left[V_e\right] - \left[V_e\right]^T\left[T^{(e)}\right]\left[\rho_e\right]$$

- where

$$C_{ij}^{(e)} = \int\left[\nabla\alpha_i \bullet \nabla\alpha_j\right] ds; T_{ij}^{(e)} = \int\left[\alpha_i\alpha_j\right] ds$$

- It can be shown that

$$T_{ij}^{(e)} = \begin{cases} A/12, & i \neq j \\ A/6, & i = j \end{cases}$$

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- Hence

$$F(V)=\sum_{e=1}^{N}F(V_e)=\frac{1}{2}\varepsilon[V]^T[C][V]-[V]^T[T][\rho]$$

- Solving the resulting equation:
- Iteration method:

$$\frac{\partial F(V)}{\partial V_k}=0,k=1,2,\cdots,n$$

- minimize the energy

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- For example for 5 node case

$$\frac{\partial F(V)}{\partial V_1} = \varepsilon\left[V_1 C_{11} + V_2 C_{21} + \cdots + V_5 C_{51}\right]$$

$$-\left[T_{11}\rho_1 + \rho_2 T_{21} + \cdots + T_{51}\rho_5\right] = 0$$

- In general, for a free node k

$$V_k = -\frac{1}{C_{kk}} \sum_{i=1, j\neq k}^{n} V_i C_{ki} + \boxed{\frac{1}{\varepsilon C_{kk}} \sum_{i=1}^{n} \rho_i T_{ki}}$$

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- Band matrix method:

$$F(V)=\frac{1}{2}\varepsilon\begin{bmatrix} V_f & V_p \end{bmatrix}\begin{bmatrix} C_{ff} & C_{fp} \\ C_{pf} & C_{pp} \end{bmatrix}\begin{bmatrix} V_f \\ V_p \end{bmatrix}-\begin{bmatrix} V_f & V_p \end{bmatrix}\begin{bmatrix} T_{ff} & T_{fp} \\ T_{pf} & T_{pp} \end{bmatrix}\begin{bmatrix} \rho_f \\ \rho_p \end{bmatrix}$$

- Minimizing F(V) w.r.t. V_f, we have,

$$\frac{\partial F}{\partial V_f}=0$$

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- gives

$$\varepsilon\left(C_{ff}V_f + C_{pf}V_p\right) - \left(T_{ff}\rho_f + T_{fp}\rho_p\right) = 0$$

$$\Rightarrow \left[C_{ff}\right]\left[V_f\right] = -\left[C_{fp}\right]\left[V_p\right] + \frac{1}{\varepsilon}\left(\left[T_{ff}\right]\left[\rho_f\right] + \left[T_{fp}\right]\left[\rho_p\right]\right)$$

- which can be written as

$$\left[A\right]\left[V\right] = \left[B\right]$$

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- where

$$\left[A\right]=\left[C_{ff}\right]$$
$$\left[V\right]=\left[V_f\right]$$
$$\left[B\right]=-\left[C_{fp}\right]\left[V_p\right]+\frac{1}{\varepsilon}\left(\left[T_{ff}\right]\left[\rho_f\right]+\left[T_{fp}\right]\left[\rho_p\right]\right)$$

- Solution of the wave equation:
- A typical wave equation is the inhomogeneous scalar Helmholtz equation

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- **Solution of inhomogeneous wave equation**

$$\nabla^2\Phi + k^2\Phi = g$$

- where Φ is the field quantity (for waveguide problem, $\Phi=H_z$ for TE mode or $\Phi=E_z$ for TM mode) to be determined, g is the source function and $k = \omega\sqrt{\mu\varepsilon}$
- is the wavenumber of the medium
- (1) k=0=g; Laplace's equation
- (2) k=0; Poisson's equation
- (3) g=0, Helmholtz wave equation

Introduction

- Construction of functionals from PDEs
- For LΦ=g is obtained by extremizing the functional

$$I(\Phi)=\frac{1}{2}\iint\left[\left|\nabla\Phi\right|^2-k^2\Phi^2+2\Phi g\right]ds$$

- Now express potential Φ and source function g in terms of shape functions α_i over a triangular element as

$$\Phi_e(x,y)=\sum_{i=1}^{3}\alpha_i\Phi_{ei}$$

$$g_e(x,y)=\sum_{i=1}^{3}\alpha_i g_{ei}$$

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- where Φ_{ei} and g_{ei} are respectively the values of Φ and g at nodal element i of element e
- Hence

$$I(\Phi_e) = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\Phi_{ei}\Phi_{ej}\int\left[\nabla\alpha_i \bullet \nabla\alpha_j\right]ds$$

$$-\frac{k^2}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\Phi_{ei}\Phi_{ej}\int\left[\alpha_i\alpha_j\right]ds + \sum_{i=1}^{3}\sum_{j=1}^{3}\Phi_{ei}g_{ej}\int\left[\alpha_i\alpha_j\right]ds$$

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$$I(\Phi_e) = \frac{1}{2}[\Phi_e]^T \left[C^{(e)}\right][\Phi_e] - \frac{k^2}{2}[\Phi_e]^T \left[T^{(e)}\right][\Phi_e] + [\Phi_e]^T \left[T^{(e)}\right][G_e]$$

- where

$$[\Phi_e] = \begin{bmatrix} \Phi_{e1} \\ \Phi_{e2} \\ \Phi_{e3} \end{bmatrix}; [G_e] = \begin{bmatrix} g_{e1} \\ g_{e2} \\ g_{e3} \end{bmatrix}$$

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- Now

$$I(\Phi)=\sum_{e=1}^{n} I(\Phi_e)=\frac{1}{2}[\Phi]^T[C][\Phi]-\frac{k^2}{2}[\Phi]^T[T][\Phi]+[\Phi]^T[T][G]$$

- where

$$[\Phi]=\begin{bmatrix}\Phi_1\\ \vdots\\ \Phi_N\end{bmatrix};[G_e]=\begin{bmatrix}g_1\\ \vdots\\ g_N\end{bmatrix}$$

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- [C] and [T] are global matrices consisting of local matrices
- $[C^{(e)}]$ and $[T^{(e)}]$ respectively
- When the source function g=0, if the free nodes are numbered first and prescribed nodes last, we have,

$$I(\Phi)=\frac{1}{2}\begin{bmatrix}\Phi_f & \Phi_p\end{bmatrix}\begin{bmatrix}C_{ff} & C_{fp}\\ C_{pf} & C_{pp}\end{bmatrix}\begin{bmatrix}\Phi_f\\ \Phi_p\end{bmatrix}-\frac{k^2}{2}\begin{bmatrix}\Phi_f & \Phi_p\end{bmatrix}\begin{bmatrix}T_{ff} & T_{fp}\\ T_{pf} & T_{pp}\end{bmatrix}\begin{bmatrix}\Phi_f\\ \Phi_p\end{bmatrix}$$

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- Setting $\dfrac{\partial I}{\partial \Phi_f} = 0$

$$\begin{bmatrix} C_{ff} & C_{fp} \end{bmatrix} \begin{bmatrix} \Phi_f \\ \Phi_p \end{bmatrix} - k^2 \begin{bmatrix} T_{ff} & T_{fp} \end{bmatrix} \begin{bmatrix} \Phi_f \\ \Phi_p \end{bmatrix} = 0$$

- For TM modes in a perfectly conducting waveguide, $\Phi_{\text{p}} = 0$ and hence

$$\left[C_{ff} - k^2 T_{ff}\right]\Phi_f = 0$$
$$\Rightarrow \left[T_{ff}^{-1} C_{ff} - k^2 I\right]\Phi_f = 0$$
$$\Rightarrow \left(A - \lambda I\right)x = 0$$

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- where $$A = T_{ff}^{-1} C_{ff}; \lambda = k^2; x = \Phi_f$$
- It looks like an eignevalue problem
- The eigenvalues are real since C and Y matrices are symmetric
- The above equation give eigenvalues and eigenfunctions of the Helmholtz problem