FEM

- **Weighted residual method**
  - PDE is written as
  - $\mathcal{L}\Phi = f$ (1)
    - where $\mathcal{L}$ denotes differential operator
    - Φ is the unknown solution to be found
    - f denote the source functions
  - To find the solution Φ, we first express it in terms of known basis functions
  - $\Phi = \sum_{j=1}^{N} c_j\, b_j$ (2)
    - where b_j is the jth basis function and c_j is the unknown constant

FEM

- Substituting (2) in (1) and integrating w.r.t. weighting functions w_i
- $\int_{\Omega} w_i \, \mathcal{L}\left(\sum_{j=1}^{N} c_j \, b_j\right) d\Omega = \int_{\Omega} w_i f \, d\Omega$
- In Galerkin's method, $w_i = b_i$
- $\sum_{j=1}^{N} c_j \int_{\Omega} b_i \, \mathcal{L}\left(b_j\right) d\Omega = \int_{\Omega} b_i f \, d\Omega \, , i = 1,2, \cdots , N$
- Or,
- $\sum_{j=1}^{N} Z_{ij} c_j = k_i , i = 1,2, \cdots , N$
- where $Z_{ij} = \int_{\Omega} b_i \, \mathcal{L}\left(b_j\right) d\Omega$ and $k_i = \int_{\Omega} b_i f \, d\Omega$

FEM

- **Weighted residual method (1-D example)**
  - Consider 1-D BVP defined by Helmholtz wave equation
  - $\frac{d^2\Phi}{dx^2} + k^2\Phi = f(x), 0 < x < L$ (1)
    - with the boundary conditions
    - $\Phi|_{x=0} = p$
    - $\left[\frac{d\Phi}{dx} + \gamma\Phi\right]_{x=L} = q$
    - Nemann BC is a special case when $\gamma = 0$
  - Like in MoM, Φ is expressed in terms of basis functions
  - $\Phi = \sum_{j=0}^{N} c_j \, b_j$ (2)
    - where b_j is the jth basis function and c_j is the unknown constant

FEM

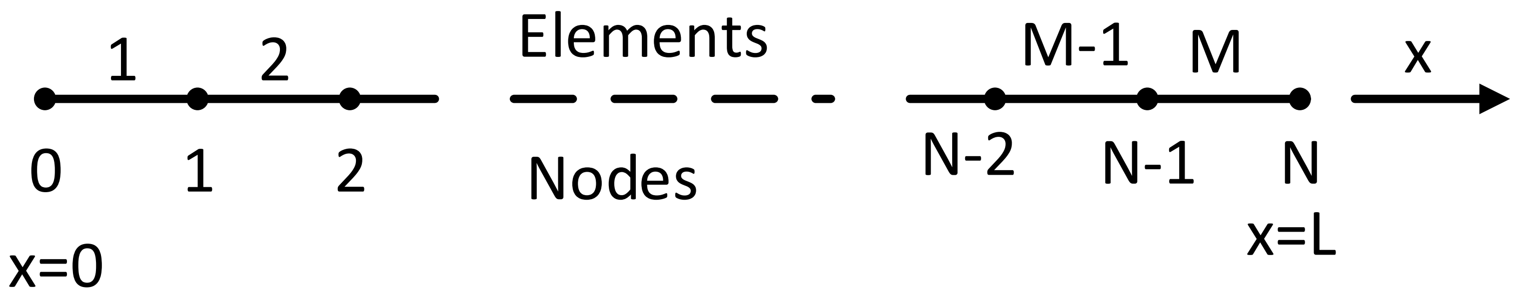

Fig. (a) One-dimensional domain subdivided into linear elements

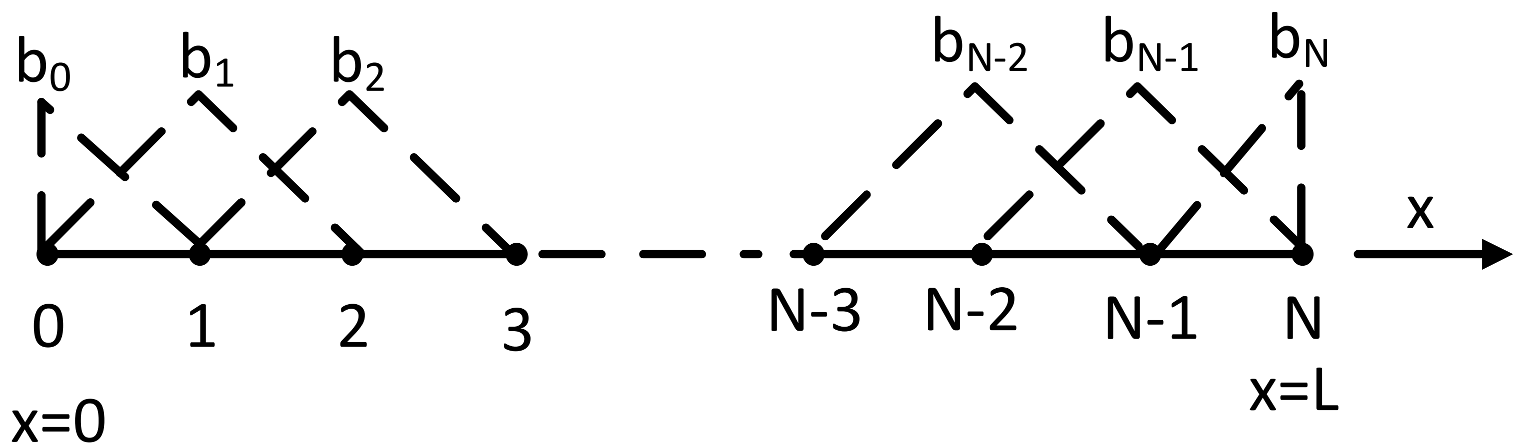

Fig. (b) One-dimensional linear basis function

FEM

- Because of BC
  - $\Phi = \sum_{j=1}^{N} c_j\, b_j + c_0 b_0 = \sum_{j=1}^{N} c_j\, b_j + p b_0$ (2)
- Applying weighting functions for Galerkin's case with i=1,2,…,N
- $\int_0^L b_i \left[\frac{d^2\Phi}{dx^2} + k^2\Phi\right] dx = \int_0^L b_i\, [f(x)] dx$ $\int v du = vu - \int u dv$
- Using integration by parts of the first term in the LHS
- $\int_0^L \left[\frac{db_i}{dx}\frac{d\Phi}{dx} - k^2\Phi b_i\right] dx - \left[b_i \frac{d\Phi}{dx}\right]_{x=L} = -\int_0^L b_i\, [f(x)] dx$
  - We have also used the fact that $b_i = 0$ at x=0 for i=1,2,…,N
- Also application of BC yields
- $\int_0^L \left[\frac{db_i}{dx}\frac{d\Phi}{dx} - k^2\Phi b_i\right] dx - [b_i(q - \gamma\Phi)]_{x=L} =$

$-\int_0^L b_i\, [f(x)] dx$

FEM

- Now we can write in matrix form as
  - $\sum_{j=1}^{N} Z_{ij} c_j = k_i, i = 1,2,\cdots,N$ (2)
- where
- $Z_{ij} = \int_0^L \left[\frac{db_i}{dx}\frac{db_j}{dx} - k^2 b_j b_i\right] dx + \gamma \delta_{iN} \delta_{jN}$
- $k_i = q\delta_{iN} - \int_0^L \left[\frac{db_i}{dx}\frac{db_0}{dx} - k^2 b_i b_0\right] dx - \int_0^L b_i \,[f(x)] dx$
- where $\delta_{iN} = 1$ for $i = N$ and zero for $i \neq N$
- Note that b_i and b_j overlap only for $j = i \pm 1$
- Hence Z_{ij} is non zero only for $Z_{ii}, Z_{i+1,i}, Z_{i+1,i}$
- which is a tridiagonal matrix
- and it is an important property of FEM which can be solved efficiently

Introduction

- **Finite Element Time-Domain Method**
- method combines the advantages of a time-domain technique
- with the versatile spatial discretization options of the finite element method
- A variety of FETD methods have been proposed

Introduction

- In FETD, we will solve second-order vector wave equation
  - by eliminating one of the field variables from Maxwell's equations
- Consider Maxwell's equations in space-time:

$$\nabla \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}; \nabla \times \vec{H} = \vec{J}_{imp} + \varepsilon \frac{\partial \vec{E}}{\partial t} + \sigma \vec{E}$$

- Dividing the second equation by ε, we have,

$$\frac{\nabla \times \vec{H}}{\varepsilon} = \frac{\vec{J}_{imp}}{\varepsilon} + \frac{\partial \vec{E}}{\partial t} + \frac{\sigma}{\varepsilon} \vec{E}$$

Introduction

- Hence, taking curl again, we have,

$$\nabla\times\left(\frac{1}{\varepsilon}\nabla\times\vec{H}\right)=\frac{\nabla\times\vec{J}_{imp}}{\varepsilon}+\frac{\partial\left(\nabla\times\vec{E}\right)}{\partial t}+\frac{\sigma}{\varepsilon}\nabla\times\vec{E}$$

- Substituting $\nabla\times\vec{E}=-\mu\frac{\partial\vec{H}}{\partial t}$ in the 2^{nd} and 3^{rd} term of the RHS, we have,

$$\nabla\times\left(\frac{1}{\varepsilon}\nabla\times\vec{H}\right)=\frac{\nabla\times\vec{J}_{imp}}{\varepsilon}-\mu\frac{\partial^2\vec{H}}{\partial t^2}+\frac{\sigma}{\varepsilon}\left(-\mu\frac{\partial\vec{H}}{\partial t}\right)$$

Introduction

- Transferring 2nd and 3rd term in the RHS to LHS, we have,

$$\nabla \times \left(\frac{1}{\varepsilon} \nabla \times \vec{H} \right) + \mu \frac{\partial^2 \vec{H}}{\partial t^2} + \frac{\sigma \mu}{\varepsilon} \frac{\partial \vec{H}}{\partial t} = \frac{\nabla \times \vec{J}_{imp}}{\varepsilon}$$

- Since

$$\nabla \times \left(\frac{1}{\varepsilon} \nabla \times \vec{H} \right) = \frac{1}{\varepsilon} \nabla \left(\nabla \bullet \vec{H} \right) - \frac{1}{\varepsilon} \nabla^2 \vec{H} = -\frac{1}{\varepsilon} \nabla^2 \vec{H}$$

Introduction

- Hence, putting the above equation and noting the sign change,

$$\frac{1}{\varepsilon}\nabla^2\vec{H} - \mu\frac{\partial^2\vec{H}}{\partial t^2} - \frac{\sigma\mu}{\varepsilon}\frac{\partial\vec{H}}{\partial t} = -\frac{\nabla\times\vec{J}_{imp}}{\varepsilon}$$

- For 2-D region, the scalar wave equation for the longitudinal component of the magnetic field for TE^z case, we have,

$$\frac{1}{\varepsilon}\nabla^2 H_z - \frac{\sigma\mu}{\varepsilon}\frac{\partial H_z}{\partial t} - \mu\frac{\partial^2 H_z}{\partial t^2} = -\frac{\left(\nabla\times\vec{J}_{imp}\right)_z}{\varepsilon}$$

Introduction

- By defining the carrier frequency ω_c, the field component and the current density can be written as

$$H_z(t) = V(t)e^{j\omega_c t}; \vec{J}(t) = \vec{j}(t)e^{j\omega_c t}$$

- where V(t) is the time-varying complex envelope of the field at the carrier frequency
- Substituting the above relation, we have,

$$\frac{1}{\varepsilon_r\varepsilon_0}\nabla^2\left(V(t)e^{j\omega_c t}\right) - \frac{\sigma\mu_r\mu_0}{\varepsilon_r\varepsilon_0}\frac{\partial\left(V(t)e^{j\omega_c t}\right)}{\partial t} - \mu_r\mu_0\frac{\partial^2\left(V(t)e^{j\omega_c t}\right)}{\partial t^2} = -\frac{\left(\nabla\times\left(\vec{j}(t)e^{j\omega_c t}\right)\right)_z}{\varepsilon_r\varepsilon_0}$$

Introduction

- Since

$$\frac{\partial\left(V(t)e^{j\omega_c t}\right)}{\partial t}=e^{j\omega_c t}\frac{\partial\left(V(t)\right)}{\partial t}+V(t)(j\omega_c)e^{j\omega_c t}$$

$$\frac{\partial^2\left(V(t)e^{j\omega_c t}\right)}{\partial t^2}=e^{j\omega_c t}\frac{\partial^2\left(V(t)\right)}{\partial t^2}+(j\omega_c)e^{j\omega_c t}\frac{\partial\left(V(t)\right)}{\partial t}$$

$$+V(t)(j\omega_c)^2 e^{j\omega_c t}+(j\omega_c)e^{j\omega_c t}\frac{\partial V(t)}{\partial t}$$

Introduction

$$\nabla^2\left(V(t)e^{j\omega_c t}\right)=e^{j\omega_c t}\nabla^2\left(V(t)\right)$$

$$\left(\nabla\times\left(j(t)e^{j\omega_c t}\right)\right)_z=e^{j\omega_c t}\left(\nabla\times\left(j(t)\right)\right)_z$$

- Therefore

$$\frac{1}{\varepsilon_r}\nabla^2\left(V(t)\right)-\frac{\mu_r}{c_0^2}\left[\left(\omega_c^2-j\omega_c\alpha\right)-\left(\alpha+2j\omega_c\right)\frac{\partial}{\partial t}-\frac{\partial^2}{\partial t^2}\right]V(t)=\frac{\left(\nabla\times\left(j(t)\right)\right)_z}{\varepsilon_r}$$

- where

$$\alpha=\frac{\sigma}{\varepsilon_r\varepsilon_0}=\frac{\sigma}{\varepsilon}$$

Introduction

- We will call the above equation as envelope equation
- The envelope equation reduces to a scalar Helmholtz equation
  - when V is time independent on which the frequency domain FEM is based
- The inner product with a testing function leads to the weak form

$$\left[\iint_S \frac{1}{\varepsilon_r}\nabla^2\left(V\left(t\right)\right)T-\frac{\mu_r}{c_0^2}\left[\left(\omega_c^2-j\omega_c\alpha\right)-\left(\alpha+2j\omega_c\right)\frac{\partial}{\partial t}-\frac{\partial^2}{\partial t^2}\right]V\left(t\right)T\right]ds$$

$$=\iint_S \frac{\left(\nabla\times\left(\vec{j}\left(t\right)\right)\right)_z}{\varepsilon_r}Tds$$

Introduction

- Since

$$\iint_S \nabla\left(T \bullet \nabla V\right) ds = \iint_S \left(\nabla T \bullet \nabla V\right) ds - \iint_S \left(T \nabla^2 V\right) ds$$

$$\Rightarrow \iint_S \left(T \nabla^2 V\right) ds = \iint_S \left(\nabla T \bullet \nabla V\right) ds - \iint_S \nabla\left(T \bullet \nabla V\right) ds$$

$$\Rightarrow \iint_S \left(T \nabla^2 V\right) ds = \iint_S \left(\nabla T \bullet \nabla V\right) ds - \oint_\Gamma \left(T \bullet \nabla V\right) dl$$

Introduction

- Therefore

$$\left[\iint_S \frac{1}{\varepsilon_r}\nabla T \bullet \nabla V - \frac{\mu_r}{c_0^2}\left[\left(\omega_c^2 - j\omega_c\alpha\right) - \left(\alpha + 2j\omega_c\right)\frac{\partial}{\partial t} - \frac{\partial^2}{\partial t^2}\right]V(t)T\right]ds$$

$$= \iint_S \frac{\left(\nabla \times \left(j(t)\right)\right)_z}{\varepsilon_r} T ds + \oint_\Gamma \frac{1}{\varepsilon_r} T \frac{\partial V}{\partial n} dl$$

Introduction

- Assuming PEC or PMC, the path integral term on the RHS vanishes
- The spatial discretization, the envelope variable is expanded in terms of 2-D FEM basis function w_j ($T_i = w_j$ for Galerkin's)
- Hence it results in a system of ODE

$$\bar{T}\frac{d^2V}{dt^2} + \bar{B}\frac{dV}{dt} + \bar{S}V + \bar{F} = 0$$

$$\left[\iint_S \frac{1}{\varepsilon_r}\nabla T \bullet \nabla V + \frac{\mu_r}{c_0^2}\left[\left(\omega_c^2 - j\omega_c\alpha\right) - \left(\alpha + 2j\omega_c\right)\frac{\partial}{\partial t} - \frac{\partial^2}{\partial t^2}\right]V(t)T\right]ds = \iint_S \frac{\left(\nabla \times \left(\vec{j}(t)\right)\right)_z}{\varepsilon_r} T ds$$

Introduction

$$\left[\iint_S \frac{1}{\varepsilon_r}\nabla T \bullet \nabla V + \frac{\mu_r}{c_0^2}\left[\left(\omega_c^2 - j\omega_c\alpha\right) - \left(\alpha + 2j\omega_c\right)\frac{\partial}{\partial t} - \frac{\partial^2}{\partial t^2}\right]V(t)T\right]ds = \iint_S \frac{\left(\nabla\times\left(\vec{j}(t)\right)\right)_z}{\varepsilon_r}Tds$$

- where

$$\overline{T}_{ij} = \iint_S \frac{\mu_r}{c_0^2} w_i w_j ds$$

$$\overline{B}_{ij} = \iint_S \frac{\mu_r}{c_0^2}\left(2jw_c + \alpha\right) w_i w_j ds$$

$$\overline{S}_{ij} = \iint_S \frac{1}{\varepsilon_r}\left(-\nabla w_i \bullet \nabla w_j - \frac{\mu_r}{c_0^2}\left(w_c^2 - j\alpha w_c\right)\right) w_i w_j ds$$

$$F_i = \iint_S \frac{1}{\varepsilon_r} w_i \left(\nabla\times\vec{j}\right)\Big|_z ds$$

Introduction

- Note that instabilities observed in the time domain FEM and
- that the Newmark-Beta formulation was suggested to solve them

$$\frac{d^2V}{dt^2} = \frac{1}{\Delta t^2}\left[V(n+1) - 2V(n) + V(n-1)\right]$$

$$\frac{dV}{dt} = \frac{1}{2\Delta t}\left[V(n+1) - V(n-1)\right]$$

- The solution of a linear system of equations is required at each time step
- but this implicit method can be formulated to be unconditionally stable

Introduction

- We can use

$$V(n) = \beta V(n+1) + (1-2\beta)V(n) + \beta V(n-1)$$

- where V(n)=V(nΔt) is the discrete time representation of V(t)
- β is a constant that has to be carefully chosen to guarantee stability
- β =1/4 leads to an unconditionally stable two-step update scheme

Introduction

- Substituting the above discretized version of V(t) and its time derivatives and taking β=1/4

$$\left(\frac{1}{\Delta t^2}\bar{T}\left[V(n+1)-2V(n)+V(n-1)\right]\right)+\left(\frac{1}{2\Delta t}\bar{B}\left[V(n+1)-V(n-1)\right]\right)$$

$$+\bar{S}\left(\frac{1}{4}V(n+1)+\frac{1}{2}V(n)+\frac{1}{4}V(n-1)\right)+\bar{F}=0$$

- Hence,

$$\left[\frac{\bar{T}}{\Delta t^2}+\frac{\bar{B}}{2\Delta t}+\frac{\bar{S}}{4}\right]V(n+1)=\left[\frac{2\bar{T}}{\Delta t^2}-\frac{\bar{S}}{2}\right]V(n)+\left[-\frac{\bar{T}}{\Delta t^2}+\frac{\bar{B}}{2\Delta t}-\frac{\bar{S}}{4}\right]V(n-1)-f(n)$$

Introduction

- To solve these equations,
  - we need to invert the matrix on the LHS
- Since this matrix is time independent,
  - it needs to be filled and
  - solved only once
- **Vector edge elements**
- Field can be approximated as

$$\vec{E}_e \approx \sum_{i=1}^{4} \vec{N}_i^e E_i^e$$