10.3 Introductory examples from electrostatics

(c) We may write the above equations in matrix form as

$$\begin{bmatrix} Z_{11} & Z_{12} & \cdots & Z_{1N} \\ Z_{21} & Z_{22} & \cdots & Z_{2N} \\ Z_{31} & Z_{32} & \cdots & Z_{3N} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{N1} & Z_{N2} & \cdots & Z_{NN} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ \vdots \\ I_N \end{bmatrix} = \begin{bmatrix} V_0 \\ V_0 \\ V_0 \\ \vdots \\ V_0 \end{bmatrix} \Rightarrow [Z_{mn}][I_n] = [V_m]$$

- where $[V_m] = [4\pi\varepsilon_0]$

$$Z_{mn} = \int_0^l \frac{b_n(y')dy'}{\sqrt{(y_m - y')^2 + a^2}} = \int_{y_{n-1}}^{y_n} \frac{dy'}{\sqrt{(y_m - y')^2 + a^2}}$$

$$\cong \int_{y_{n-1}}^{y_n} \frac{dy'}{\sqrt{(y_m - y')^2}} = \int_{y_{n-1}}^{y_n} \frac{dy'}{y_m - y'} \approx \frac{\Delta}{|y_m - y_n|} \qquad for \quad m \neq n$$

10.3 Introductory examples from electrostatics

- Special care for calculating the Z_{mn} for m=n case
  - since the expression for Z_{mn} is infinite for this case
- Extraction of this singularity
- Substitute $y_m - y' = \xi \Rightarrow d\xi = -dy'$

$$Z_{mn} = -\int_{\Delta}^{0} \frac{d\xi}{\sqrt{(\xi)^2 + a^2}} = \int_{0}^{\Delta} \frac{d\xi}{\sqrt{(\xi)^2 + a^2}} = \log\left(\xi + \sqrt{(\xi)^2 + a^2}\right)\Big|_0^{\Delta}$$

$$= \ln\left[\frac{\Delta + \sqrt{\Delta^2 + a^2}}{a}\right]$$

10.3 Introductory examples from electrostatics

- Self or diagonal terms are the
  - most dominant elements in the [Z] matrix
- Note that linear geometry of this problem
  - yields a matrix that is symmetric toeplitz, i.e.,

$$
[Z_{mn}] = \begin{bmatrix} Z_{11} & Z_{12} & \cdots & Z_{1N} \\ Z_{12} & Z_{11} & \cdots & Z_{1N-1} \\ . & . & . & . \\ Z_{1N} & Z_{1N-1} & \cdots & Z_{11} \end{bmatrix}
$$

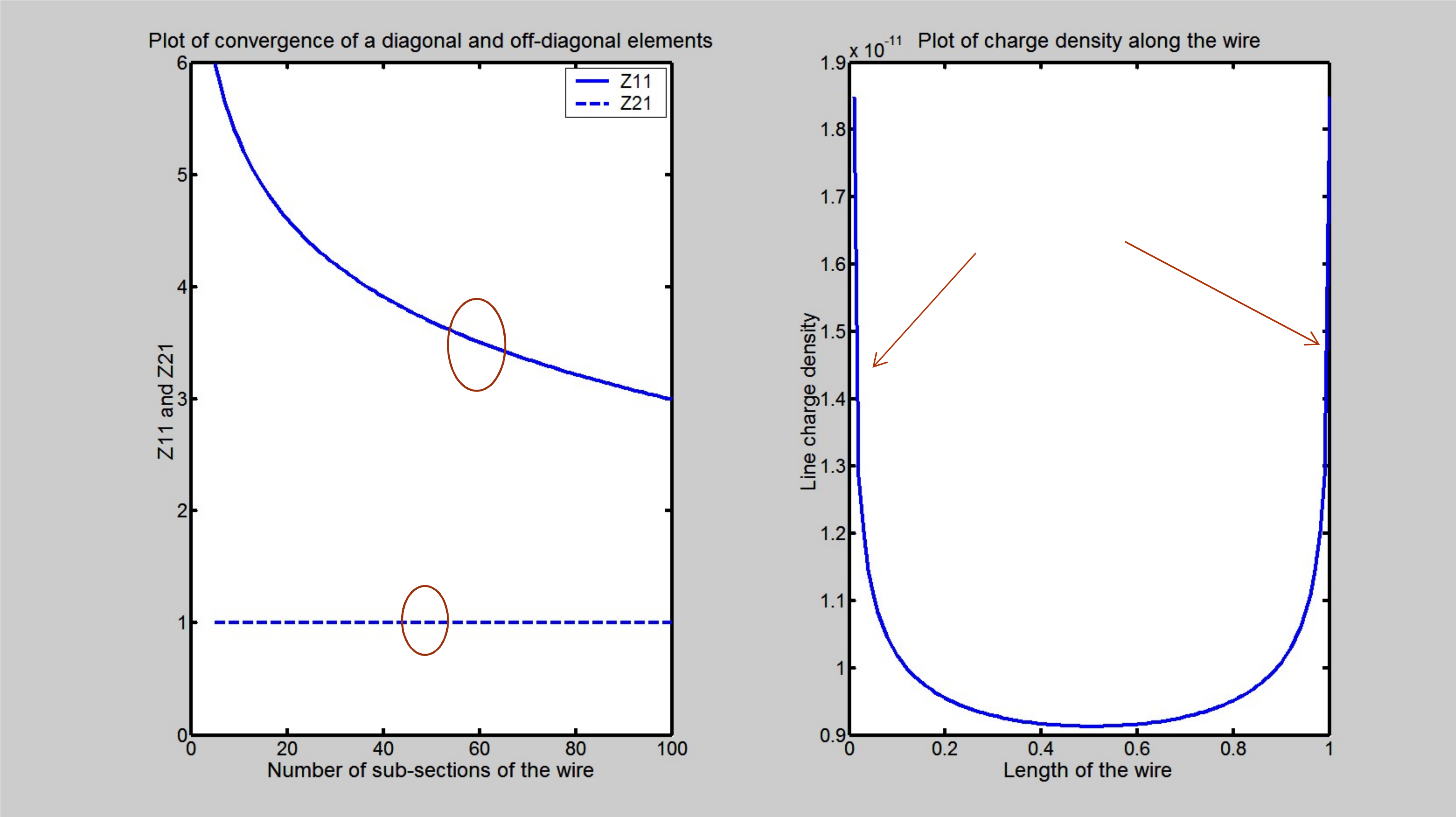
10.3 Introductory examples from electrostatics

- All other rows are a rearranged version of the first row
- Required to calculate the first row of the matrix only
- Remaining elements can be obtained by the rearrangement formula:

$$Z_{mn} = Z_{1,|m-n|+1}, \qquad m \geq 2, \quad n \geq 1$$

- Therefore the unknown [I] matrix could be solved as

$$[I_n] = [Z_{mn}]^{-1}[V_m]$$



- Fig. 10.3 (a) Convergence plot of Z_{11} and Z_{21} (b) Plot of line charge density of the wire (MATLAB program provided in the book)

10.3 Introductory examples from electrostatics

- Let us see the convergence of these two types of elements of the Z matrix say,
  - Z_{11} and Z_{21}
- Fig. 10.3 (a) shows the convergence plot of two elements of the Z matrix
  - for number of sub-sections varying from 5 to 100
- The graph of Z_{21} (dashed line) versus number of sub-sections is a straight line
  - so any number of sub-sections between 5 and 100 should give the same result

10.3 Introductory examples from electrostatics

- But the graph of Z_{11} versus number of sub-sections is
  - decreasing quite fast at the initial values of number of sub-sections and
  - it is decreasing more slowly for larger values of number of sub-sections
- This shows that at
  - higher values of number of sub-sections,
  - we will get a more convergent result
- Choose the maximum number of sub-sections and
  - plot the line charge density as depicted in the Fig. 10.3 (b)

10.3 Introductory examples from electrostatics

- See the condition number of the [Z] matrix in order to see
  - whether the [Z] matrix is well-behaved or not
- The **condition number** of [Z] matrix
  - (=7.1409) for maximum number of sub-sections is good
- No problem in taking the inverse
- Fig. 10.3 (b) line charge density is
  - maximum at the two end points of the wire and
  - minimum at the center of the wire
- 2-D Electrostatic case: Charge density of a square conducting plate discussed in the book

10.4 Some commonly used basis functions

- The weighted sum of basis functions is
  - used to represent the unknown function in MoM
- Choose a basis function that reasonably approximates
  - the unknown function over the given interval
- **Basis functions** commonly used in antenna or scattering problems are of two types:
  - entire domain functions and
  - sub-domain functions

10.4 Some commonly used basis functions

10.4.1 Entire domain basis functions

- The entire domain functions exist over the full domain

$-l/2<x<l/2$

- Some examples are:
- Fourier (is well known) $b_n(x) = \cos\left\{(\frac{n-1}{2})\frac{2x}{l}\right\}$
- Chebyshev (will discuss briefly) $b_n(x) = T_{2n-2}(\frac{2x}{l})$
- Legendre (will discuss briefly) $b_n(x) = P_{2n-2}(\frac{2x}{l})$
- where n=1,2,3,...,N.

10.4 Some commonly used basis functions

- Chebyshev's differential equation

$$\left(1-x^2\right)y'' - xy' + n^2 y = 0$$

- where n is a real number
- Solutions Chebyshev functions of degree n
- n is a non-negative integer, i.e., n=0,1,2,3,…,
  - the Chebyshev functions are called Chebyshev polynomials denoted by $T_n(x)$

10.4 Some commonly used basis functions

- A Chebyshev polynomial at one point can be
  - expressed by neighboring Chebyshev polynomials at the same point

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \qquad T_2(x) = 2xT_1(x) - T_0(x)$$

- where $T_0(x)=1, T_1(x)=x$
- Legendre's differential equation

$$\left(1-x^2\right)y'' - 2xy' + n(n+1)y = 0$$

- where n is a real number

10.4 Some commonly used basis functions

- Solutions of this equation are called Legendre functions of degree n
- When n is a non-negative integer, i.e., n=0,1,2,3,…,
  - the Legendre functions are called Legendre polynomials denoted by $P_n(x)$
- Legendre polynomial at one point can be
  - expressed by neighboring Legendre polynomials at the same point

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$$

  - where $P_0(x)=1$, $P_1(x)=x$

$$2P_2(x) = 3xP_1(x) - P_0(x)$$

10.4 Some commonly used basis functions

- Disadvantage: entire domain basis function may not be applicable of any general problem
  - Choose a particular basis function for a particular problem
  - Crucial and only experts in the area could do it efficiently
- Developing a general purpose MoM based software,
  - software for analyzing almost every problem in electromagnetics
  - this is not feasible
- Sub-domain basis functions could achieve this purpose

10.4 Some commonly used basis functions

10.4.2 Sub-domain basis functions

- Sub-domain basis functions exist only on one of the N overlapping segments
  - into which the domain is divided
- Some examples are:
- Piecewise constant function (pulse)

$$b_n(x) = \begin{cases} 1 & x[n-1] < x < x[n] \\ 0 & otherwise \end{cases}$$

10.4 Some commonly used basis functions

- Piecewise triangular function

$$b_n(x) = \begin{cases} \dfrac{\Delta - |x - x_n|}{\Delta} & x[n-1] < x < x[n+1] \\ 0 & otherwise \end{cases}$$

$$= \begin{cases} \dfrac{x - x_{n-1}}{x_n - x_{n-1}} & x[n-1] < x < x[n] \\ \dfrac{x_{n+1} - x}{x_{n+1} - x_n} & x[n] < x < x[n+1] \\ 0 & otherwise \end{cases}$$

10.4 Some commonly used basis functions

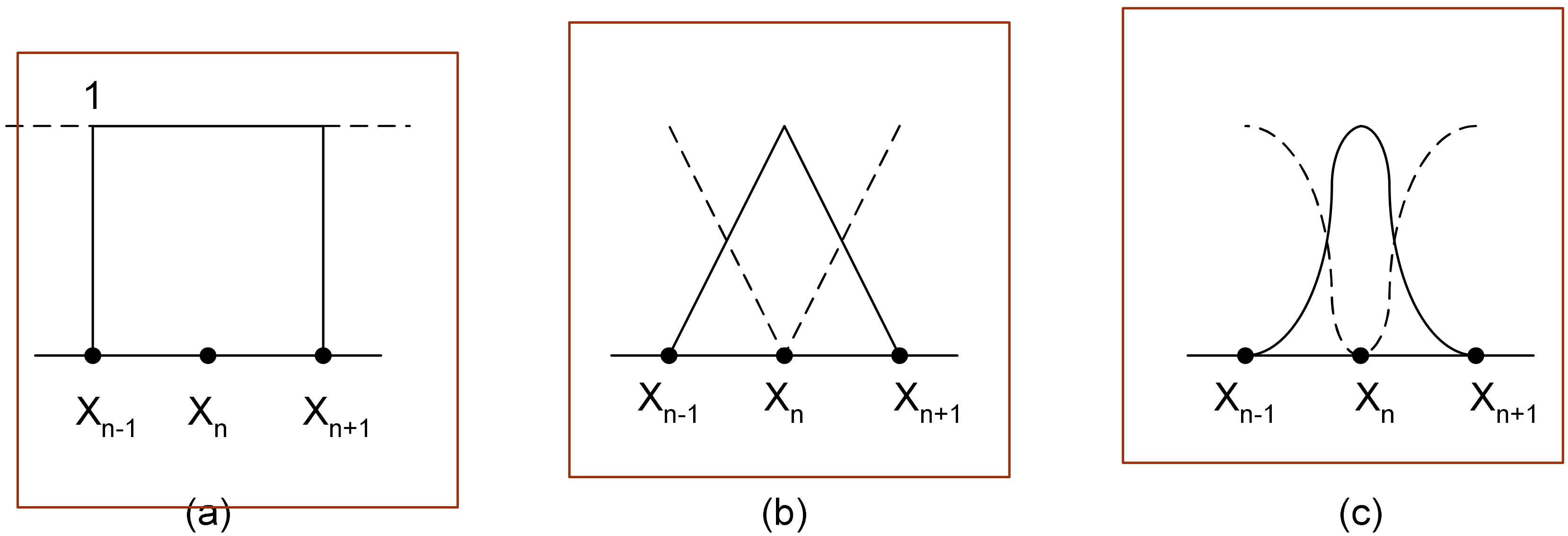
- Piecewise sinusoidal function

$$b_n(x) = \begin{cases} \dfrac{\sin\left\{k\left(\Delta - |x - x_n|\right)\right\}}{\sin\left(k\Delta\right)} & x[n-1] < x < x[n+1] \\ 0 & otherwise \end{cases}$$

$$= \begin{cases} \dfrac{\sin\left\{k\left(x - x_n\right)\right\}}{\sin\left\{k\left(x_n - x_{n-1}\right)\right\}} & x[n-1] < x < x[n] \\ \dfrac{\sin\left\{k\left(x_{n+1} - x\right)\right\}}{\sin\left\{k\left(x_n - x_{n-1}\right)\right\}} & x[n] < x < x[n+1] \\ 0 & otherwise \end{cases}$$

- where $\Delta = l/N$, assuming equal subintervals but it is not mandatory and k is a constant

10.4 Some commonly used basis functions



- Fig. 10.5 Sub-domain basis functions (a) Piecewise constant function (b) Piecewise triangular function (c) Piecewise sinusoidal function

10.4 Some commonly used basis functions

- Since the derivative of the pulse function is impulsive
  - we cannot employ it for MoM problems
    - where the linear operator L consists of derivatives
- Piecewise triangular and sinusoidal functions
  - may be used for such kinds of problems
- Piecewise sinusoidal functions are generally used
  - for analysis of wire antennas since
  - they can approximate sinusoidal currents in the wire antennas

10.5 Wire Antennas and Scatterers

- For Piece-wise triangular and sinusoidal functions
- when we have N points in an interval
- we will have N-1 sub-sections and
- N-2 basis functions may be used

Programming exercise 2 (Homework)

- Plot the following entire domain and sub-domain basis functions

10.5 Wire Antennas and Scatterers

- Chebyshev function (order n=5)
- Legendre function (order n=3)
- Piece wise constant function
- Piece wise triangular function
- Piece wise sinusoidal function

10.5 Wire Antennas and Scatterers

- Consider application of MoM techniques
  - to wire antennas and scatterers

10.5 Wire Antennas and Scatterers

- Antennas can be distinguished from scatterers
  - in terms of the location of the source
- If the source is on the wire
  - it is regarded as antenna
- When the wire is far from the source
  - it acts as scatterer
- For the wire objects (antenna or scatterer)
  - we require to know the current distribution accurately

10.5 Wire Antennas and Scatterers

- Integral equations are derived and
  - solved for this purpose

Wire antennas

- Feed voltage to an antenna is known
  - and the current distribution could be calculated
- other antenna parameters such as
  - impedance,
  - radiation pattern, etc.
- can be calculated

10.5 Wire Antennas and Scatterers

Wire scatterers

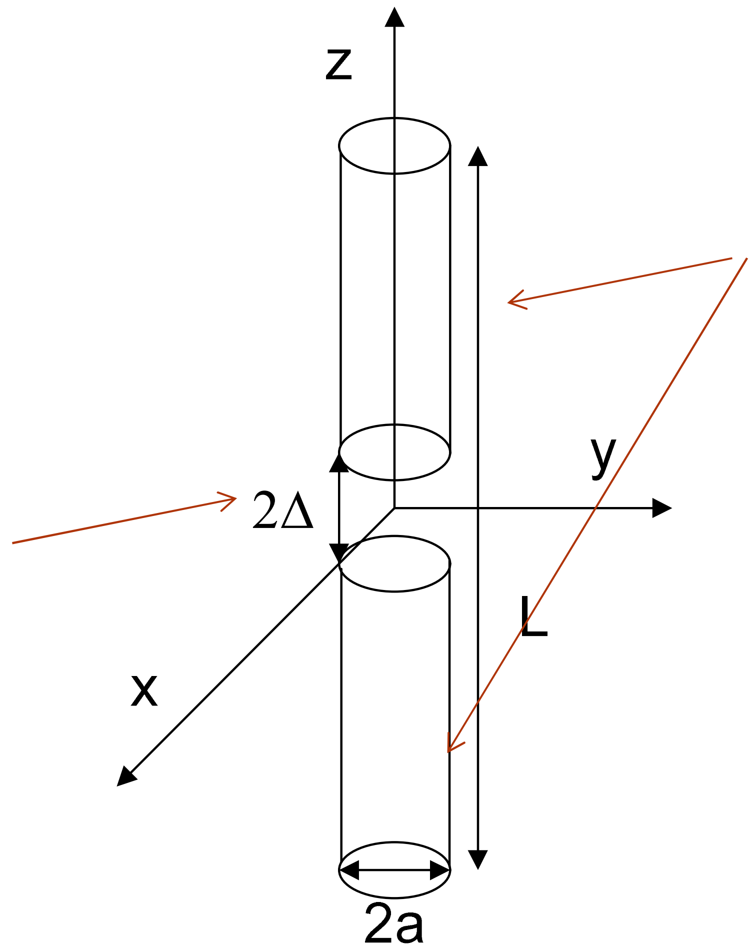
- Wave impinges upon surface of a wire scatterer
  - it induces current density
  - which in turn is used to generate the scattered fields
- We will consider
- how to find the current distribution on a
  - thin wire or
  - cylindrical antenna
- using the MoM

10.5 Wire Antennas and Scatterers

10.5.1 Electric field integral equation (EFIE)

- On perfect electric conductor like metal
  - the total tangential electric field is zero
- Centrally excited cylindrical antenna (Fig. 10.6)
- have two kinds of electric fields viz.,
  - incident and
  - scattered electric fields

$$\vec{E}_{\tan}^{tot} = 0 \Rightarrow \vec{E}_{\tan}^{inc} + \vec{E}_{\tan}^{scat} = 0 \Rightarrow \vec{E}_{\tan}^{inc} = -\vec{E}_{\tan}^{scat}$$



- Fig. 10.6 A thin wire antenna of length L, radius a (a<<L) and feed gap 2Δ

10.5 Wire Antennas and Scatterers

- where the $\vec{E}^{inc}$ is the source or impressed field and
- $\vec{E}^{scat}$ can be computed from the
- current density induced on the cylindrical wire antenna due to the
  - incident or
  - impressed field

10.5.2 Hallen's and Pocklington's Integro-differential equation

- Let us consider a perfectly conducting wire of
  - length L and
  - radius a such that a<<L and λ, the wavelength corresponding to the operating frequency

10.5 Wire Antennas and Scatterers

- Consider the wire to be a hollow metal tube
  - open at both ends
- Let us assume that an incident wave $\vec{E}^{inc}(\vec{r})$
  - impinges on the surface of a wire
- When the wire is an antenna
  - the incident field is produced by the feed at the gap (see Fig. 10.6)
- The impressed field E_z^{inc} is required
  - to be known on the surface of the wire

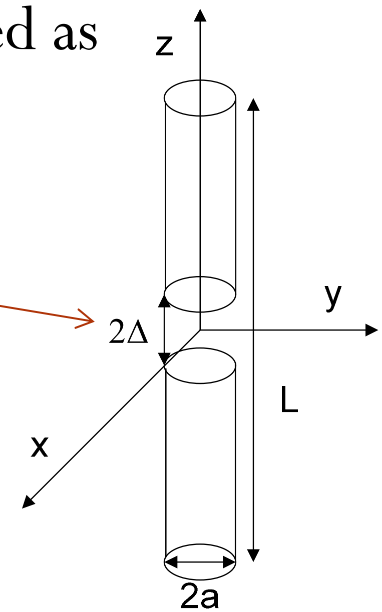
10.5 Wire Antennas and Scatterers

- Simplest excitation
  - delta-gap excitation
- For delta gap excitation (assumption)
  - excitation voltage at the feed terminal is constant and
  - zero elsewhere
- Implies incident field
  - constant over the feed gap and
  - zero elsewhere

10.5 Wire Antennas and Scatterers

- $2V_0$ (from $+V_0$ to $-V_0$) voltage source applied
  - across the feed gap 2Δ,
- Incident field on the wire antenna can be expressed as

$$\vec{E}_z^{inc} = \begin{cases} \dfrac{V_0}{\Delta}; & |z| < \Delta \\ 0; & \Delta < |z| < \dfrac{L}{2} \end{cases}$$



- Induced current density
  - due to the incident or impressed electric field
- produces the scattered electric field $\vec{E}^{scat}(\vec{r})$

10.5 Wire Antennas and Scatterers

- The total electric field is given by

$$\vec{E}^{tot}(\vec{r}) = \vec{E}^{inc}(\vec{r}) + \vec{E}^{scat}(\vec{r})$$

- Since the wire is assumed to be perfectly conducting,
  - tangential component of the total electric field on the surface of the wire is zero
- For a cylindrical wire placed along z-axis, we can write,

$$\vec{E}_z^{tot}(\vec{r}) = \vec{E}_z^{inc}(\vec{r}) + \vec{E}_z^{scat}(\vec{r}) = 0; \quad \textit{on the wire antenna}$$

10.5 Wire Antennas and Scatterers

- that is,

$$\vec{E}_z^{scat}(\vec{r}) = -\vec{E}_z^{inc}(\vec{r})$$

- Find the electric field from the potential functions using

$$\vec{E} = -j\omega\vec{A} - \nabla V$$

- Lorentz Gauge condition,

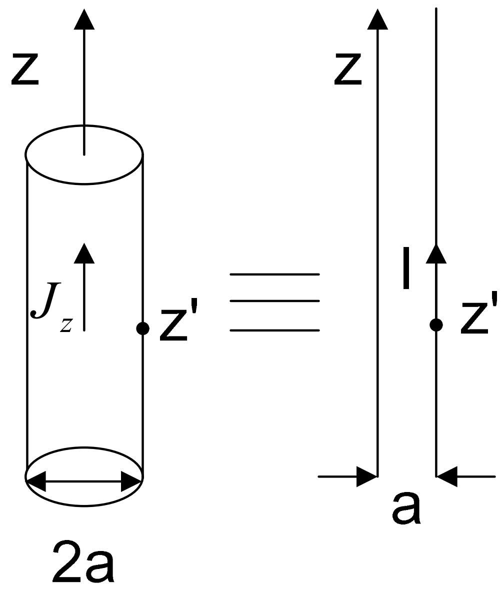
$$\nabla \bullet \vec{A} = -j\omega\mu_0\varepsilon_0 V$$

10.5 Wire Antennas and Scatterers

- For a thin cylinder,
- current density considered to be independent of ϕ

$$J_z(z') = \frac{1}{2\pi a} I(z')$$

- where is $J_z(z')$ the surface current density
  - at a point on the conductor z'
- skin depth of the perfect conductor is almost zero
  - and therefore all the currents flow on the surface of the wire



- Fig. 10.7 Cylindrical conductor of radius a with surface current density $J_z(z')\left(\frac{A}{m}\right)$
- and its equivalence to the case of the conductor replaced by current filament $I(z') = 2\pi a J_z(z')(A)$ at a distance a from the z-axis

10.5 Wire Antennas and Scatterers

- The current $I(z')$ may be assumed to be
- a filamentary current located parallel to z-axis
- at a distance *a* (a is a very small number) as shown in the Fig. 10.7
- For the current flowing only in the z direction,

$$E_z = -j\omega A_z - \frac{\partial V}{\partial z}$$

- From Lorentz Gauge condition for time harmonic case,

$$\frac{\partial A_z}{\partial z} = -j\omega\mu_0\varepsilon_0 V \Rightarrow \frac{\partial^2 A_z}{\partial z^2} = -j\omega\mu_0\varepsilon_0 \frac{\partial V}{\partial z} \Rightarrow -\frac{\partial V}{\partial z} = \frac{1}{j\omega\mu_0\varepsilon_0}\frac{\partial^2 A_z}{\partial z^2}$$

10.5 Wire Antennas and Scatterers

- Therefore,

$$E_z = -j\omega A_z + \frac{1}{j\omega\mu_0\varepsilon_0}\frac{\partial^2 A_z}{\partial z^2} = \frac{1}{j\omega\mu_0\varepsilon_0}\left(\omega^2\mu_0\varepsilon_0 A_z + \frac{\partial^2 A_z}{\partial z^2}\right) = \frac{1}{j\omega\mu_0\varepsilon_0}\left(\beta_0^2 A_z + \frac{\partial^2 A_z}{\partial z^2}\right)$$

- Magnetic vector potential can be expressed as

$$A_z = \mu_0 \iint_S J_z \frac{e^{-j\beta_0 r}}{4\pi r} ds'$$

- Putting the J_z expression from (10.20), we have,

$$A_z = \mu_0 \int_{-L/2}^{L/2} \int_0^{2\pi} \frac{I(z')}{2\pi a} \frac{e^{-j\beta_0 r}}{4\pi r} a d\phi' dz'$$

10.5 Wire Antennas and Scatterers

- where

$$r=\sqrt{\left(x-x'\right)^2+\left(y-y'\right)^2+\left(z-z'\right)^2}$$

- For ρ =a

$$A_z(\rho=a)=\mu_0\int_{-L/2}^{L/2}I(z')\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{-j\beta_0 r}}{4\pi r}d\phi'\right)dz'$$

- where

$$r(\rho=a)=\sqrt{4a^2\sin^2\left(\frac{\varphi'}{2}\right)+\left(z-z'\right)^2}$$

10.5 Wire Antennas and Scatterers

- Therefore, we can write

$$A_z(\rho = a) = \mu_0 \int_{-L/2}^{L/2} I(z')G(z,z')dz'$$

- where

$$G(z,z') = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-j\beta_0 r}}{4\pi r} d\phi'$$

- $G(\vec{r},\vec{r}\,')$ is the field at the observation point caused by a unit point source placed at $\vec{r}\,'$

10.5 Wire Antennas and Scatterers

- The field at $\vec{r}$ by a source distribution $J(\vec{r}^{\,'})$
  - is the integral of $J(\vec{r}^{\,'})G(\vec{r},\vec{r}^{\,'})$ over the range of $\vec{r}^{\,'}$ occupied by the source
- The function G is called the Green's function
- We have,

$$E_z = \frac{1}{j\omega\varepsilon_0\mu_0}\left(\frac{\partial^2 A_z}{\partial z^2} + \beta_0^2 A_z\right)$$

- and

$$A_z(\rho = a) = \mu_0 \int_{-L/2}^{L/2} I(z^{'})\left(\frac{1}{2\pi}\int_0^{2\pi}\frac{e^{-j\beta_0 r}}{4\pi r}d\phi^{'}\right)dz^{'}$$

10.5 Wire Antennas and Scatterers

- From the above two equations we can write, two equations:

  (a)

$$E_z = \frac{1}{j\omega\varepsilon_0}\left(\frac{\partial^2}{\partial z^2} + \beta_0^2\right)\int_{-L/2}^{L/2} I(z^{'})G(z,z^{'})dz^{'}$$

- This electric field is the field due to current $I(z^{'})$
  - [which results because of the impressed or source field] and
  - this field can be written as the scattered field
- Therefore,

$$E_z^{scat} = \frac{1}{j\omega\varepsilon_0}\left(\frac{\partial^2}{\partial z^2} + \beta_0^2\right)\int_{-L/2}^{L/2} I(z^{'})G(z,z^{'})dz^{'}$$

10.5 Wire Antennas and Scatterers

- Since from the EFIE on the surface of the wire,

$$E_z^{scat}\left(\rho=a\right)=-E_z^{inc}\left(\rho=a\right)$$

$$\frac{1}{j\omega\varepsilon_0}\left(\frac{\partial^2}{\partial z^2}+\beta_0^2\right)\int_{-L/2}^{L/2} I(z^{'})G(z,z^{'})dz^{'}=-E_z^{inc}(\rho=a)$$

- This equation is called the *Hallen's Integro-differential equation*
- In this case, differential is outside the integral

10.5 Wire Antennas and Scatterers

(b)
$$E_z = \frac{1}{j\omega\varepsilon_0} \int_{-L/2}^{L/2} \left(\frac{\partial^2}{\partial z^2} + \beta_0^2 \right) I(z') G(z, z') dz'$$

- This electric field is the field due to current $I(z')$
  - [which results because of the impressed or source field] and
  - this field can be written as the scattered field
- Therefore,

$$E_z^{scat} = \frac{1}{j\omega\varepsilon_0} \int_{-L/2}^{L/2} \left(\frac{\partial^2}{\partial z^2} + \beta_0^2 \right) I(z') G(z, z') dz'$$

10.5 Wire Antennas and Scatterers

- Since from EFIE on the surface of the wire,

$$E_z^{scat}(\rho = a) = -E_z^{inc}(\rho = a)$$

$$\frac{1}{j\omega\varepsilon_0}\int_{-L/2}^{L/2}\left(\frac{\partial^2}{\partial z^2}+\beta_0^2\right)I(z')G(z,z')dz' = -E_z^{inc}(\rho = a)$$

- This equation is the *Pocklington's Integro-differential equation*
- In this case, the differential has moved inside the integral
- Richmond has simplified the above equation as follows:

10.5 Wire Antennas and Scatterers

- (c) In cylindrical coordinates,

$$r = \left|\vec{r} - \vec{r}^{'}\right| = \left|\sqrt{\left(z - z^{'}\right)^{2} + \left|\vec{\rho} - \vec{\rho}^{'}\right|^{2}}\right|$$

$$\because \rho^{'} = a \therefore \left|\vec{\rho} - \vec{\rho}^{'}\right| = \rho^{2} + a^{2} - 2\vec{\rho} \bullet \vec{\rho}^{'} = \rho^{2} + a^{2} - 2\rho a \cos\left(\phi - \phi^{'}\right)$$

$$\Rightarrow r = \left|\vec{r} - \vec{r}^{'}\right| = \sqrt{\rho^{2} + a^{2} - 2\rho a \cos\left(\phi - \phi^{'}\right) + \left(z - z^{'}\right)^{2}}$$

- Problem under analysis has cylindrical symmetry and
  - observation for any values of ϕ won't make any difference
  - we may assume without loss of generality $\phi = 0$
  - hence $\phi - \phi^{'} = \phi^{'}$

10.5 Wire Antennas and Scatterers

$$A_z = \mu_0 \int_{-L/2}^{L/2} \frac{I(z^{'})}{2\pi} \int_0^{2\pi} \frac{e^{-j\beta_0 r}}{4\pi r} d\phi^{'} dz^{'}$$

- where $r = \sqrt{\rho^2 + a^2 - 2\rho a \cos\left(\phi^{'}\right) + \left(z - z^{'}\right)^2}$
- and the inner integration $\int_0^{2\pi} \frac{e^{-j\beta_0 r}}{4\pi r} d\phi^{'}$
  - is also referred to as cylindrical wire kernel
- Thin wire approximation $r \cong \sqrt{\rho^2 + \left(z - z^{'}\right)^2}$
- If we assume a<<λ and is very small, we have, $\phi^{'}$
- Inner integrand is no more dependent on the variable

10.5 Wire Antennas and Scatterers

- Therefore

$$A_z = \mu_0 \int_{-L/2}^{L/2} \frac{I(z^{'})e^{-j\beta_0 r}}{4\pi r} dz^{'}$$

- Also called as thin wire approximation
  - with the reduced kernel
- For this case, we can write

$$G\left(z, z^{'}\right) \cong \frac{e^{-j\beta_0 r}}{4\pi r} = G\left(r\right)$$

10.5 Wire Antennas and Scatterers

- Now in the light of this simplification of the magnetic vector potential,
- we can simplify equation 10.29c (see example 10.4) as follows:

$$\frac{1}{j\omega\varepsilon_0 4\pi}\int_{-L/2}^{L/2} I_z(z')\frac{e^{-j\beta_0 r}}{r^5}\left[\left(1+j\beta_0 r\right)\left(2r^2-3a^2\right)+\left(\beta_0 a r\right)^2\right]dz' = -E_z^{inc}(\rho = a)$$

- This form of the Pocklington's integro-differential is more suitable for MoM formulation
  - since it does not involve any differentiation.