Introduction

- **Finite element analysis of vector fields**
- FEM can be extended to deal BVP involving vector fields
- It is very important for electrodynamic problems
  - which deal with vector electromagnetic fields
- **BVP:**
- Consider the problem of finding the electric field intensity $\vec{E}$
  - due to impressed electric current density $\vec{J}_{imp}$ in a domain Ω
    - characterized by permittivity ε and permeability μ
- To calculate $\vec{E}$ we need to solve Maxwell's equations

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- $\nabla \times \vec{E} = -j\omega\mu\vec{H}$ (1)
- $\nabla \times \vec{H} = j\omega\epsilon\vec{E} + \vec{J}_{imp}$ (2)
- $\nabla \cdot (\epsilon\vec{E}) = -\frac{1}{j\omega}\nabla \cdot \vec{J}_{imp}$ $\because \nabla \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t} = -j\omega\rho_v$
- $\nabla \cdot (\mu\vec{H}) = 0$
  - subject to certain BCs
- In first equation divide by μ_r we have
  - $\left(\frac{1}{\mu_r}\nabla \times \vec{E}\right) = -j\omega\mu_0\vec{H}$

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- Taking another curl
  - $\nabla \times \left(\frac{1}{\mu_r} \nabla \times \vec{E}\right) = -j\omega\mu_0(\nabla \times \vec{H})$
- Putting second equation
  - $\nabla \times \vec{H} = j\omega\epsilon\vec{E} + \vec{J}_{imp}$
- in
  - $\nabla \times \left(\frac{1}{\mu_r} \nabla \times \vec{E}\right) = -j\omega\mu_0(\nabla \times \vec{H})$
- we have
  - $\nabla \times \left(\frac{1}{\mu_r} \nabla \times \vec{E}\right) = -j\omega\mu_0(j\omega\epsilon\vec{E} + \vec{J}_{imp})$

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- $\nabla \times \left(\frac{1}{\mu_r}\nabla \times \vec{E}\right) = \omega^2\mu_0\epsilon_0\epsilon_r\vec{E} - j\omega\mu_0\vec{J}_{imp}$
- $\nabla \times \left(\frac{1}{\mu_r}\nabla \times \vec{E}\right) = k_0^2\epsilon_r\vec{E} - j\omega\mu_0\vec{J}_{imp}$
  - where $k_0^2 = \omega^2\mu_0\epsilon_0, Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$ and
- hence $Z_0k_0 = \omega\sqrt{\mu_0\epsilon_0}\sqrt{\frac{\mu_0}{\epsilon_0}} = \omega\mu_0$
- So
  - $\nabla \times \left(\frac{1}{\mu_r}\nabla \times \vec{E}\right) - k_0^2\epsilon_r\vec{E} = -jZ_0k_0\vec{J}_{imp}$

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- The functional for such electrodynamic problems[1] is
  - $F(\vec{E}) = \int_{\Omega} \left[\frac{1}{\mu_r}(\nabla \times \vec{E}) \cdot (\nabla \times \vec{E}) - k_0^2 \epsilon_r \vec{E} \cdot \vec{E}\right] d\Omega + jZ_0 k_0 \int \vec{E} \cdot \vec{J}_{imp} d\Omega$
- For source free vector wave equation
  - $\nabla \times \left(\frac{1}{\mu_r} \nabla \times \vec{E}\right) - k_i^2 \epsilon_r \vec{E} = 0$
- The functional is
  - $F(\vec{E}) = \int_{\Omega} \left[\frac{1}{\mu_r}(\nabla \times \vec{E}) \cdot (\nabla \times \vec{E}) - k_i^2 \epsilon_r \vec{E} \cdot \vec{E}\right] d\Omega$
- [1] J.-M. Jin, *The Finite Element Method in Electromagnetics*, John Wiley & Sons, 2002

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- The solution is the set of eignevalues k_i and associated eigenvectors $\vec{E}_i$
- **High frequency variational function**
- The 2-D variational functional analysis for a homogeneous waveguide
  - For homogeneously filled waveguide, eigenmode splits into TE and TM modes
- The functional for the transverse field components subject to the prescribed boundary condition
  - $\hat{n} \times \vec{E}_t = 0$
- is given on next slide

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- $F(\vec{E}_t) = \int\int_S \left[\frac{1}{\mu_r}(\nabla_t \times \vec{E}_t)\cdot(\nabla_t \times \vec{E}_t) - k_0^2\epsilon_r\vec{E}_t \cdot \vec{E}_t\right] dS$
- Considering TE modes
  - $E_z = 0$
- The vectors of unknowns $\{e\}$ in the generalized eigenvalue problem is found as
  - $[S]\{e\} = k^2[T]\{e\}$
- which will also include prescribed values also

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- In order include only contribution from free edges
  - $[S_{ff}]\{e_f\} = k^2[T_{ff}]\{e_f\}$
- which will also include prescribed nodes also
- This can be obtained from the functional
  - $F\{e\} = \{e_f e_p\}^T \begin{bmatrix} S_{ff} & S_{fp} \\ S_{pf} & S_{pp} \end{bmatrix} \{e_f e_p\}$
  - Differentiate $F\{e\}$ w.r.t. $\{e_f\}$
  - then applying the prescribed conditions $\{e_p\} = 0$
- We can get the above relation
  - $[S_{ff}]\{e_f\} = k^2[T_{ff}]\{e_f\}$

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- Explicit formula for the elemental matrix entries
  - $S_{ij} = \int \int_S \left(\nabla_t \times \vec{N}_i\right) \cdot \left(\nabla_t \times \vec{N}_j\right) dS$
- and
  - $T_{ij} = \int \int_S \; \vec{N}_i \cdot \vec{N}_j \, dS$
- For wave propagating along z-axis,
  - ∇_t and $\nabla_t \times$ are operators in the x-y plane
- The three simplex coordinates are given by
  - $\lambda_i = a_i + b_i x + c_i y$

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- Therefore,
  - $\lambda_1 = a_1 + b_1 x + c_1 y$
  - $\lambda_2 = a_2 + b_2 x + c_2 y$
  - $\lambda_3 = a_3 + b_3 x + c_3 y$
- In matrix form
- $\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} b_1 & c_1 & a_1 \\ b_2 & c_2 & a_2 \\ b_3 & c_3 & a_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$
- Also we know that λ_i is equal to 1 for node *i* and zero for other nodes

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- In matrix form
- $$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}b_1 & c_1 & a_1\\b_2 & c_2 & a_2\\b_3 & c_3 & a_3\end{bmatrix}\begin{bmatrix}x_1 & x_2 & x_3\\y_1 & y_2 & y_3\\1 & 1 & 1\end{bmatrix}$$
- Hence we can find (b_i, c_i, a_i) as follows
- $$\begin{bmatrix}b_1 & c_1 & a_1\\b_2 & c_2 & a_2\\b_3 & c_3 & a_3\end{bmatrix} = \begin{bmatrix}x_1 & x_2 & x_3\\y_1 & y_2 & y_3\\1 & 1 & 1\end{bmatrix}^{-1}$$
- Let us define
- $\vec{v}_{ij} = \nabla\lambda_i \times \nabla\lambda_j = (b_i\hat{x} + c_i\hat{y}) \times (b_j\hat{x} + c_j\hat{y}) = (b_i c_j - b_j c_i)\hat{z} = -\vec{v}_{ji}$

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- Let us also define
- $\phi_{ij} = \nabla\lambda_i \cdot \nabla\lambda_j = (b_i\hat{x} + c_i\hat{y}) \cdot \left(b_j\hat{x} + c_j\hat{y}\right) = b_i b_j + c_i c_j$
- Now we can find curl of $\vec{N}_i = \vec{w}_{i1,i2} = \lambda_{i1}\nabla\lambda_{i2} - \lambda_{i2}\nabla\lambda_{i1}$ where i_1 and i_2 are the endpoints of edge *i* as follows
- $\nabla \times \vec{N}_i = \nabla \times (\lambda_{i1}\nabla\lambda_{i2} - \lambda_{i2}\nabla\lambda_{i1}) = \nabla \times (\lambda_{i1}\nabla\lambda_{i2}) - \nabla \times (\lambda_{i2}\nabla\lambda_{i1})$
- Note that $\nabla \times \left(\phi\vec{A}\right) = \phi\nabla \times \left(\vec{A}\right) + \nabla\phi \times \left(\vec{A}\right)$
- And since $\nabla \times (\nabla\Psi) = 0$ where Ψ is scalar function, we have,
- $\nabla \times (\lambda_{i1}\nabla\lambda_{i2}) = \nabla\lambda_{i1} \times \nabla\lambda_{i2}$ and $\nabla \times (\lambda_{i2}\nabla\lambda_{i1}) = \nabla\lambda_{i2} \times \nabla\lambda_{i1} = -\nabla\lambda_{i1} \times \nabla\lambda_{i2}$
- $\therefore \nabla \times \vec{N}_i = 2\nabla\lambda_{i1} \times \nabla\lambda_{i2} = 2\vec{v}_{i1,i2}$

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- Hence
- $S_{ij} = \int\int_S \left(\nabla_t \times \vec{N}_i\right) \cdot \left(\nabla_t \times \vec{N}_j\right) dS =$

$4\int\int_S \left(\vec{v}_{i1,i2}\right) \cdot \left(\vec{v}_{j1,j2}\right) dS = 4A\left(\vec{v}_{i1,i2}\right) \cdot \left(\vec{v}_{j1,j2}\right)$

- where $A = |A'|$ and A' is defined as
- $A' = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$
- Also note that $\vec{N}_i \cdot \vec{N}_j = \left(\lambda_{i1}\nabla\lambda_{i2} - \lambda_{i2}\nabla\lambda_{i1}\right) \cdot \left(\lambda_{j1}\nabla\lambda_{j2} - \lambda_{j2}\nabla\lambda_{j1}\right)$

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- $\vec{N}_i \cdot \vec{N}_j = (\lambda_{i1}\nabla\lambda_{i2} - \lambda_{i2}\nabla\lambda_{i1}) \cdot \left(\lambda_{j1}\nabla\lambda_{j2} - \lambda_{j2}\nabla\lambda_{j1}\right)$
- $= \lambda_{i1}\lambda_{j1}\nabla\lambda_{i2} \cdot \nabla\lambda_{j2} - \lambda_{i1}\lambda_{j2}\nabla\lambda_{i2} \cdot \nabla\lambda_{j1} - \lambda_{i2}\lambda_{j1}\nabla\lambda_{i1} \cdot \nabla\lambda_{j2} + \lambda_{i2}\lambda_{j2}\nabla\lambda_{i1} \cdot \nabla\lambda_{j1}$
- $= \lambda_{i1}\lambda_{j1}\phi_{i2,j2} - \lambda_{i1}\lambda_{j2}\phi_{i2,j1} - \lambda_{i2}\lambda_{j1}\phi_{i1,j2} + \lambda_{i2}\lambda_{j2}\phi_{i1,j1}$
- Hence,
- $T_{ij} = \int\int_S \ \vec{N}_i \cdot \vec{N}_j \, dS$
- $= \int\int_S \ \left(\lambda_{i1}\lambda_{j1}\phi_{i2,j2} - \lambda_{i1}\lambda_{j2}\phi_{i2,j1} - \lambda_{i2}\lambda_{j1}\phi_{i1,j2} + \lambda_{i2}\lambda_{j2}\phi_{i1,j1}\right) dS$

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- Hence, $T_{ij} = \int\int_S \vec{N}_i \cdot \vec{N}_j \, dS$
- $= \int\int_S (\lambda_{i1}\lambda_{j1}\phi_{i2,j2} - \lambda_{i1}\lambda_{j2}\phi_{i2,j1} - \lambda_{i2}\lambda_{j1}\phi_{i1,j2} + \lambda_{i2}\lambda_{j2}\phi_{i1,j1}) \, dS$
- $= \phi_{i2,j2}\int\int_S \lambda_{i1}\lambda_{j1} \, dS - \phi_{i2,j1}\int\int_S \lambda_{i1}\lambda_{j2} \, dS$
- $-\phi_{i1,j2}\int\int_S \lambda_{i2}\lambda_{j1} \, dS + \phi_{i1,j1}\int\int_S \lambda_{i2}\lambda_{j2} \, dS$
- It has been shown that $M_{ij} = \int\int_S \lambda_i\lambda_j \, dS$
- $T_{ij} = \phi_{i2,j2}M_{i1,j1} - \phi_{i2,j1}M_{i1,j2}$
- $-\phi_{i1,j2}M_{i2,j1} + \phi_{i1,j1}M_{i2,j2}$