10.5 Wire Antennas and Scatterers

- Since from the EFIE on the surface of the wire,

$$E_z^{scat}\left(\rho = a\right) = -E_z^{inc}\left(\rho = a\right)$$

$$\frac{1}{j\omega\varepsilon_0}\left(\frac{\partial^2}{\partial z^2} + \beta_0^2\right)\int_{-L/2}^{L/2} I(z^{'})G(z,z^{'})dz^{'} = -E_z^{inc}(\rho = a)$$

- This equation is called the *Hallen's Integro-differential equation*
- In this case, differential is outside the integral

10.5 Wire Antennas and Scatterers

(b) $$E_z = \frac{1}{j\omega\varepsilon_0} \int_{-L/2}^{L/2} \left(\frac{\partial^2}{\partial z^2} + \beta_0^2 \right) I(z') G(z,z') dz'$$

- This electric field is the field due to current $I(z')$
  - [which results because of the impressed or source field] and
  - this field can be written as the scattered field
- Therefore,

$$E_z^{scat} = \frac{1}{j\omega\varepsilon_0} \int_{-L/2}^{L/2} \left(\frac{\partial^2}{\partial z^2} + \beta_0^2 \right) I(z') G(z,z') dz'$$

10.5 Wire Antennas and Scatterers

- Since from EFIE on the surface of the wire,

$$E_z^{scat}\left(\rho = a\right) = -E_z^{inc}\left(\rho = a\right)$$

$$\frac{1}{j\omega\varepsilon_0}\int_{-L/2}^{L/2}\left(\frac{\partial^2}{\partial z^2} + \beta_0^2\right) I(z^{'})G(z,z^{'})dz^{'} = -E_z^{inc}(\rho = a)$$

- This equation is the *Pocklington's Integro-differential equation*
- In this case, the differential has moved inside the integral
- Richmond has simplified the above equation as follows:

10.5 Wire Antennas and Scatterers

- (c) In cylindrical coordinates,

$$r = \left|\vec{r} - \vec{r}'\right| = \left|\sqrt{\left(z - z'\right)^2 + \left|\vec{\rho} - \vec{\rho}'\right|^2}\right|$$

$$\because \rho' = a \therefore \left|\vec{\rho} - \vec{\rho}'\right| = \rho^2 + a^2 - 2\vec{\rho} \bullet \vec{\rho}' = \rho^2 + a^2 - 2\rho a \cos\left(\phi - \phi'\right)$$

$$\Rightarrow r = \left|\vec{r} - \vec{r}'\right| = \sqrt{\rho^2 + a^2 - 2\rho a \cos\left(\phi - \phi'\right) + \left(z - z'\right)^2}$$

- Problem under analysis has cylindrical symmetry and
  - observation for any values of ϕ won't make any difference
  - we may assume without loss of generality $\phi = 0$
  - hence $\phi - \phi' = \phi'$

10.5 Wire Antennas and Scatterers

$$A_z = \mu_0 \int_{-L/2}^{L/2} \frac{I(z')}{2\pi} \int_0^{2\pi} \frac{e^{-j\beta_0 r}}{4\pi r} d\phi' dz'$$

- where $r = \sqrt{\rho^2 + a^2 - 2\rho a \cos(\phi') + (z - z')^2}$
- and the inner integration $\int_0^{2\pi} \frac{e^{-j\beta_0 r}}{4\pi r} d\phi'$
  - is also referred to as cylindrical wire kernel
- Thin wire approximation $r \cong \sqrt{\rho^2 + (z - z')^2}$
- If we assume a<<λ and is very small, we have, ϕ'
- Inner integrand is no more dependent on the variable

10.5 Wire Antennas and Scatterers

- Therefore

$$A_z = \mu_0 \int_{-L/2}^{L/2} \frac{I(z')e^{-j\beta_0 r}}{4\pi r} dz'$$

- Also called as thin wire approximation
  - with the reduced kernel
- For this case, we can write

$$G\left(z, z'\right) \cong \frac{e^{-j\beta_0 r}}{4\pi r} = G\left(r\right)$$

10.5 Wire Antennas and Scatterers

- Now in the light of this simplification of the magnetic vector potential,
- we can simplify equation 10.29c (see example 10.4) as follows:

$$\frac{1}{j\omega\varepsilon_0 4\pi}\int_{-L/2}^{L/2} I_z(z')\frac{e^{-j\beta_0 r}}{r^5}\left[\left(1+j\beta_0 r\right)\left(2r^2-3a^2\right)+\left(\beta_0 a r\right)^2\right]dz' = -E_z^{inc}(\rho=a)$$

- This form of the Pocklington's integro-differential is more suitable for MoM formulation
  - since it does not involve any differentiation.

10.5 Wire Antennas and Scatterers

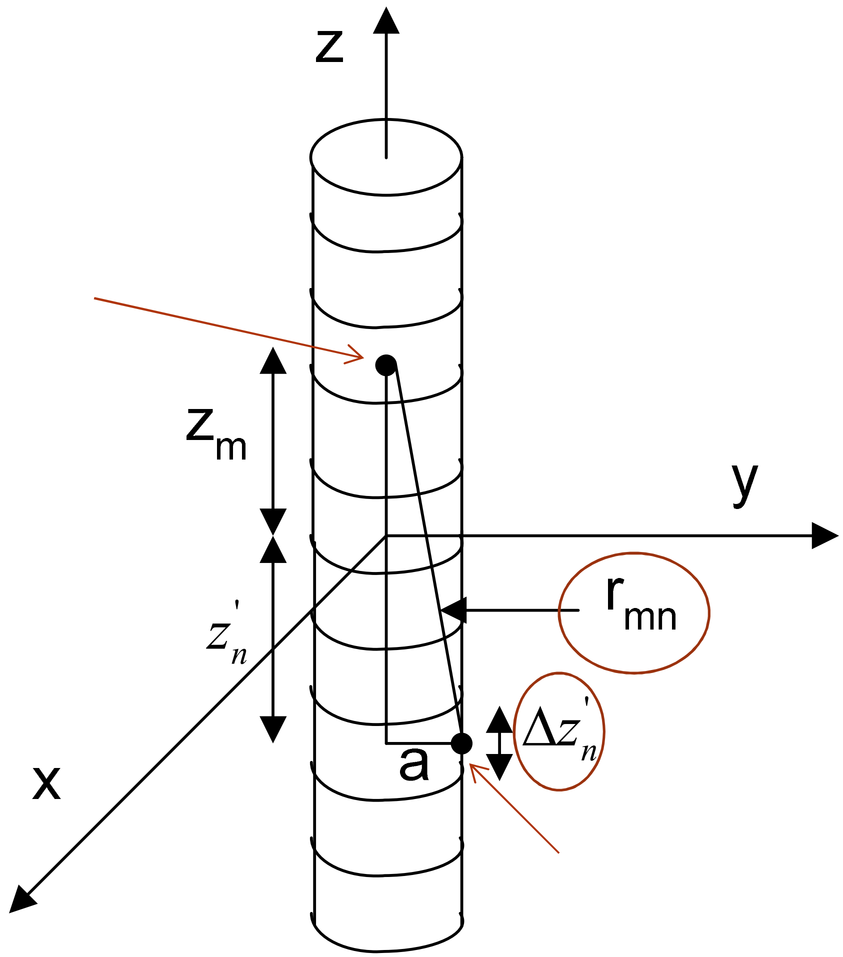
10.5.3 MoM Formulation of Pocklington's Integro-differential equation

- Applying MoM formulation to above integral equation
- Divide the wire in to N segments
- Consider pulse basis function and
  - express the current as a series expansion
  - in the form of a staircase approximation as

$$I(z') = \sum_{n=1}^{N} I_n b_n(z')$$

where

$$b_n(z') = \begin{cases} 1 & for \quad \Delta z'_n \\ 0 & otherwise \end{cases}$$



- Fig. 10.8 Thin wire dipole is divided into N equal segments

10.5 Wire Antennas and Scatterers

- $\Delta z_n^{'}$ is the length of the n[th] segment, expressed as

$$-\frac{L}{2}+\frac{L}{N}(n-1)<z\leq-\frac{L}{2}+\frac{L}{N}n$$

$$-E_z^{inc}=\int_{-L/2}^{L/2} I(z^{'})F(z,z^{'})dz^{'}$$

- where $$F(z,z^{'})=\frac{1}{j\omega\varepsilon}\left(\frac{\partial^2}{\partial z^2}+\beta_0^2\right)G\left(z,z^{'}\right)$$

10.5 Wire Antennas and Scatterers

- Substituting I(z′) and
  - evaluating at $z=z_m$ (middle of the m^{th} segment) as shown in the Fig. 10.7
- for point matching with weighting functions as $w_m(z)=\delta(z-z_m)$
- where z_m is the center of the segment m

$$z_m = -\frac{L}{2}+\frac{L}{N}(m-0.5)$$

- and m=1,2,3,…,M, we can write,

$$-E_z^i(z_m)=\sum_{n=1}^{N} I_n \int_{\Delta z_n'} F\left(z_m, z'\right)dz'$$

10.5 Wire Antennas and Scatterers

- To overcome the singularity
  - for the self term or diagonal elements of the [Z] matrix
- we have assumed that the source is on the surface of the wire
  - whereas the observation is the axis of the wire
- Using mid-point integration, we have,

$$-E_z^i(z_m) = \sum_{n=1}^{N} I_n F_{mn}$$

- where $F_{mn} = \int_{\Delta z_n'} F(z_m, z')dz' \cong F(z_m, z_n')\Delta z_n'$

10.5 Wire Antennas and Scatterers

- For m=1,2,…,M,

$$\begin{bmatrix} F_{11} & F_{12} & \cdots & F_{1N} \\ F_{21} & F_{22} & \cdots & F_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ F_{M1} & F_{M2} & \cdots & F_{MN} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_N \end{bmatrix} = \begin{bmatrix} -E_z^{inc}(z_1) \\ -E_z^{inc}(z_2) \\ \\ -E_z^{inc}(z_N) \end{bmatrix}$$

- In the compact form, we can write

$$[F_{mn}][I_n] = -[E_m]$$

- Multiplying both sides by Δz, we can write,

$$[Z_{mn}][I_n] = -[V_m]$$

10.5 Wire Antennas and Scatterers

- $[I_n]$ can be computed by matrix inversion
  - Can find the approximate current distribution on the antenna
- Other important antenna parameters are
  - input impedance of the antenna and
  - the total radiated fields
- which can be obtained as:

$$Z_{input} = \frac{2V_0}{I_{\frac{N}{2}}}$$

$$\vec{E}^{tot}(\vec{r}) = \vec{E}^{inc}(\vec{r}) + \vec{E}^{scat}(\vec{r}) = \vec{E}^{inc}(\vec{r}) + \int_{-L/2}^{L/2} \sum_{n=1}^{N} I_n b_n(z') \frac{1}{j\omega\varepsilon 4\pi} \left(\frac{\partial^2}{\partial z^2} + \beta_0^2 \right) \frac{e^{-j\beta_0 r}}{r} dz'$$

10.5 Wire Antennas and Scatterers

Example 10.5

- Consider a short dipole (thin wire antenna) of length 0.3λ and radius 0.01λ.
- Find the current distribution on the short dipole and input impedance using MoM.
- Assume frequency of operation of the antenna is at 1 MHz.
- Choose the number of discretizations on the thin wire as three segments only.

10.5 Wire Antennas and Scatterers

- For three segments discretization ($\Delta z = 0.1\lambda$) on the thin wire antenna
- the MoM matrix equation will be

$$\begin{bmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{21} & Z_{22} & Z_{23} \\ Z_{31} & Z_{32} & Z_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = -\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

$$\Rightarrow \Delta z \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = -\Delta z \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

10.5 Wire Antennas and Scatterers

- where

$$Z_{mn} = F_{mn}\Delta z = \frac{e^{-j\beta_0 r_{mn}}\left[\left(1+j\beta_0 r_{mn}\right)\left(2r_{mn}^2-3a^2\right)+\left(\beta_0 a r_{mn}\right)^2\right]\Delta z}{j\omega\varepsilon_0 4\pi r_{mn}{}^5}$$

- Simplifying the above expression of the Z_{mn} (see book)

$$Z_{mn} = \frac{-jZ_0\frac{\Delta z}{\lambda}\left[\left(1+j2\pi\frac{r_{mn}}{\lambda}\right)\left(2-3\left(\frac{a}{r_{mn}}\right)^2\right)+\left(2\pi\frac{a}{\lambda}\right)^2\right]\left\{\cos\left(2\pi\frac{r_{mn}}{\lambda}\right)-j\sin\left(2\pi\frac{r_{mn}}{\lambda}\right)\right\}}{8\pi^2\lambda\left(\frac{r_{mn}}{\lambda}\right)^3}$$

10.5 Wire Antennas and Scatterers

- For thin wire approximations,

$$r_{mn} \cong \sqrt{\rho^2 + (z_m - z_n)^2}$$
$$\therefore r_{11} = \sqrt{a^2} = a = r_{22} = r_{33} = 0.01\lambda;$$
$$r_{12} = \sqrt{a^2 + \Delta z^2} = r_{21} = r_{23} = r_{32} = 0.1\lambda;$$
$$r_{13} = \sqrt{a^2 + (2\Delta z)^2} = r_{31} = 0.2\lambda$$

- [Z] symmetric Toeplitz matrix (need to calculate the first row of the matrix only)

$$[Z] = \begin{bmatrix} Z_1 & Z_2 & Z_3 \\ Z_2 & Z_1 & Z_3 \\ Z_3 & Z_2 & Z_1 \end{bmatrix}$$

10.5 Wire Antennas and Scatterers

- About the [V] matrix, for delta gap excitation,

$$[V] = -\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

- since a voltage 1V exists at the feed gap only
- Solve the [I] matrix and get the current distribution
- and the input impedance can be calculated as

$$Z_{input} = \frac{2V_0}{I_{\frac{N}{2}}} = \frac{1}{I_2}$$

10.6 Software language for implementation of electromagnetics codes

- *Choice of software languages* for implementing electromagnetics code
- FORTRAN language complex numbers are a built-in datatype
- Many computational electromagnetics programmer prefer
  - to use FORTRAN language
  - for implementing their algorithms
- Earlier versions of FORTRAN were a functional language

10.6 Software language for implementation of electromagnetics codes

- New versions of FORTRAN are object-oriented languages
- C++, another object-oriented language,
  - is also widely used for many numerical methods
- FORTRAN and C++ are efficient in implementation
  - since the computational time is less
- MATLAB is also convenient environments since
  - it accepts complex numbers,
  - graphics are very easy to create and
  - many in-built functions are readily available for use

10.6 Software language for implementation of electromagnetics codes

- MATLAB any additional "for" loop in the program,
  - the time it takes to run the program increases drastically
- Good to consider the advantages and disadvantages
  - for employment of any software language
- For instance,
  - drawing graphics in C is somewhat involved,
  - but MATLAB is convenient for such things.
  - program written in C runs faster than MATLAB and so on

10.6 Software language for implementation of electromagnetics codes

- Computational electromagnetics is a topic
  - which you can learn only by doing
- Some simulation exercises are given at the end of the chapter,
- you should always write down a
  - MATLAB or
  - any other software language program
- in which you are comfortable and
- see those results

10.7 Summary

- Summarize the three steps involved in MoM:

(a) Derivation of appropriate integral equations

(b) Conversion or discretization of the integral equation into

- *a matrix equation using*
  - *basis or expansion functions and*
  - *weighting or testing functions*
- *as well as evaluation of the matrix elements*

(c) Solving the matrix equation and

- *obtaining the desired parameters*

Dipole (Electrostatics)

- Programming Exercise 1 (In-class)
- Assume feed at the centre
- Radius of cylindrical dipole, $a = 0.00065$ m
- Length of dipole, $2l = 0.005$ m
- Permittivity of free space, $\varepsilon_o = 8.8541878176 \times 10^{-12}$ F/m
- Relative permittivity, $\varepsilon_r = 1$ (air)
- Number of segments (pulses), $N = 30$
- Segment length, $\Delta z = 2l/N$
- Match point, z_m at segment m
- $z_m = (m - 0.5)\Delta z - l$ $\quad m = 1, 2, 3, \ldots, N$

Dipole (Electrostatics)

- Discretization of Z_{mn}

$$Z_{mn} = \begin{cases} 2\ln\left(\dfrac{(\Delta z/2) + \sqrt{a^2 + (\Delta z/2)^2}}{a}\right) & m = n \\ \ln\left(\dfrac{z_m + (\Delta z/2) - z_n + \sqrt{a^2 + (z_m - z_n + \Delta z/2)^2}}{z_m - (\Delta z/2) - z_n + \sqrt{a^2 + (z_m - z_n - \Delta z/2)^2}}\right) & m \neq n \end{cases}$$

- To create a complete matrix, $[Z_{mn}]$:

$$\begin{bmatrix} Z_{11} & Z_{12} & \cdots & Z_{1,30} \\ Z_{21} & Z_{22} & \cdots & Z_{2,30} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{30,1} & Z_{30,2} & \cdots & Z_{30,30} \end{bmatrix}$$

Dipole (Electrostatics)

- Discretization voltage, V
- $V_o = 1$
- $V_m = 4\pi\varepsilon_o\varepsilon_r V_o$
- To create transpose matrix, $[V_m]$:

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_{30} \end{bmatrix}$$

Dipole (Electrostatics)

- The unknown charge densities, $[I_n]$ along the dipole are determined
  - by inverting the matrix, $[Z_{mn}]$ and multiplying with the voltage, $[V_m]$

$$\begin{bmatrix} I_1 \\ I_2 \\ \vdots \\ I_{30} \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & \cdots & Z_{1,30} \\ Z_{21} & Z_{22} & \cdots & Z_{2,30} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{30,1} & Z_{30,2} & \cdots & Z_{30,30} \end{bmatrix}^{-1} \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_{30} \end{bmatrix}$$

- Write a MATLAB program to plot line charge density (pC) versus distance (mm) of the centre-fed dipole

Programming Exercise 3 (Homework)

- Write a MATLAB program to find the current distribution and input impedance of a centre-fed dipole
- Hints:
  - Pocklington's equation (Richmond's expression)

$$\frac{1}{j\omega\varepsilon_0 4\pi}\int_{-L/2}^{L/2} I_z(z')\frac{e^{-jkr}}{r^5}\left[(1+jkr)\left(2r^2-3a^2\right)+(kar)^2\right]dz' = -E_z^{inc}(\rho=a)$$

  - Piece-wise constant function
  - Magnetic frill source
  - Point-matching method

Dipole (Pocklington's Equation)

Electrodynamic case

- Define z coordinate as symbol
- Radius cylindrical dipole, $a = 0.00065$ m
- Outer radius coaxial line, $b = 0.00205$ m
- Half length dipole, $h = 0.005$ m
- Operation frequency, $f = 1$ GHz
- Light velocity, $c = 3\times10^8$ m/s

Dipole (Pocklington's Equation)

- Permittivity of free space, $\varepsilon_o = 8.8541878176 \times 10^{-12}$ F/m
- Relative permittivity, $\varepsilon_r = 1$ (air)
- Relative permittivity of coaxial line medium, $\varepsilon_r = 2.06$ (Teflon)
- Angular frequency, $\omega = 2\pi f$
- Propagation constant, $k = 2\pi f (\varepsilon_r)^{0.5} / c$

Dipole (Pocklington's Equation)

- Number segment (pulses), $N = 9$ (odd number)
- Segment length, $\Delta z = 2h/N$
- Match point, z_m at segment m
- $z_m = (m - 0.5)\Delta z - h \qquad m = 1, 2, 3, \ldots, N$

Dipole (Pocklington's Equation)

- Discretization of Z_{mn}

$$Z_{mn} = \frac{1}{4\pi} \times$$

$$\int_{z_n - \Delta z/2}^{z_n + \Delta z/2} \frac{e^{-jkr_m}}{r_m^5} \left[\begin{array}{l} (1 + jkr_m)(2r_m^2 - 3a^2) \\ + (kar_m)^2 \end{array} \right] dz'$$

- where

$$r_m = \sqrt{a^2 + (z_m - z')^2}$$

Dipole (Pocklington's Equation)

- The integral Z_{mn} can be solved by numerical integration (*quadv* function in MATLAB toolbox) for point $z1$ at $m = 1$ segment
- We assume that $z_m = z_n$ and solve the integral from z_n-($\Delta z/2$) to z_n+($\Delta z/2$) and yield the matrix
- $[Z_{11}, Z_{12}, Z_{13}, Z_{14}, Z_{15}, Z_{16}, Z_{17}, Z_{18}, Z_{19}]$

Dipole (Pocklington's Equation)

- To create a matrix:

$$\begin{bmatrix} Z_{11}/2 & Z_{12} & Z_{13} & Z_{14} & \cdots & Z_{19} \\ 0 & Z_{11}/2 & Z_{12} & Z_{13} & \cdots & Z_{18} \\ 0 & 0 & Z_{11}/2 & Z_{12} & \cdots & Z_{17} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & Z_{11}/2 \end{bmatrix}$$

Dipole (Pocklington's Equation)

- To create a complete matrix, $[Z_{mn}]$:

$$\begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} & \cdots & Z_{19} \\ Z_{12} & Z_{11} & Z_{12} & Z_{13} & \cdots & Z_{18} \\ Z_{13} & Z_{12} & Z_{11} & Z_{12} & \cdots & Z_{17} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ Z_{19} & Z_{18} & Z_{17} & Z_{16} & \cdots & Z_{11} \end{bmatrix}$$

- Magnetic frill source[+]
- the feed gap is replaced with a circumferentially directed magnetic current density existing over an annular aperture

[+]https://ieeexplore.ieee.org/stamp/stamp.jsp?arnumber=5948354

Dipole (Pocklington's Equation)

- The inner radius of the aperture is that of the wire and outer radius is found from the characteristic impedance of the transmission line feeding the antenna
- For this frill source the electric field is found on the surface of the wire
- Discretization electric field, E_z

$$E_{zm}^{i} = \frac{-j\omega\varepsilon_o V_o}{\ln\left(b/a\right)}\left[\frac{e^{-jkr_1}}{r_1} - \frac{e^{-jkr_2}}{r_2}\right]$$

- where $r_1 = \sqrt{z_m^2 + a^2}$ $\quad r_2 = \sqrt{z_m^2 + b^2}$

Dipole (Pocklington's Equation)

- assume Vo=1.
- To create transpose matrix, $[E_m]$:

$$\begin{bmatrix} E_{z1}^{i} \\ E_{z2}^{i} \\ E_{z3}^{i} \\ \vdots \\ E_{z9}^{i} \end{bmatrix}$$

- The unknown current vector, $[I_n]$ along the dipole is determined by
  - inverting the matrix, $[Z_{mn}]$ and multiplying with the electric field vector, $[E_m]$

Dipole (Pocklington's Equation)

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ \vdots \\ I_9 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} & \cdots & Z_{19} \\ Z_{12} & Z_{11} & Z_{12} & Z_{13} & \cdots & Z_{18} \\ Z_{13} & Z_{12} & Z_{11} & Z_{12} & \cdots & Z_{17} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ Z_{19} & Z_{18} & Z_{17} & Z_{16} & \cdots & Z_{11} \end{bmatrix}^{-1} \begin{bmatrix} E^i_{z1} \\ E^i_{z2} \\ E^i_{z3} \\ \vdots \\ E^i_{z9} \end{bmatrix}$$

- The input impedance, Z_{in} at coaxial fed point (at $z_5 = 0$).
- We assume v0=1 V.
- Z_{in}=v0/I_5