MoM Advances

- **Brief review of Dyadic analysis:**
- Like vector analysis, dyadic analysis is for dyads
- Dyadic operations and theorems provide an effective tool for manipulation of field quantities (*Tai, C. T., "Dyadic Green's Functions in Electromagnetic Theory," New York: IEEE Press, 2nd ed., 1993*)
- Dyad notation was first introduced by Gibbs in 1884 (*Gibbs, J. W., "The scientific papers of J. Willard Gibbs" Vol. 2, pp. 84-90, New York: Dover, 1961.*)

MoM Advances

- *Dyads are extension of vectors*
- Consider a vector $\vec{D}$ in Cartesian coordinates represented as
- $\vec{D} = D_1\hat{x}_1 + D_2\hat{x}_2 + D_3\hat{x}_3 = \sum_{i=1}^{3} D_i\hat{x}_i$
- It is just a compact and convenient notation of a vector and its components in which
- $D_1 = D_x, \hat{x}_1 = \hat{x},$
- $D_2 = D_y, \hat{x}_2 = \hat{y},$
- $D_3 = D_z, \hat{x}_3 = \hat{z}$

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- Now consider three such different vectors $\vec{D}_1, \vec{D}_2$ *and* $\vec{D}_3$
- where
- $\vec{D}_1 = \sum_{i=1}^{3} D_{i1}\hat{x}_i,$
- $\vec{D}_2 = \sum_{i=1}^{3} D_{i2}\hat{x}_i$ *and*
- $\vec{D}_3 = \sum_{i=1}^{3} D_{i3}\hat{x}_i$
- *Looks like a column vector*

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- In compact notation,
- $\vec{D}_j = \sum_{i=1}^{3} D_{ij}\hat{x}_i, j = 1,2,3$
- which constitute a dyad $\overleftrightarrow{D}$ with two-sided arrow head like this
- $\overleftrightarrow{D} = \sum_{j=1}^{3} \vec{D}_j\hat{x}_j$
- $= \sum_{j=1}^{3}\left(\sum_{i=1}^{3} D_{ij}\,\hat{x}_i\right)\hat{x}_j$
- $= \sum_{j=1}^{3}\sum_{i=1}^{3} D_{ij}\,\hat{x}_i\hat{x}_j$

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- The doublets $\hat{x}_i\hat{x}_j$ form the nine unit dyad basis in dyadic analysis
- $\hat{x}_1\hat{x}_1 = \hat{x}\hat{x}, \hat{x}_1\hat{x}_2 = \hat{x}\hat{y}, \hat{x}_1\hat{x}_3 = \hat{x}\hat{z}$
- $\hat{x}_2\hat{x}_1 = \hat{y}\hat{x}, \hat{x}_2\hat{x}_2 = \hat{y}\hat{y}, \hat{x}_2\hat{x}_3 = \hat{y}\hat{z}$
- $\hat{x}_3\hat{x}_1 = \hat{z}\hat{x}, \hat{x}_3\hat{x}_2 = \hat{z}\hat{y}, \hat{x}_3\hat{x}_3 = \hat{z}\hat{z}$
- which is an extension of three unit basis vectors in vector analysis
- $\hat{x}_1 = \hat{x},$
- $\hat{x}_2 = \hat{y}\ and$
- $\hat{x}_3 = \hat{z}$
- Note that $\hat{x}_i\hat{x}_j \neq \hat{x}_j\hat{x}_i, i \neq j$ so the ordering is important

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- Matrix notation of a dyad $\overleftrightarrow{D}$
- $\overleftrightarrow{D} = (\vec{D}_1 \vec{D}_2 \vec{D}_3) = \begin{pmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{pmatrix}$
- In general **dyads** an be formed by product of two vectors $\vec{A}\ and\ \vec{B}$ where $\vec{A}$ is 3×1 matrix and $\vec{B}$ is a 1×3 matrix
- which we usually call as juxtaposition of two vectors side by side without any operation
- $\overleftrightarrow{D} = \vec{A}\vec{B}$

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- We can also find the transpose of dyad $\overleftrightarrow{D}$
- $\overleftrightarrow{D} = \sum_{j=1}^{3} \vec{D}_j \hat{x}_j = \sum_{j=1}^{3} \sum_{i=1}^{3} D_{ij} \, \hat{x}_i \hat{x}_j$
- Or, in terms of x, y and z
- $\overleftrightarrow{D} = D_{xx}\hat{x}\hat{x} + D_{xy}\hat{x}\hat{y} + D_{xz}\hat{x}\hat{z}$
- $+D_{yx}\hat{y}\hat{x} + D_{yy}\hat{y}\hat{y} + D_{yz}\hat{y}\hat{z} + D_{zx}\hat{z}\hat{x} + D_{zy}\hat{z}\hat{y} + D_{zz}\hat{z}\hat{z}$
- *as*
- $\left[\overleftrightarrow{D}\right]^T = \sum_{j=1}^{3} \hat{x}_j \vec{D}_j$
- $= \sum_{j=1}^{3} \sum_{i=1}^{3} D_{ij} \, \hat{x}_j \hat{x}_i$

MoM Advances

- Or, in terms of x, y and z
- $\left[\overleftrightarrow{D}\right]^T = D_{xx}\hat{x}\hat{x} + D_{yx}\hat{x}\hat{y} + D_{zx}\hat{x}\hat{z}$
- $+D_{xy}\hat{y}\hat{x} + D_{yy}\hat{y}\hat{y} + D_{zy}\hat{y}\hat{z} + D_{xz}\hat{z}\hat{x} + D_{yz}\hat{z}\hat{y} + D_{zz}\hat{z}\hat{z}$
- For a symmetric dyad $\left[\overleftrightarrow{D}\right]^T = \overleftrightarrow{D}$
- One very important symmetric dyad is "idemfactor" or "unit" dyad for which $D_{ij} = \delta_{ij}$

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- Note that
- $D_{ij} = \delta_{ij} = 1\ for\ i = j\ and$
- $D_{ij} = \delta_{ij} = 0\ for\ i \neq j$
- Hence $\delta_{11} = \delta_{22} = \delta_{33} = 1\ implies\ D_{11} = D_{22} = D_{33} = 1$
- $and\ all\ other\ values\ are\ zero$

$$\overleftrightarrow{I} = \begin{pmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Therefore, unit dyad is given by
- $\overleftrightarrow{I} = \hat{x}_1\hat{x}_1 + \hat{x}_2\hat{x}_2 + \hat{x}_3\hat{x}_3 = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z}$

MoM Advances

- Dyad itself does not have any physical interpretation
- When it acts on a vector, it has meaningful interpretation
- (a) Scalar product with a vector gives another vector
- For example: *Anterior scalar product* with vector $\vec{C}$
- $\vec{C}\cdot\overleftrightarrow{D} = \left(C_x\hat{x} + C_y\hat{y} + C_z\hat{z}\right)\cdot\overleftrightarrow{D} = C_x\hat{x}\cdot\overleftrightarrow{D}+C_y\hat{y}\cdot\overleftrightarrow{D}+C_z\hat{z}\cdot\overleftrightarrow{D}$
- $C_x\hat{x}\cdot\overleftrightarrow{D}$
- $= C_x\hat{x}\cdot\left(D_{xx}\hat{x}\hat{x}+D_{xy}\hat{x}\hat{y} + D_{xz}\hat{x}\hat{z} + D_{yx}\hat{y}\hat{x} + D_{yy}\hat{y}\hat{y} + D_{yz}\hat{y}\hat{z}+D_{zx}\hat{z}\hat{x} + D_{zy}\hat{z}\hat{y} + D_{zz}\hat{z}\hat{z}\right)$
- $= C_x\hat{x}\cdot\left(D_{xx}\hat{x}\hat{x}+D_{xy}\hat{x}\hat{y} + D_{xz}\hat{x}\hat{z}\right)$
- $= C_xD_{xx}\hat{x}+C_xD_{xy}\hat{y} + C_xD_{xz}\hat{z}$

MoM Advances

- $C_y\hat{y}\ \cdot \overleftrightarrow{D}$
- $= C_y\hat{y} \cdot \left(D_{xx}\hat{x}\hat{x}+D_{xy}\hat{x}\hat{y} + D_{xz}\hat{x}\hat{z} + D_{yx}\hat{y}\hat{x} + D_{yy}\hat{y}\hat{y} + D_{yz}\hat{y}\hat{z}+D_{zx}\hat{z}\hat{x} + D_{zy}\hat{z}\hat{y} + D_{zz}\hat{z}\hat{z}\right)$
- $= C_y\hat{y}\ \cdot \left(D_{yx}\hat{y}\hat{x} + D_{yy}\hat{y}\hat{y} + D_{yz}\hat{y}\hat{z}\right)$
- $= C_yD_{yx}\hat{x}+C_yD_{yy}\hat{y} + C_yD_{yz}\hat{z}$
- $C_z\hat{z}\ \cdot \overleftrightarrow{D}$
- $= C_z\hat{z} \cdot \left(D_{xx}\hat{x}\hat{x}+D_{xy}\hat{x}\hat{y} + D_{xz}\hat{x}\hat{z} + D_{yx}\hat{y}\hat{x} + D_{yy}\hat{y}\hat{y} + D_{yz}\hat{y}\hat{z}+D_{zx}\hat{z}\hat{x} + D_{zy}\hat{z}\hat{y} + D_{zz}\hat{z}\hat{z}\right)$
- $= C_z\hat{z}\ \cdot \left(D_{zx}\hat{z}\hat{x} + D_{zy}\hat{z}\hat{y} + D_{zz}\hat{z}\hat{z}\right)$
- $= C_zD_{zx}\hat{x}+C_zD_{zy}\hat{y} + C_zD_{zz}\hat{z}$
- Or in compact notation
- $\vec{C} \cdot \overleftrightarrow{D} = \sum_{i=1}^{3}\sum_{j=1}^{3} C_iD_{ij}\hat{x}_j$
- gives another vector

MoM Advances

- *Posterior scalar product* with vector $\vec{C}$
- $\overleftrightarrow{D} \cdot \vec{C} = \overleftrightarrow{D} \cdot (C_x\hat{x} + C_y\hat{y} + C_z\hat{z}) = \overleftrightarrow{D} \cdot C_x\,\hat{x}$
  $+\overleftrightarrow{D} \cdot C_y\,\hat{y}+\overleftrightarrow{D} \cdot C_z\hat{z}$
- $(D_{xx}\hat{x}\hat{x}+D_{xy}\hat{x}\hat{y} + D_{xz}\hat{x}\hat{z} + D_{yx}\hat{y}\hat{x} + D_{yy}\hat{y}\hat{y} +$
  $D_{yz}\hat{y}\hat{z}+D_{zx}\hat{z}\hat{x} + D_{zy}\hat{z}\hat{y} + D_{zz}\hat{z}\hat{z}) \cdot C_x\hat{x} =$
  $(D_{xx}\hat{x}\hat{x}+D_{yx}\hat{y}\hat{x} + D_{zx}\hat{z}\hat{x}) \cdot C_x\,\hat{x} =$
  $D_{xx}C_x\hat{x}+D_{yx}C_x\hat{y} + D_{zx}C_x\hat{z}$

MoM Advances

- Similarly,
- $\left(D_{xx}\hat{x}\hat{x}+D_{xy}\hat{x}\hat{y}+D_{xz}\hat{x}\hat{z}+D_{yx}\hat{y}\hat{x}+D_{yy}\hat{y}\hat{y}+D_{yz}\hat{y}\hat{z}+D_{zx}\hat{z}\hat{x}+D_{zy}\hat{z}\hat{y}+D_{zz}\hat{z}\hat{z}\right)\cdot C_y\hat{y}=\left(D_{xy}\hat{x}\hat{y}+D_{yy}\hat{y}\hat{y}+D_{zy}\hat{z}\hat{y}\right)\cdot C_y\,\hat{y}=D_{xy}C_y\hat{x}+D_{yy}C_y\hat{y}+D_{zy}C_y\hat{z}$
- $\left(D_{xx}\hat{x}\hat{x}+D_{xy}\hat{x}\hat{y}+D_{xz}\hat{x}\hat{z}+D_{yx}\hat{y}\hat{x}+D_{yy}\hat{y}\hat{y}+D_{yz}\hat{y}\hat{z}+D_{zx}\hat{z}\hat{x}+D_{zy}\hat{z}\hat{y}+D_{zz}\hat{z}\hat{z}\right)\cdot C_z\hat{z}=\left(D_{xz}\hat{x}\hat{z}+D_{yz}\hat{y}\hat{z}+D_{zz}\hat{z}\hat{z}\right)\cdot C_z\,\hat{z}=D_{xz}C_z\hat{x}+D_{yz}C_z\hat{y}+D_{zz}C_z\hat{z}$
- Or in compact notation
- $\vec{C}\cdot\overleftrightarrow{D}=\sum_{i=1}^{3}\sum_{j=1}^{3}D_{ji}C_i\hat{x}_j$

MoM Advances

- Dyad $\overleftrightarrow{D}$ anterior and posterior scalar product with vector $\vec{C}$
  - gives different vectors
- *Anterior vector product* with vector $\vec{C}$
- $\vec{C} \times \overleftrightarrow{D} = (\vec{C} \times \vec{A})\vec{B}$
- *Posterior vector product* with vector $\vec{C}$
- $\overleftrightarrow{D} \times \vec{C} = \vec{A}(\vec{B} \times \vec{C})$
- Dyad $\overleftrightarrow{D}$ anterior and posterior vector product with vector $\vec{C}$
  - gives different dyads

MoM Advances

- How to get this equation?

$$\tilde{\overset{\leftrightarrow}{G}}_{EJ}\left(\vec{k}_t, z, z'\right) = -V_{horizon}^{TE}\left(z, z'\right)\left(\hat{k}_t \times \hat{z}\right)\left(\hat{k}_t \times \hat{z}\right) - V_{horizon}^{TM}\left(z, z'\right)\left(\hat{k}_t\right)\left(\hat{k}_t\right)$$

- S.-G. Pan and I. Wolff, "Scalarization of Dyadic Spectral Green's Functions and Network Formalism for Three-Dimensional Full-Wave Analysis of Planar Lines and Antennas," IEEE Trans. Microw. Theory and Tech., Vol. 42, no. 11, Nov. 1994, pp. 2118-2127
- Steps:
- Maxwell's equations for fields,
- Green's function dyadic version of Maxwell's equations,
- spectral domain Green's function dyadic version of Maxwell's equations in spectral domain

MoM Advances

- Scalarization of dyadic spectral Green's functions so that they can be determined from two sets of z-dependent in-homogenous transmission line equations
- How to convert multi-layered structure to TE/TM circuit models? How to find the length of the transmission lines?
- Height of the substrate for every layer decides the length of that substrate
- For example the substrate height is h_1 for layer 1 then the transmission line length would be h_1

MoM Advances

- In the equivalent circuit model current source is 1 A. Why?
- We are interested in finding the Green's function
- $$\vec{E}\left(\vec{r}\right)=\int_V \overleftrightarrow{G}_{EJ}\left(\vec{r},\vec{r}^{'}\right)\bullet\vec{J}_e\left(\vec{r}^{'}\right)d\vec{r}^{'}$$
- What is D_{TE} and D_{TM}?
- Denominator of the equivalent TE and TM impedance

$$Z_{eq}^{TE}=k_0\eta_0\frac{1}{D_{TE}}\qquad Z_{eq}^{TM}=\frac{\eta_0\beta_0}{k_0 D_{TM}}$$

- How do we find them?

$$Z_{eq}^{TE}=\frac{Z_d^{TE}Z_u^{TE}}{Z_d^{TE}+Z_u^{TE}};Z_{eq}^{TM}=\frac{Z_d^{TM}Z_u^{TM}}{Z_d^{TM}+Z_u^{TM}}$$

MoM Advances

- The far field radiation pattern of rectangular PMA after transforming to spherical coordinates may be obtained as follows:

$$E_\theta = \frac{j\exp\left(-j\beta_0 r\right)}{\lambda r}\left[\cos\phi\tilde{E}_x + \sin\phi\tilde{E}_y\right]$$

$$E_\phi = \frac{j\exp\left(-j\beta_0 r\right)}{\lambda r}\left[-\sin\phi\cos\theta\tilde{E}_x + \cos\phi\cos\theta\tilde{E}_y\right]$$

$$\tilde{E}_x = \tilde{G}_{EJ}^{xx}\tilde{J}_x + \tilde{G}_{EJ}^{xy}\tilde{J}_y \qquad \tilde{E}_y = \tilde{G}_{EJ}^{yx}\tilde{J}_x + \tilde{G}_{EJ}^{yy}\tilde{J}_y$$

$$\vec{E}\left(\vec{r}\right) = \int_V \overleftrightarrow{G}_{EJ}\left(\vec{r},\vec{r}^{\,'}\right)\bullet\vec{J}_e\left(\vec{r}^{\,'}\right)d\vec{r}^{\,'}$$

MoM Advances

- Note that spectral dyadic Green's functions are functions of $\vec{k}_t = k_x \hat{x} + k_y \hat{y}$.
- It can be shown that

$$k_x = k_0 \sin\theta\cos\phi; k_y = k_0 \sin\theta\sin\phi; \left|\vec{k}_t\right| = k_0 \sin\theta$$

- The directivity of PRMA may be obtained as

$$D(\theta,\phi) = \frac{U(\theta,\phi)}{U_{avg}} = \frac{4\pi U(\theta,\phi)}{\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} U(\theta,\phi)\sin\theta d\theta d\phi}$$

$$U(\theta,\phi) = \frac{\left|E(\theta,\phi)\right|^2}{\eta_0} r^2 = \frac{\left|E_\theta\right|^2 + \left|E_\phi\right|^2}{\eta_0} r^2$$

MoM Advances

- Electric Field Integral Equation

$$\hat{z}\times\left(\vec{E}^{source}\left(\vec{r}\right)+\vec{E}^{radiated}\left(\vec{r}\right)\right)=0$$

$$\hat{z}\times\left(\vec{E}^{source}\left(\vec{r}\right)+\int_{patch}\overleftrightarrow{G}_{EJ}\left(\vec{r},\vec{r}^{'}\right)\bullet\vec{J}_{e}\left(\vec{r}^{'}\right)d\vec{r}^{'}\right)=0$$

$$\vec{J}_{e}\left(x^{'},y^{'}\right)=J_{x}\left(x^{'},y^{'}\right)\hat{x}+J_{y}\left(x^{'},y^{'}\right)\hat{y}$$

$$\overleftrightarrow{G}_{EJ}\left(x,y;x^{'},y^{'}\right)=G_{EJ}^{xx}\left(x,y;x^{'},y^{'}\right)\hat{x}\hat{x}+G_{EJ}^{xy}\left(x,y;x^{'},y^{'}\right)\hat{x}\hat{y}+G_{EJ}^{yx}\left(x,y;x^{'},y^{'}\right)\hat{y}\hat{x}+G_{EJ}^{yy}\left(x,y;x^{'},y^{'}\right)\hat{y}\hat{y}$$

MoM Advances

- Using dyadic analysis, one may convert the vector EFIE into scalar EFIE as follows

$$E_x^{source}(x,y) = -\iint_{patch} G_{EJ}^{xx}\left(x,y;x^{'},y^{'}\right)J_x\left(x^{'},y^{'}\right)dx^{'}dy^{'} - \iint_{patch} G_{EJ}^{xy}\left(x,y;x^{'},y^{'}\right)J_y\left(x^{'},y^{'}\right)dx^{'}dy^{'}$$

$$E_y^{source}(x,y) = -\iint_{patch} G_{EJ}^{yx}\left(x,y;x^{'},y^{'}\right)J_x\left(x^{'},y^{'}\right)dx^{'}dy^{'} - \iint_{patch} G_{EJ}^{yy}\left(x,y;x^{'},y^{'}\right)J_y\left(x^{'},y^{'}\right)dx^{'}dy^{'}$$

MoM Advances

- Since we have derived spectral dyadic Green's functions in the previous section, we may take its inverse Fourier transform as follows

$$G_{EJ}^{pq}\left(x,y;x^{'},y^{'}\right)=\frac{1}{\left(2\pi\right)^{2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{G}_{EJ}^{pq}\left(k_{x},k_{y}\right)e^{-jk_{x}\left(x-x^{'}\right)}e^{-jk_{y}\left(y-y^{'}\right)}dk_{x}dk_{y}$$

where variables p,q may be either x or y

MoM Advances

- Substituting this in the scalar EFIE, we have,

$$E_x^{source}(x,y) = -\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\iint_{patch}\tilde{G}_{EJ}^{xx}(k_x,k_y)J_x(x',y')dx'dy' + \iint_{patch}\tilde{G}_{EJ}^{xy}(k_x,k_y)J_y(x',y')dx'dy'\right)e^{-jk_x(x-x')}e^{-jk_y(y-y')}dk_x dk_y$$

$$E_y^{source}(x,y) = -\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\iint_{patch}\tilde{G}_{EJ}^{yx}(k_x,k_y)J_x(x',y')dx'dy' + \iint_{patch}\tilde{G}_{EJ}^{yy}(k_x,k_y)J_y(x',y')dx'dy'\right)e^{-jk_x(x-x')}e^{-jk_y(y-y')}dk_x dk_y$$

MoM Advances

- As usual in MoM, we may approximate the unknown current density in terms of known basis functions

$$J_x(x,y)=\sum_{n=1}^{N} I_n^x B_n^x(x,y); J_y(x,y)=\sum_{n=1}^{N} I_n^y B_n^y(x,y)$$

- where piecewise sinusoidal (PWS) basis functions used are

$$B_n^x(x,y)=\frac{\sin\left[k_s\left(\Delta x-\left|x-x_n\right|\right)\right]}{\sin\left(k_s\Delta x\right)}; \left|y-y_n\right|\le\frac{\Delta y}{2}, \left|x-x_n\right|\le\Delta x$$

$$B_n^y(x,y)=\frac{\sin\left[k_s\left(\Delta y-\left|y-y_n\right|\right)\right]}{\sin\left(k_s\Delta y\right)}; \left|x-x_n\right|\le\frac{\Delta x}{2}, \left|y-y_n\right|\le\Delta y$$

MoM Advances

- Putting this in the scalar EFIE, we have,

$$-4\pi^2 E_x^{source}(x,y) =$$
$$\sum_{n=1}^{N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\iint_{S_{source}} \tilde{G}_{EJ}^{xx}(k_x,k_y) I_n^x B_n^x(x',y')dx'dy' + \iint_{S_{source}} \tilde{G}_{EJ}^{xy}(k_x,k_y) I_n^y B_n^y(x',y')dx'dy'\right) e^{-jk_x(x-x')}e^{-jk_y(y-y')}dk_x dk_y$$

$$-4\pi^2 E_y^{source}(x,y) =$$
$$\sum_{n=1}^{N}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\iint_{S_{source}} \tilde{G}_{EJ}^{yx}(k_x,k_y) I_n^x B_n^x(x',y')dx'dy' + \iint_{S_{source}} \tilde{G}_{EJ}^{yy}(k_x,k_y) I_n^y B_n^y(x',y')dx'dy'\right) e^{-jk_x(x-x')}e^{-jk_y(y-y')}dk_x dk_y$$