MoM Advances

- Spectral domain MoM
- It could be expressed in the MoM matrix form as follows

$$V_m^x = \sum_{n=1}^{N} \left(I_n^x Z_{nm}^{xx} + I_n^y Z_{nm}^{xy} \right), m = 1, 2, \ldots, N$$

$$V_m^y = \sum_{n=1}^{N} \left(I_n^x Z_{nm}^{yx} + I_n^y Z_{nm}^{yy} \right), m = 1, 2, \ldots, N$$

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- where

$$Z_{nm}^{pq} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\iint_{S_{source}}\iint_{S_{test}} \tilde{G}_{EJ}^{pq}\left(k_x,k_y\right)B_n^p\left(x,y\right)B_m^q\left(x',y'\right)e^{-jk_x\left(x-x'\right)}e^{-jk_y\left(y-y'\right)}dxdydx'dy'dk_xdk_y$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\left\{\iint_{S_{test}} B_n^p\left(x,y\right)e^{jk_xx}e^{jk_yy}dxdy\right\}^{*}\left\{\tilde{G}_{EJ}^{pq}\left(k_x,k_y\right)\right\}\left\{\iint_{S_{source}} B_n^q\left(x',y'\right)e^{jk_xx'}e^{jk_yy'}dx'dy'\right\}dk_xdk_y\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\tilde{B}_n^p\left(k_x,k_y\right)\right\}^{*}\tilde{G}_{EJ}^{pq}\left(k_x,k_y\right)\tilde{B}_n^q\left(k_x,k_y\right)dk_xdk_y$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{B}_n^p\left(-k_x,-k_y\right)\tilde{G}_{EJ}^{pq}\left(k_x,k_y\right)\tilde{B}_n^q\left(k_x,k_y\right)dk_xdk_y$$

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and

$$V_m^p = -4\pi^2 \iint_{S_{test}} E_p^{source}(x,y) B_m^p(x,y)\, dxdy$$

$$= -\iint_{S_{test}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{E}_p^{source}(k_x,k_y) e^{-jk_x x} e^{-jk_y y} B_m^p(x,y)\, dxdy dk_x dk_y$$

$$= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{ \iint_{S_{test}} B_m^p(x,y) e^{jk_x x} e^{jk_y y} dxdy \right\}^* \tilde{E}_p^{source}(k_x,k_y) dk_x dk_y$$

$$= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{ \tilde{B}_m^p(k_x,k_y) \right\}^* \tilde{E}_p^{source}(k_x,k_y) dk_x dk_y$$

$$= -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tilde{B}_m^p(-k_x,-k_y) \tilde{E}_p^{source}(k_x,k_y) dk_x dk_y$$

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- First we have a double infinite integration which may be converted into a single infinite integration by the following transformation in the variables

$$k_x = k_\rho \cos\alpha, k_y = k_\rho \sin\alpha$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\bullet)dk_x dk_y = \int_{0}^{2\pi}\int_{0}^{\infty}(\bullet)k_\rho dk_\rho d\alpha$$

- In the RHS k_ρ can be considered as a part of the integrand itself

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- When the integrand has a pole say for

$$k_\rho = k_{\rho 0}; k_0 \le k_{\rho 0} \le \sqrt{\varepsilon_r} k_0$$

- Special care must be taken while integrating
- We may divide the integration into three parts as follows

$$\int_0^{2\pi}\int_0^{\infty}(\bullet)dk_\rho d\alpha = \int_0^{2\pi}\int_0^{k_\rho^0-\delta}(\bullet)dk_\rho d\alpha + \int_0^{2\pi}\int_{k_\rho^0-\delta}^{k_\rho^0+\delta}(\bullet)dk_\rho d\alpha + \int_0^{2\pi}\int_{k_\rho^0+\delta}^{\infty}(\bullet)dk_\rho d\alpha = I_1 + I_2 + I_3$$

δ (typical value is 0.001 k_0) is small shift from the pole location

- I_1 can be integrated as usual

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- I_2 includes the pole in the integrand and hence can be integrated as follows

$$\int_{k_\rho^0-\delta}^{k_\rho^0+\delta} (\bullet) dk_\rho = \int_{k_\rho^0-\delta}^{k_\rho^0+\delta} \frac{f(k_\rho)}{g(k_\rho)} dk_\rho$$

- where $f(k_\rho), g(k_\rho)$ are the numerator and denominator function of the integrand
- Assume the pole is complex then it will no longer on the real line but shifted from it hence

$$k_\rho = k_\rho^0 - j\gamma$$

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- Using Taylor series expansion of the denominator of the above integrand, we have,

$$\because g\left(k_\rho^0 - j\gamma\right) = 0$$

$$g\left(k_\rho\right) \cong g\left(k_\rho^0 - j\gamma\right) + \left(k_\rho - k_\rho^0 + j\gamma\right) g^{'}\left(k_\rho^0 - j\gamma\right) = \left(k_\rho - k_\rho^0 + j\gamma\right) g^{'}\left(k_\rho^0 - j\gamma\right)$$

- Hence the integration becomes

$$I_2 = \int_{k_\rho^0 - j\gamma - \delta}^{k_\rho^0 - j\gamma + \delta} \frac{f\left(k_\rho\right)}{g\left(k_\rho\right)} dk_\rho = \frac{1}{g^{'}\left(k_\rho^0 - j\gamma\right)} \int_{k_\rho^0 - \delta}^{k_\rho^0 + \delta} \frac{f\left(k_\rho\right)}{\left(k_\rho - k_\rho^0 + j\gamma\right)} dk_\rho$$

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- Also

$$\because f\left(k_\rho\right) \approx f\left(k_\rho^0 - j\gamma\right)$$

$$I_2 = \frac{f\left(k_\rho^0 - j\gamma\right)}{g'\left(k_\rho^0 - j\gamma\right)} \int_{k_\rho^0 - j\gamma - \delta}^{k_\rho^0 - j\gamma + \delta} \frac{1}{\left(k_\rho - k_\rho^0 + j\gamma\right)} dk_\rho$$

- For real root, the above equation reduces to

$$\int_{k_\rho^0 - \delta}^{k_\rho^0 + \delta} \frac{f\left(k_\rho\right)}{g\left(k_\rho\right)} dk_\rho = \frac{j\pi f\left(k_\rho^0\right)}{g'\left(k_\rho^0\right)}$$

- *How do you find this?*

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- Digression:
- Improper integral along real axis:

$$I = \int_{-\infty}^{\infty} f(x)dx$$

- where the function f(x) has pole at $x=x_0$

$$I = \lim_{R \to \infty} \int_{-R}^{R} f(x)dx$$

- The integral can be carried out over a contour in complex plane

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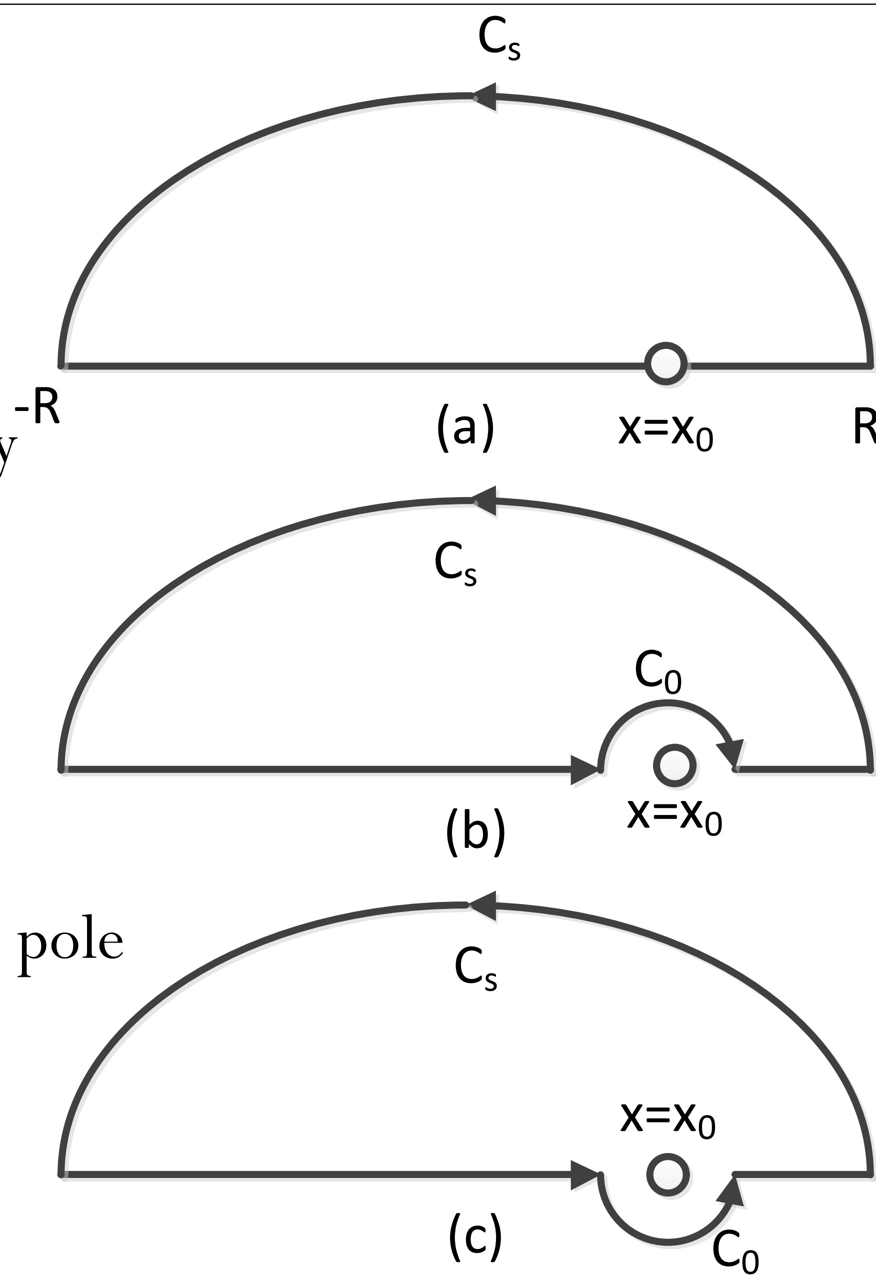
- Fig. An improper integral along the real axis, analytically continued in the upper half z-plane for integration as a contour integral

(a) Contour

(b) Deformed to exclude the pole

(c) Deformed to include the pole



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Wirtinger Calculus

- Complex derivative of a complex function f(z)
- For a function f(z) of a complex variable z=x+jy ∈ C, x,y ∈ R,
- its derivative w.r.t. z and z* are defined as

$$\frac{\partial f(z)}{\partial z}=\frac{1}{2}\left(\frac{\partial f(z)}{\partial x}-j\frac{\partial f(z)}{\partial y}\right);\frac{\partial f(z)}{\partial z^*}=\frac{1}{2}\left(\frac{\partial f(z)}{\partial x}+j\frac{\partial f(z)}{\partial y}\right)$$

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- For example,

$$f(z)=az, \frac{\partial f(z)}{\partial z}=a, \frac{\partial f(z)}{\partial z^*}=0;$$

$$f(z)=az^*, \frac{\partial f(z)}{\partial z}=0, \frac{\partial f(z)}{\partial z^*}=a;$$

$$f(z)=zz^*, \frac{\partial f(z)}{\partial z}=z^*, \frac{\partial f(z)}{\partial z^*}=z$$

- Analytic function
- A function f(z) is said to be analytic at a point $z=z_0$ if the derivative $f'(z)$ exists at z_0 and in some small region around z_0

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- Similar to definition of complex variable f(z) may be written as a sum of two functions, each of which is a function of two real variables x and y

$$f(z) = u(x,y) + jv(x,y)$$

- Note that $f^{'}(z) = \frac{\partial f(z)}{\partial z} = \frac{1}{2}\left(\frac{\partial f(z)}{\partial x} - j\frac{\partial f(z)}{\partial y}\right)$ exists if

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

- It is also called as Cauchy-Riemann condition

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- For example,

$$f(z)=az, \frac{\partial f(z)}{\partial z}=a$$

- Cauchy-Riemann condition is satisfied

$$u=ax, v=ay; \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=a; \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}=0$$

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- The function f(x) is analytic is continued to the upper half plane Im(z)≥0
- Its analytic continuation is called f(z) and is obtained by replacing the real variable x by the complex variable z
- Therefore,

$$\oint_C f(z)dz = \lim_{R\to\infty}\left[\int_{-R}^{R} f(x)dx + \int_{C_S} f(z)dz\right]$$

- where C_s is a semicircle of very large radius (see Fig. (a))

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- Conditions to be satisfied for the above formula to work are:
- *Assumption 1:*

  F(z) should be analytic everywhere in the upper half plane defined by Im(z)≥0 except for a finite number of isolated singular points
- *Assumption 2:*

  F(z) should vanish strongly as $1/z^2$ for $|z| \to \infty, 0 \le \theta \le \pi$ which means that the integrand approach zero over the semicircle Cs and the contribution of the arc Cs to the integral vanishes

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- To evaluate the integral the integration along the closed contour is now deformed to exclude the pole at $z=z_0=x_0$ (see Fig. (b))
- In the vicinity of the isolated pole the integrand is analytic so that the deformation around the pole is in the form of semicircle C_0 of vanishingly small radius

$$\oint_C f(z)dz = \lim_{R\to\infty}\left[\int_{-R}^{z_0-\rho} f(x)dx + \int_{C_0} f(z)dz + \int_{z_0+\rho}^{R} f(x)dx + \int_{C_S} f(z)dz\right]$$

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- Cauchy integral formula:
- If a function f(z) is analytic on a closed contour C and within the interior region bounded by it then

$$\oint_C \frac{f(z)}{z-z_0}dz = 2\pi j \times residue \quad of \quad f(z)$$

$$= \begin{cases} 0, & z_0 \quad exterior \quad of \quad C \\ 2\pi j f(z_0), & z_0 \quad interior \quad of \quad C \end{cases}$$

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- Here our pole is located outside the closed contour, hence
- LHS is zero
- Because of assumption 2, we have, $\int_{C_S} f(z)\,dz = 0$
- Hence

$$0 = \lim_{R\to\infty}\left[\int_{-R}^{z_0-\rho} f(x)\,dx + \int_{C_0} f(z)\,dz + \int_{z_0+\rho}^{R} f(x)\,dx\right]$$

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- In the limit $\rho \to 0$ $\quad - \lim\limits_{\rho \to 0} \int\limits_{C_0} f(z)\,dz = \int\limits_{-\infty}^{\infty} f(x)\,dx$

- f(z) is a function which has a pole at $z=z_0$
- Hence we may write

$$f(z) = \frac{n(z)}{d(z)}; d(z)\Big|_{z=z_0} = 0$$

- Using Taylor Series expansion of d(z) for z near z_0

$$d(z) = d(z_0) + (z - z_0)d'(z_0) = (z - z_0)d'(z_0)$$

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- We have $f(z)=\dfrac{n(z)}{(z-z_0)d'(z_0)}$
- Therefore

$$-\lim_{\rho\to 0}\int_{C_0}\frac{n(z)}{(z-z_0)d'(z_0)}dz=-\lim_{\rho\to 0}\frac{1}{d'(z_0)}\int_{C_0}\frac{n(z)}{(z-z_0)}dz$$

- Substitute

$$z=z_0+\rho e^{j\theta}$$

$$dz=j\rho e^{j\theta}d\theta$$

$$-\lim_{\rho\to 0}\frac{1}{d'(z_0)}\int_{\pi}^{0}\frac{n(z_0+\rho e^{j\theta})}{(\rho e^{j\theta})}j\rho e^{j\theta}d\theta=-j\frac{n(z_0)}{d'(z_0)}\int_{\pi}^{0}d\theta=j\pi\frac{n(z_0)}{d'(z_0)}$$

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- We can also consider the Fig. (c) case also
- In this case the pole $z=x_0$ is also included in the closed contour, hence

$$\oint_C f(z)dz = 2\pi f(z_o)$$

$$= \lim_{R\to\infty}\left[\int_{-R}^{z_0-\rho} f(x)dx + \int_{C_0} f(z)dz + \int_{z_0+\rho}^{R} f(x)dx + \int_{C_S} f(z)dz\right]$$

- For

$$f(z)=\frac{n(z)}{(z-z_0)d'(z_0)};\mathrm{Re}sidue\left(f(z)\right)\Big|_{z=z_0}=\frac{n(z_0)}{d'(z_0)}$$

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- Therefore $$2\pi j\frac{n(z_0)}{d'(z_0)} = \lim_{R\to\infty}\left[\int_{-R}^{z_0-\rho} f(x)dx + \int_{C_0} f(z)dz + \int_{z_0+\rho}^{R} f(x)dx\right]$$

$$= \int_{-R}^{R} f(x)dx + \lim_{R\to\infty}\left[\int_{C_0} f(z)dz\right]$$

- Let us find the second term on RHS

$$\lim_{\rho\to 0}\int_{C_0}\frac{n(z)}{(z-z_0)d'(z_0)}dz = \lim_{\rho\to 0}\frac{1}{d'(z_0)}\int_{C_0}\frac{n(z)}{(z-z_0)}dz$$

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- Hence

$$z = z_0 + \rho e^{j\theta}$$

$$dz = j\rho e^{j\theta} d\theta$$

$$\underset{\rho \to 0}{Lim} \frac{1}{d'(z_0)} \int_{\pi}^{2\pi} \frac{n(z_0 + \rho e^{j\theta})}{(\rho e^{j\theta})} j\rho e^{j\theta} d\theta = j \frac{n(z_0)}{d'(z_0)} \int_{\pi}^{2\pi} d\theta = j\pi \frac{n(z_0)}{d'(z_0)}$$

- Finally

$$\int_{-R}^{R} f(x) dx = \pi j \frac{n(z_0)}{d'(z_0)}$$

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- For real root, the above equation reduces to

$$\int_{k_\rho^0-\delta}^{k_\rho^0+\delta} \frac{f(k_\rho)}{g(k_\rho)} dk_\rho = \frac{j\pi f(k_\rho^0)}{g^{'}(k_\rho^0)}$$

- Since for our case $Integrand = \frac{n(z)}{d(z)} = \frac{f(k_\rho)}{g(k_\rho)}$

- The inner integral of is an infinite integral but it may be truncated at around 200 k_0
- This value may be decided by doing convergence analysis