MoM Advances

- **Spatial domain MPIE MoM**
- Spectral domain Mixed potential dyadic Green's functions
- It has been reported in the literature that MPIE MoM is more efficient than EFIE
- since the Green's functions in the MPIE formulation are less singular.
- The magnetic field intensity can be expressed in terms of magnetic vector potential

$$\mu \vec{H} = \nabla \times \vec{A}$$

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- For time harmonic fields the relation between the electric field and electric potential and magnetic vector potential is given as follows

$$\vec{E} = -j\omega\vec{A} - \nabla\Phi$$

- For x-directed current source, the two mixed potential spectral Green's function components are

$$\tilde{G}_{xx}^{A} = \frac{1}{j\omega\mu_0} V_{horizon}^{TE} = \frac{1}{j\omega\mu_0} Z_{eq}^{TE} = \frac{k_0\eta_0}{j\omega\mu_0} \frac{1}{D_{TE}} = \frac{1}{jD_{TE}}$$

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$$\tilde{G}^{\Phi} = \frac{j\omega\varepsilon_0}{k_\rho^2}\left(V_{horizon}^{TM} - V_{horizon}^{TE}\right) = \frac{j\omega\varepsilon_0}{k_\rho^2}\left(Z_{eq}^{TM} - V_{eq}^{TE}\right)$$

$$= \frac{j\omega\varepsilon_0\eta_0}{k_\rho^2}\left(\frac{k_0}{D_{TE}} - \frac{\beta_0}{k_0 D_{TM}}\right) = \frac{j}{k_\rho^2}\left(\frac{k_0^2 D_{TM} - \beta_0 D_{TE}}{D_{TE} D_{TM}}\right)$$

- Spatial domain dyadic Green's functions
- In order to obtain the spatial domain mixed potential Green's functions, we may apply the Fourier-Bessel transform of spectral domain mixed potential Green's functions as follows

$$G_{xx}^{A} = \int_0^\infty \tilde{G}_{xx}^{A} J_0\left(k_\rho \rho\right) k_\rho dk_\rho ; G^{\Phi} = \int_0^\infty \tilde{G}^{\Phi} J_0\left(k_\rho \rho\right) k_\rho dk_\rho$$

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- where J_0 is the Bessel function
- How?
- we may take its inverse Fourier transform as follows

$$G_{EJ}^{pq}\left(x,y;x',y'\right)=\frac{1}{\left(2\pi\right)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tilde{G}_{EJ}^{pq}\left(k_x,k_y\right)e^{-jk_x\left(x-x'\right)}e^{-jk_y\left(y-y'\right)}dk_x dk_y$$

where variables p,q may be either x or y

- First we have a double infinite integration which may be converted into a single infinite integration by the following transformation in the variables

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- Substitute

$$k_x = k_\rho \cos\alpha, k_y = k_\rho \sin\alpha; \vec{k}_\rho = k_\rho \cos\alpha \hat{x} + k_\rho \sin\alpha \hat{y} = k_x \hat{x} + k_y \hat{y}$$

- Hence

$$G_{EJ}^{pq}\left(\vec{r},\vec{r}'\right) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\tilde{G}_{EJ}^{pq}\left(k,z,z'\right) e^{-j\vec{k}\bullet\left(\vec{r}-\vec{r}'\right)} \right) d\vec{k}$$

$$= \frac{1}{2\pi}\int_{0}^{\infty} \tilde{G}_{EJ}^{pq}\left(k_\rho,z,z'\right) \left(\frac{1}{2\pi}\int_{0}^{2\pi} e^{-j\vec{k}_\rho\bullet\left(\vec{r}-\vec{r}'\right)} d\alpha \right) k_\rho dk_\rho$$

$$= \frac{1}{2\pi}\int_{0}^{\infty} \tilde{G}_{EJ}^{pq}\left(k_\rho,z,z'\right) J_0\left(k_\rho\left|\vec{r}-\vec{r}'\right|\right) k_\rho dk_\rho$$

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- Spatial domain MPIE MoM
- Mixed potential integral equation (MPIE) for x-directed horizontal electric dipole can be expressed in terms of the mixed potential Green's functions
- It is basically an expression of the continuity of the tangential component of the electric field
- $\hat{z} \times \vec{E}^{inc} = -\hat{z} \times \int_S \overleftrightarrow{G}_{EJ}(\vec{r}, \vec{r}') \cdot \vec{J}_S(\vec{r}') d\vec{s}'$
- We need to replace the scattered field in the RHS by using the following relation with the potentials

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- Since
- $\vec{E} = -j\omega\vec{A} - \nabla\Phi$
- and $\vec{A}$ satisfies Helmholtz wave equation
- $(\nabla^2 + k^2)\vec{A} = -\mu\vec{J}$
- and hence $\vec{A}(\vec{r}) = \int_S \overleftrightarrow{G}^A(\vec{r},\vec{r}') \cdot \vec{J}_S(\vec{r}')d\vec{s}'$
- In electrostatics, $V(\vec{r}) = \int_S G^\Phi(\vec{r},\vec{r}')q_S(\vec{r}')d\vec{s}'$

$$\hat{z}\times\vec{E}^{inc} = \hat{z}\times\left[j\omega\int_S \overleftrightarrow{G}^A \bullet \vec{J}_S dS' + \nabla\int_S G^\Phi q_S dS' \right]$$

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- As usual in MoM, we may approximate the unknown current and charge in terms of some known basis functions

$$\vec{J}_S \cong \sum_{i=1}^{N} \alpha_i \vec{B}_i ; q_S \cong \sum_{i=1}^{N} \alpha_i \left(-\frac{\nabla \bullet \vec{B}_i}{j\omega} \right)$$

- Note that here Bi is the basis function not to be confused with magnetic flux density
- Equation continuity $\nabla \cdot \vec{J}_S + j\omega q_s = 0$
- Then apply the same testing function for Galerkin's MoM, we have,

$$\hat{z} \times \int_S \vec{E}^{inc} \bullet \vec{B}_j dS = \hat{z} \times \sum_{i=1}^{N} \alpha_i \left[j\omega \int_S \vec{B}_j \int_S \overleftrightarrow{G}^A \bullet \vec{B}_i dS' dS - \frac{1}{j\omega} \int_S \vec{B}_j \bullet \nabla \int_S G^\Phi \nabla \bullet \vec{B}_i dS' dS \right]$$

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- **EM Absorption in the Human body**
- EM absorption is specified in terms of specific absorption rate (SAR) which is the mass normalized rate of energy absorbed by the body
- At a specific location, SAR may be defined as

$$SAR = \frac{\sigma}{\rho}\left|E\right|^2$$

- where σ is tissue conductivity, ρ is tissue mass density, E=rms value of internal field strength

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- Usual MoM steps are required:
  - Deriving the appropriate IE
  - Converting IE to matrix equation & matrix elements calculation
  - Solving the set of simultaneous equations
- We will use tensor integral-equation here
- What is this?
- When some electric field is incident on human body, the induced current in the body gives scattered electric field
  - Correspondingly the body may be replaced by an equivalent current density

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- Consider Maxwell curl equation

$$\nabla \times \vec{E} = -j\omega\mu_0 \vec{H}$$

- We can derive wave equation from Maxwell curl equation as

$$\begin{aligned}
&\nabla \times \nabla \times \vec{E} = -j\omega\mu_0 \left(\nabla \times \vec{H}\right) = -j\omega\mu_0 \left(\sigma\vec{E} + j\omega\varepsilon\vec{E}\right) \\
&= -j\omega\mu_0 \left(\sigma\vec{E} + j\omega\left(\varepsilon - \varepsilon_0\right)\vec{E} + j\omega\varepsilon_0\vec{E}\right) \\
&= -j\omega\mu_0 \left(\vec{J}_{conduction} + \vec{J}_{polarization} + j\omega\varepsilon_0\vec{E}\right) \\
&= -j\omega\mu_0 \left(\vec{J}_{eq} + j\omega\varepsilon_0\vec{E}\right) \\
&\Rightarrow \nabla \times \nabla \times \vec{E} - k_0^2\vec{E} = -j\omega\mu_0\vec{J}_{eq}
\end{aligned}$$

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- It is more like wave is propagating in free space
- And there is an equivalent current source which is effectively produced as an effect of the human body
- The equivalent current density can be expressed in terms of a tensor as follows

$$\vec{J}_{eq}\left(\vec{r}\right)=\sigma\left(\vec{r}\right)\vec{E}\left(\vec{r}\right)+j\omega\left(\varepsilon\left(\vec{r}\right)-\varepsilon_0\right)\vec{E}\left(\vec{r}\right)=\tau\left(\vec{r}\right)\vec{E}\left(\vec{r}\right)$$

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- The tensor $\tau(\vec{r})$ takes into account all the effect of a human body in terms of a 3-D matrix
- Consider a biological body of arbitratry shape with constitutive parameters $\varepsilon, \mu_0, \sigma$ illuminated by an incident (or impressed) plane EM wave
- The induced current in the body gives rise to a scattered field $\vec{E}^s$

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- For time varying electric fields $\vec{E}^s = -j\omega\vec{A} - \nabla\phi$
- where

$$\vec{A} = \mu_0 \int_{V'} G_0\left(\vec{r},\vec{r}'\right)\vec{J}_{eq}\left(\vec{r}'\right)dv'$$

- and the free space scalar Green's function is given by

$$G_0\left(\vec{r},\vec{r}'\right) = \frac{e^{-jk\left|\vec{r}-\vec{r}'\right|}}{4\pi\left|\vec{r}-\vec{r}'\right|}$$

$$\nabla \bullet \vec{A} = -j\omega\mu\varepsilon\phi$$

- From Lorentz Gaug condition

$$\Rightarrow \phi = \frac{\nabla \bullet \vec{A}}{-j\omega\mu\varepsilon}$$

- Hence, the scattered fields are

$$\vec{E}^s = -j\omega\vec{A} + \frac{\nabla\nabla \bullet \vec{A}}{j\omega\mu\varepsilon}, \vec{H}^s = \frac{1}{\mu}\nabla \times \vec{A}$$

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- Since scattered electric and magnetic field are dependent on magnetic vector potential which is dependent on the equivalent current density, hence the fields are dependent on the equivalent current density $\vec{J}_{eq}$
- Let us analyze the dependence of fields on the $\vec{J}_{eq}$
- Suppose $\vec{J}_{eq}$ is an infinitesimal elementary source at $\vec{r}^{'}$ pointed in x direction so that

$$\vec{J}_{eq} = \delta\left(\vec{r} - \vec{r}^{'}\right)\hat{x}$$

- The corresponding magnetic vector potential is

$$\vec{A} = \mu_0 G_0\left(\vec{r}, \vec{r}^{'}\right)\hat{x} \qquad \because \vec{A} = \mu_0 \int_{V^{'}} G_0\left(\vec{r}, \vec{r}^{'}\right)\vec{J}_{eq}\left(\vec{r}^{'}\right)dv^{'}$$

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- If $\vec{G}_{0x}\left(\vec{r},\vec{r}^{'}\right)$ is the electric field produced by the above mentioned elementary source, it must satisfy the wave equation

$$\nabla\times\nabla\times\vec{G}_{0x}\left(\vec{r},\vec{r}^{'}\right)-k_0^2\vec{G}_{0x}\left(\vec{r},\vec{r}^{'}\right)=-j\omega\mu_0\delta\left(\vec{r}-\vec{r}^{'}\right)$$

- whose solution is given by

$$\vec{E}^s=\vec{G}_{0x}=-j\omega\mu_0 G_0\left(\vec{r},\vec{r}^{'}\right)\hat{x}+\frac{\nabla\nabla\bullet G_0\left(\vec{r},\vec{r}^{'}\right)}{j\omega\varepsilon}\hat{x} \qquad \because \vec{E}^s=-j\omega\vec{A}+\frac{\nabla\nabla\bullet\vec{A}}{j\omega\mu\varepsilon}$$

$$\vec{A}=\mu_0 G_0\left(\vec{r},\vec{r}^{'}\right)\hat{x}$$

$$\Rightarrow\vec{G}_{0x}=-j\omega\mu_0\left(1+\frac{1}{k^2}\nabla\nabla\bullet\right)G_0\left(\vec{r},\vec{r}^{'}\right)\hat{x}$$

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- $\vec{G}_{0x}\left(\vec{r},\vec{r}^{'}\right)$ is referred to as a free space vector Green's function with a source pointed in the x-direction

- We could also find the free space vector Green's function $\vec{G}_{0y}\left(\vec{r},\vec{r}^{'}\right),\vec{G}_{0z}\left(\vec{r},\vec{r}^{'}\right)$ for a source pointed in the y-direction and z-direction respectively

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- We could now introduce a dyadic function which will store these three free space vector Green's function as

$$\overleftrightarrow{G}_0\left(\vec{r},\vec{r}^{'}\right)=\vec{G}_{0x}\left(\vec{r},\vec{r}^{'}\right)\hat{x}+\vec{G}_{0y}\left(\vec{r},\vec{r}^{'}\right)\hat{y}+\vec{G}_{0z}\left(\vec{r},\vec{r}^{'}\right)\hat{z}$$

- This is called free space dyadic Green's function
- It is a solution of the dyadic differential equation

$$\nabla\times\nabla\times\overleftrightarrow{G}_0\left(\vec{r},\vec{r}^{'}\right)-k_0^2\overleftrightarrow{G}_0\left(\vec{r},\vec{r}^{'}\right)=-j\omega\mu_0\overleftrightarrow{I}\delta\left(\vec{r}-\vec{r}^{'}\right)$$

- where unit dyad is given by

$$\overleftrightarrow{I}=\hat{x}\hat{x}+\hat{y}\hat{y}+\hat{z}\hat{z}$$

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- The physical meaning is that $\overleftrightarrow{G}_0\left(\vec{r},\vec{r}^{'}\right)$ is the electric field at a point $\vec{r}$ due to an infinitesimal source at $\vec{r}^{'}$ in any arbitrary orientation
- Then the scattered electric field due to any arbitrary equivalent current density may be expressed as

$$\vec{E}^{s}=\int_{V^{'}}\overleftrightarrow{G}_0\left(\vec{r},\vec{r}^{'}\right)\bullet\vec{J}_{eq}\left(\vec{r}^{'}\right)dv^{'}$$

- where

$$\overleftrightarrow{G}_0\left(\vec{r},\vec{r}^{'}\right)=\vec{G}_{0x}\left(\vec{r},\vec{r}^{'}\right)\hat{x}+\vec{G}_{0y}\left(\vec{r},\vec{r}^{'}\right)\hat{y}+\vec{G}_{0z}\left(\vec{r},\vec{r}^{'}\right)\hat{z}$$