Title: The TWISTED DERIVATION problem for ALGEBRAS

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Abstract: Let $\mathcal{R}$ be a ring with 1 and σ, τ be two different endomorphisms of $\mathcal{R}$. A (σ, τ)-derivation $\delta : \mathcal{R} \to \mathcal{R}$ is an additive map satisfying $\delta(ab) = \delta(a)\tau(b) + \sigma(a)\delta(b)$ for all $a, b \in \mathcal{R}$. If x is a fixed element in $\mathcal{R}$ and the (σ, τ)-derivation δ_x on $\mathcal{R}$ is such that $\delta_x(a) = x\tau(a) - \sigma(a)x$, then δ_x is called a (σ, τ)-inner derivation of $\mathcal{R}$ induced by x. If $\tau = id$, then δ and δ_x are simply called σ-derivation and σ-inner derivation respectively. If $\sigma = \tau = id$, then δ and δ_x are respectively the ordinary derivation and inner derivation (coming from x) of $\mathcal{R}$. Such twisted derivations were first mentioned by Jacobson in ([1],(Chapter 7.7)) in the study of generalization of Galois theory over division rings. Other important applications include the study of q-difference operators in number theory ([2], [3]). In the last decade some important applications of (σ, τ)-derivations were given by Hartwig et al. in their highly influential work [4]. They found a way for the study of deformations of Witt algebra and constructed generalizations of Lie algebras known as hom-Lie algebras with the help of such derivations. Just as algebras of derivations become Lie algebras, hom-Lie algebras were constructed so that they appear as algebras of twisted derivations wth some added natural conditions. Since then the study of these kinds of twisted derivations have gained a new impetus as hom-Lie algebras form very interesting mathematical objects helping in the study of deformations and discretizations of vector fields that have widespread applications to quantum physics, algebraic geometry and number theory ([5]).

Twisted derivations were introduced and studied mostly in the commuative ring setting. The most common examples of (σ, τ)-derivations are of the form $a(\tau - \sigma)$ for some suitable element a in the ring or some extension of it. It can be shown in many commutative rings *all* twisted derivations are of this form. Most of the known examples of of such twisted derivations in commutative rings are for unique factorization domains. In ([6]) we characterize such twisted derivations over rings of integers of quadratic number fields, thus providing examples of such derivations over some non-UFD's as well. We also investigate for fixed σ and τ, the module of (σ, τ)-derivations satisfying universal properties analogous to that of Kähler modules in case of ordinary derivations.

We however are more interested in twisted derivations of non-commutative rings, in particular, group rings. We provide a generalization of an application of Skolem-Noether Theorem to derivation on a finite dimensional central simple algebra for the (σ, τ)-derivation case ([7]). Then we prove the following in ([7]) and ([8]): Let R be a semiprime ring with 1 and G be a torsion group such that $[G : \mathcal{Z}(G)] < \infty$. Suppose that either char $R = 0$ or for every characteristic p of R, p does not divide $o(g)$, for all $g \in G$. If σ and τ are central R-endomorphisms of RG, there exists a ring T containing R such that $\mathcal{Z}(R) \subset \mathcal{Z}(T)$ and $\delta = \delta_x$ for some $x \in TG$. These are generalizations of the well known results on ordinary derivations of group rings by Spiegel [9] and Ferrero et al. [10]. We provide applications of the above result to integral group rings of finite groups and connect twisted derivations of integral group rings to other important problems in the field such as the Isomorphism Problem and the Zassenhaus Conjectures. We also give an example of a group G which is both locally finite and nilpotent and such that for every field F, there exists an F-linear σ-derivation of FG which is not σ-inner.

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