Rank Computation of Elliptic Curves : A pathway towards Congruent Number Problem over number fields

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A positive rational number $n \in \mathbb{Q}$ is said to be a *congruent number* if it can be represented as the area of a right angled triangle known as *a rational right triangle* whose side lengths are rational numbers. The *congruent number problem* asks for a classification of all congruent numbers. A quick calculation shows that the problem of finding congruent numbers becomes equivalent to finding $n \in \mathbb{N}$ such that there exists $a, b, c \in \mathbb{Q}$ satisfying

$$a^2 + b^2 = c^2 \text{ and } \frac{1}{2}ab = n.$$

One of the equivalent forms of the aforementioned congruent number problem is in terms of non-zero rank of elliptic curves of the form $E_n : y^2 = x^3 - n^2x$. This is the modern approach towards the *congruent number problem* and also the approach that we are most interested in.

In this talk we will mostly be focusing on the rank computation of an elliptic curve of a particular form. Rank of a congruent elliptic curve $y^2 = x^3 - n^2x$ effectively determines whether n is congruent or not. Unfortunately, there is no such algorithm till date that calculates the rank for an arbitrary elliptic curve. This, in turn, makes the problem of finding a general answer to the *congruent number problem* quite non-trivial. The modern approach towards the rank calculation for an elliptic curve is mostly dedicated to compute a group named *the Selmer Group* which contains deep arithmetic information about the elliptic curve, and whose rank is predicted to be equal to the Mordell-Weil rank by a part of the celebrated *Birch and Swinnerton-Dyer Conjecture*. Being one of the classical problems in number theory, the aforementioned problem has a vast literature dedicated towards its current development, a few of which will be mentioned in the talk. In his works, D. R. Heath Brown has shown a way to regulate the *Selmer group* of a congruent elliptic curve in a series of two papers ([1], [2]). In the second paper, there is an elaborate appendix written by P. Monsky which describes a way of calculating the 2-Selmer rank for a congruent elliptic curve in terms of rank calculation of square matrices. A detailed definition and significance of the *Selmer group* will be described in this talk. A subgroup of the aforementioned *Selmer group*, named *the Fine Selmer Group* behaves more interestingly in terms of its similarity with the behavior of ideal class groups of number fields. We will start with a brief procedure of rank computation of a general elliptic curve over number fields using the Ideal Class Groups of that number field and the *Selmer* and *Shafarevich-Tate group* of an arbitrary elliptic curve of certain form. We will finish the talk with the application of our procedure on the congruent number problem in a more general set-up.

[1] Heath-Brown, D.R., The size of Selmer groups for the congruent number problem. *Invent. Math.*, 111 (1993), 171–195.

[2] Heath-Brown, D.R., The size of Selmer groups for the congruent number problem. II, with an appendix by P. Monsky. *Invent. Math.*, 118 (1994), 331–370.