# Physics II: Electromagnetism

## PH 102

**Lecture 1**

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Thanks to
Dr. Sayan Chakrabarti

January-May 2020

# Syllabus

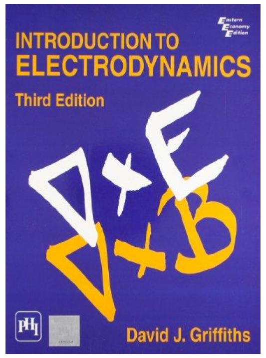
- Mathematical Preliminaries:
- Vector analysis: Gradient, Divergence and Curl; Line, Surface, and Volume integrals; Gauss's divergence theorem and Stokes' theorem in Cartesian; Spherical polar and Cylindrical polar coordinates;
- Dirac Delta function.
- Electrostatics: Gauss's law and its applications, Divergence and Curl of Electrostatic fields, Electrostatic Potential, Boundary conditions, Work and Energy, Conductors, Capacitors, Laplace's equation, Method of images, Boundary value problems in Cartesian Coordinate Systems, Dielectrics, Polarisation, Bound Charges, Electric displacement, Boundary conditions in dielectrics, Energy in dielectrics, Forces on dielectrics.

Before midsem

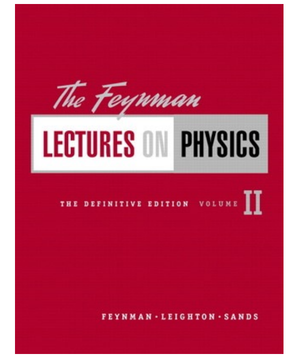
- Magnetostatics: Lorentz force, Biot-Savart and Ampere's laws and their applications, Divergence and Curl of Magnetostatic fields, Magnetic vector Potential, Force and torque on a magnetic dipole, Magnetic materials, Magnetization, Bound currents, Boundary conditions.
- Electrodynamics: Ohm's law, Motional EMF, Faraday's law, Lenz's law, Self and Mutual inductance, Energy stored in magnetic field, Maxwell's equations, Continuity Equation, Poynting Theorem, Wave solution of Maxwell Equations.
- Electromagnetic waves: Polarization, reflection & transmission at oblique incidences.

After midsem

# Texts and References

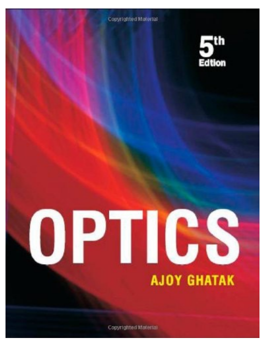



Introduction to electrodynamics : D. J. Griffiths, 3rd edition, Prentice Hall of India (2005)

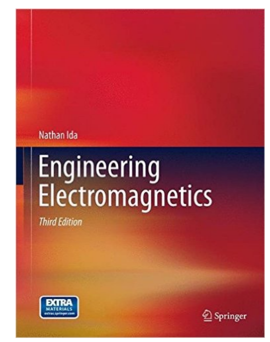



The Feynman lectures on Physics, Vol II: R. P. Feynman, R. B. Leighton and M. Sands, Narosa Publishing House (1998)

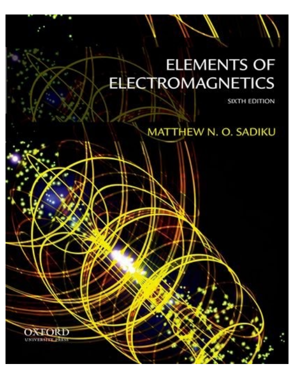
http://www.feynmanlectures.caltech.edu/



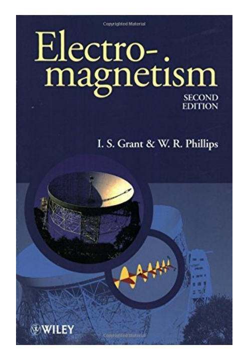

Optics : A. K. Ghatak, Tata McGraw Hill (2007)




Engineering Electromagnetics : N. Ida, Springer (2005)




Elements of Electromagnetics : M. N. O. Sadiku, Oxford (2005)




Electromagnetism : I. S. Grant and W. R. Phillips, John Wiley (1990)

Website: Lecture slides, tutorials and discussion forum on http://www.iitg.ac.in/phy/ph102.php

## Classes

- There will be two classes and a tutorial in a week (just like PH101).
- Classes: **Wednesday: 11 am** (Div. III & IV) and **4 pm** (Div. I & II) and **Thursday: 11 am** (Div. III & IV) and 4 **pm** (Div. I & II).
- Tutorial: **Tuesday (8 am)**
- **Sometimes there will be class on Saturday/Sunday at the same time as a back up class.**
- Mode of teaching: mainly using slides, black boards will be used as and when necessary.

## Tutorials

- Tutorials are for you to interact with a teacher.
- Necessarily held to clear your doubts and answer your queries regarding the course topic
- A problem sheet will be given after sufficient coverage of the course topics
- You are expected to attempt these problems before you come to tutorials
- The teacher may or may not solve all the problems in the tutorial. If you find a problem to be difficult, please ask the teacher to help.

75% attendance in class+tutorial is mandatory for appearing in end sem.

# Assessments

All questions in the exams will be subjective type with short/or long answers

| Events | Date | % Marks |
|---|---|---|
| Quiz 1 | 4th February, 2020 | 10% |
| Mid-sem | 27th Feb to 4th March | 30% |
| Quiz 2 | To be decided | 10% |
| End-sem | 2nd to 9th May, 2020 | 50% |

Have any query/doubt?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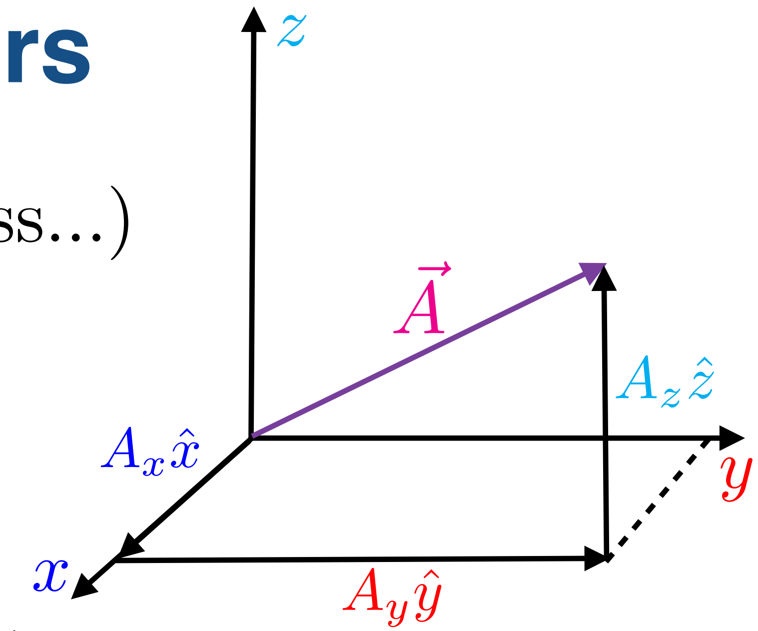
# Part I
# Scalars and Vectors

# What you already know : Algebra of vectors




A quantity with magnitude : Scalar (Temperature, Mass...)

A quantity with magnitude and direction: Vector (Position, Velocity, Force...)

(Does not have location)

A few facts about the algebra of vectors:

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$

$$\begin{aligned}
\vec{A}.\vec{B} &= \text{scalar} = A_xB_x + A_yB_y + A_zB_z = AB\cos\theta \\
\vec{A}\times\vec{B} &= \text{vector} = AB\sin\theta\hat{n} \\
(\vec{A}\times\vec{B})_z &= A_xB_y - A_yB_x \\
(\vec{A}\times\vec{B})_x &= A_yB_z - A_zB_y \\
(\vec{A}\times\vec{B})_y &= A_zB_x - A_xB_z
\end{aligned}$$

Projection of $\vec{B}$ along $\vec{A}$. or other way.

: • Direction of $\hat{n}$.
• Area of a parallelogram.

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Some more

$$\begin{aligned}
\vec{A}\times\vec{A} &= \vec{0} \\
\vec{A}.(\vec{A}\times\vec{B}) &= 0 \\
\vec{A}.(\vec{B}\times\vec{C}) &= (\vec{A}\times\vec{B}).\vec{C} = \vec{B}\cdot(\vec{C}\times\vec{A}) \\
\vec{A}\times(\vec{B}\times\vec{C}) &= \vec{B}\,(\vec{A}.\vec{C}) - \vec{C}\,(\vec{A}.\vec{B})
\end{aligned}$$

Volume of parallelepiped

• Will sometime use $\vec{A} = (A_x, A_y, A_z)$ to denote a vector

# What you may not know : Levi Civita symbol

You may skip this part and stick to usual way of calculation of cross products.
But very useful to do the vector computations!!

$$\vec{A}.\vec{B} = \sum_{i=1}^{3} A_i B_i \quad \text{also written as} \longrightarrow A_i B_i \quad \text{(Einstein's summation convention)}$$

$$(= A_1 B_1 + A_2 B_2 + A_3 B_3)$$

where $1 \to x,\ 2 \to y,\ 3 \to z$

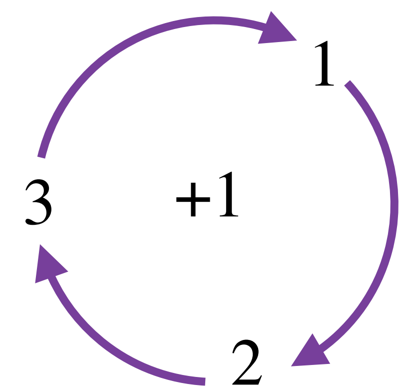
The cross product can be conveniently described by introducing a symbol $\varepsilon_{ijk}$ (Levi Civtia)

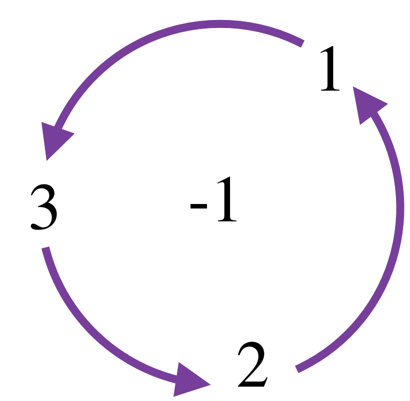
$j, k$ : dummy indices
$i$ : free index

$$(\vec{A} \times \vec{B})_i = \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} A_j B_k$$

where $\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$ and $\varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213} = -1$ and all $\varepsilon_{iij} = \varepsilon_{iii} = 0$

$$
\begin{aligned}
(\vec{A} \times \vec{B})_1 &= \varepsilon_{123} A_2 B_3 + \varepsilon_{132} A_3 B_2 = A_2 B_3 - A_3 B_2 \\
(\vec{A} \times \vec{B})_2 &= \varepsilon_{231} A_3 B_1 + \varepsilon_{213} A_1 B_3 = A_3 B_1 - A_1 B_3 \\
(\vec{A} \times \vec{B})_3 &= \varepsilon_{312} A_1 B_2 + \varepsilon_{321} A_2 B_1 = A_1 B_2 - A_2 B_1
\end{aligned}
$$




Helps to simplify cross product calculations.

# Levi Civita: An example

You may skip this part and stick to usual way of calculation of cross products

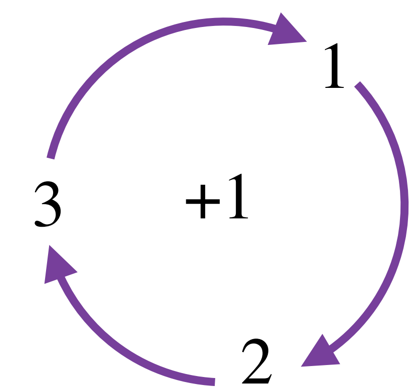
1. $\varepsilon_{ijk}$ has $3 \times 3 \times 3 = 27$ components: 3 components are equal to 1, 3 are equal to $-1$ and rest 21 are zero.

2. $\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$, where $\delta_{ij} = 1$ if $i = j$ and 0 if $i \neq j$.

An example:

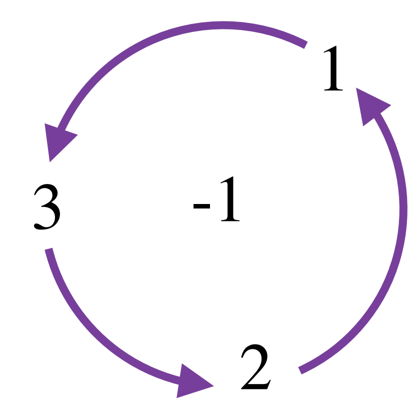
Show that $\quad \vec{A}.(\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}).\vec{C}$

$$\text{LHS } = A_i \varepsilon_{ijk} B_j C_k = \varepsilon_{ijk} A_i B_j C_k$$




By cycling the indices, we get

$$\varepsilon_{ijk} A_i B_j C_k = \varepsilon_{kij} C_k A_i B_j = \varepsilon_{jki} B_j C_k Ai.$$




Hence

$$\vec{A}.(\vec{B} \times \vec{C}) = \vec{C}.(\vec{A} \times \vec{B}) = \vec{B}.(\vec{C} \times \vec{A})$$

Little complicated example:

Example 1: Prove $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}\,(\vec{A}\cdot\vec{C}) - \vec{C}\,(\vec{A}\cdot\vec{B})$

ith comp. of LHS: $[\vec{A} \times (\vec{B} \times \vec{C})]_i$

$$= \epsilon_{ijk}\, A_j\, (\vec{B} \times \vec{C})_k$$

$$= \epsilon_{ijk}\, A_j\, \epsilon_{klm}\, B_l\, C_m$$

$$= \epsilon_{kij}\, \epsilon_{klm}\, A_j\, B_l\, C_m$$

$$= (\delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl})\, A_j\, B_l\, C_m$$

$$= A_m\, B_i\, C_m - A_l\, B_l\, C_i$$

$$= B_i\,(\vec{A}\cdot\vec{C}) - C_i\,(\vec{A}\cdot\vec{B})$$

$$= [\vec{B}\,(\vec{A}\cdot\vec{C})]_i - [\vec{C}\,(\vec{A}\cdot\vec{B})]_i$$

Home Works:

1. $\vec{A}\times(\vec{B}\times\vec{C})+\vec{B}\times(\vec{C}\times\vec{A})+\vec{C}\times(\vec{A}\times\vec{B})=\vec{0}$

2. $(\vec{A}\times\vec{B})\times(\vec{C}\times\vec{D}) = \vec{B}\,(\vec{A}\cdot\vec{C}\times\vec{D}) - \vec{A}\,(\vec{B}\cdot\vec{C}\times\vec{D})$

Example 2: Prove $(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) = (\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})$

$$LHS = (\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D})$$

$$= (\vec{A}\times\vec{B})_i\,(\vec{C}\times\vec{D})_i$$

$$= \epsilon_{ijk}\,A_j\,B_k\;\epsilon_{ilm}\,C_l\,D_m$$

$$= \epsilon_{ijk}\, \epsilon_{ilm}\, A_j\, B_k\, C_l\, D_m$$

$$= (\delta_{jl}\, \delta_{km} - \delta_{jm}\, \delta_{kl})\, A_j\, B_k\, C_l\, D_m$$

$$= A_l\, B_m\, C_l\, D_m - A_m\, B_l\, C_l\, D_m$$

$$= (\vec{A}\cdot\vec{C})\,(\vec{B}\cdot\vec{D}) - (\vec{A}\cdot\vec{D})\,(\vec{B}\cdot\vec{C})$$

<u>Home works:</u> 1. $(\vec{A}\times\vec{B})\cdot(\vec{B}\times\vec{C})\times(\vec{C}\times\vec{A})$

$$= \left[\vec{A}\cdot(\vec{B}\times\vec{C})\right]^2$$

2. $(\vec{A}\times\vec{B})\cdot(\vec{C}\times\vec{D}) + (\vec{B}\times\vec{C})\cdot(\vec{A}\times\vec{D})$

$+ (\vec{C}\times\vec{A})\cdot(\vec{B}\times\vec{D}) = 0.$

More Homeworks :

1. $(\vec{A}+\vec{B})\cdot(\vec{B}+\vec{C})\times(\vec{C}+\vec{A}) = 2\,\vec{A}\cdot(\vec{B}\times\vec{C})$

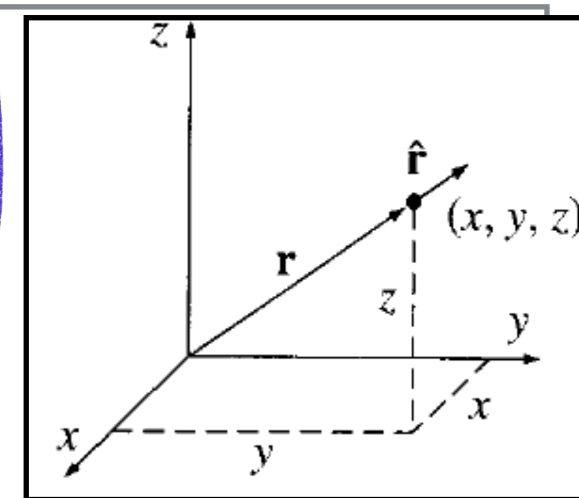
2. $|\vec{A}\times\vec{B}|^2 + |\vec{A}\cdot\vec{B}|^2 = |\vec{A}|^2|\vec{B}|^2$

# Position, Displacement and Separation vectors

**Position:** $\mathbf{r} \equiv x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}.$

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{r} = \frac{x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}}$$

Magnitude (Distance from origin)

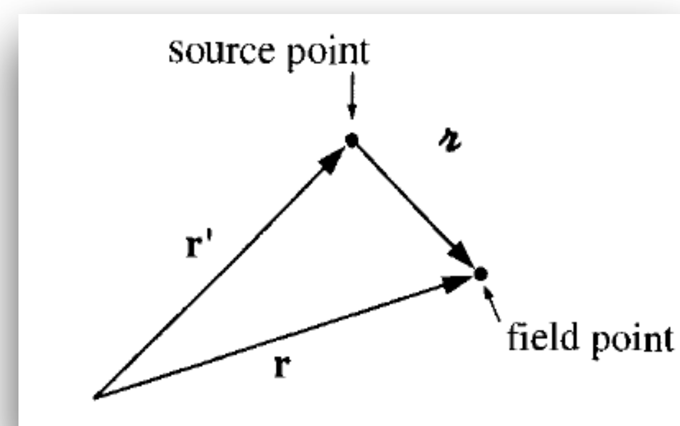



**Infinitesimal displacement vector:** From $(x, y, z)$ to $(x + dx, y + dy, z + dz)$ is

$$d\vec{r} = dx\hat{x} + dy\hat{y} + dz\hat{z}$$

**Separation vector:** $\boldsymbol{\mathscr{r}} \equiv \mathbf{r} - \mathbf{r}'.$

$$\hat{\boldsymbol{\mathscr{r}}} = \frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}} = \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}.$$

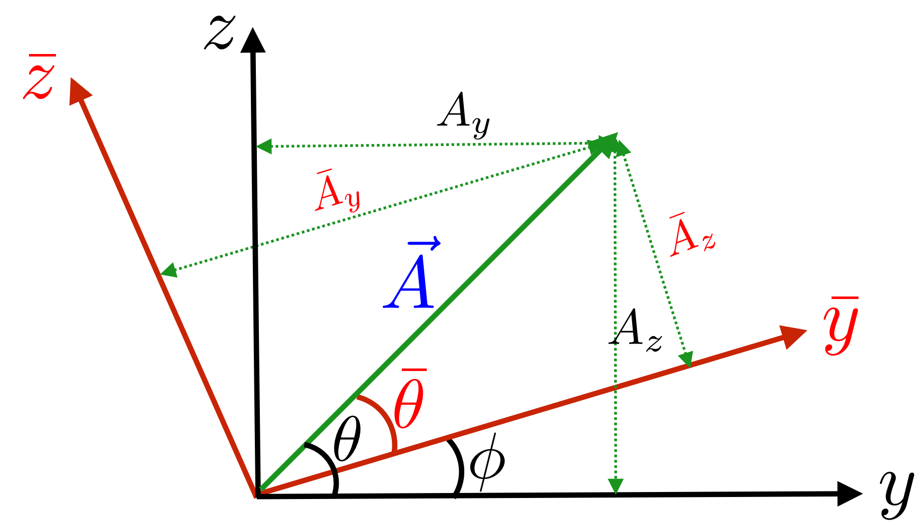


# How vectors transform

A quantity with magnitude and direction : Vector $\longrightarrow$ Not satisfactory Why?

Components of a vector has to transform "properly" under a coordinate change.

Suppose $\bar{x}, \bar{y}, \bar{z}$ coordinate system is rotated by angle $\phi$ relative to $x, y, z$ about a common axis $x = \bar{x}$ which is perpendicular to the page.




$$A_y = A\cos\theta, \;\; A_z = A\sin\theta$$

$$\bar{A}_y = A\cos\bar{\theta} = A\cos(\theta-\phi) = A(\cos\theta\cos\phi+\sin\theta\sin\phi) = \cos\phi A_y+\sin\phi A_z$$

$$\bar{A}_z = A\sin\bar{\theta} = A\sin(\theta-\phi) = A(\sin\theta\cos\phi-\cos\theta\sin\phi) = -\sin\phi A_y+\cos\phi A_z$$

In a compact form
$$\begin{pmatrix}\bar{A}_y\\ \bar{A}_z\end{pmatrix} = \begin{pmatrix}\cos\phi & \sin\phi\\ -\sin\phi & \cos\phi\end{pmatrix}\begin{pmatrix}A_y\\ A_z\end{pmatrix}$$

A scalar remains invariant under a change in coord. system

In general, for rotation about arbitrary axis in 3-D

$$\begin{pmatrix}\bar{A}_x\\ \bar{A}_y\\ \bar{A}_z\end{pmatrix} = \begin{pmatrix}R_{xx} & R_{xy} & R_{xz}\\ R_{yx} & R_{yy} & R_{yz}\\ R_{zx} & R_{zy} & R_{zz}\end{pmatrix}\begin{pmatrix}A_x\\ A_y\\ A_z\end{pmatrix} \Longrightarrow \bar{A}_i = \sum_{j=1}^{3} R_{ij}A_j$$

A vector is any set of 3 components that has the above transformation property

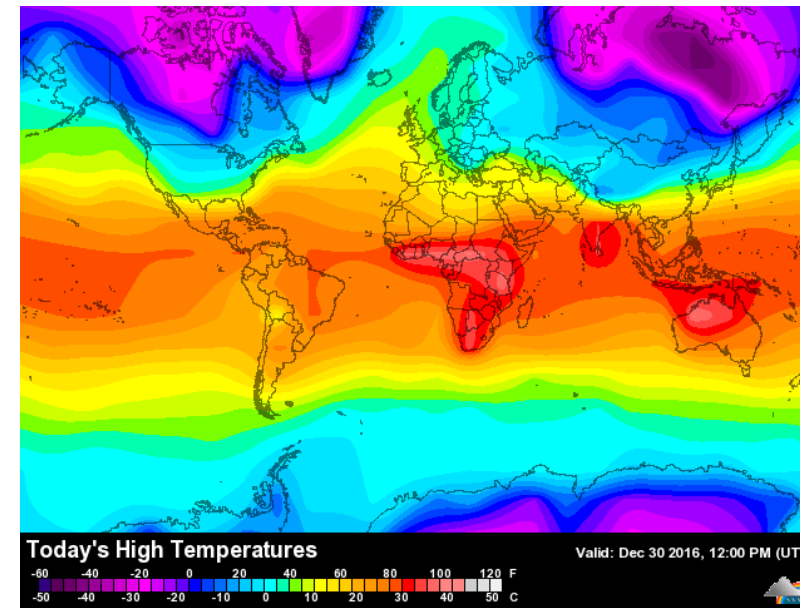
# Part II
# Scalar and Vector Fields

# Concept of a field

By a field we mean to associate a scalar or vector property to each point in a region of space.

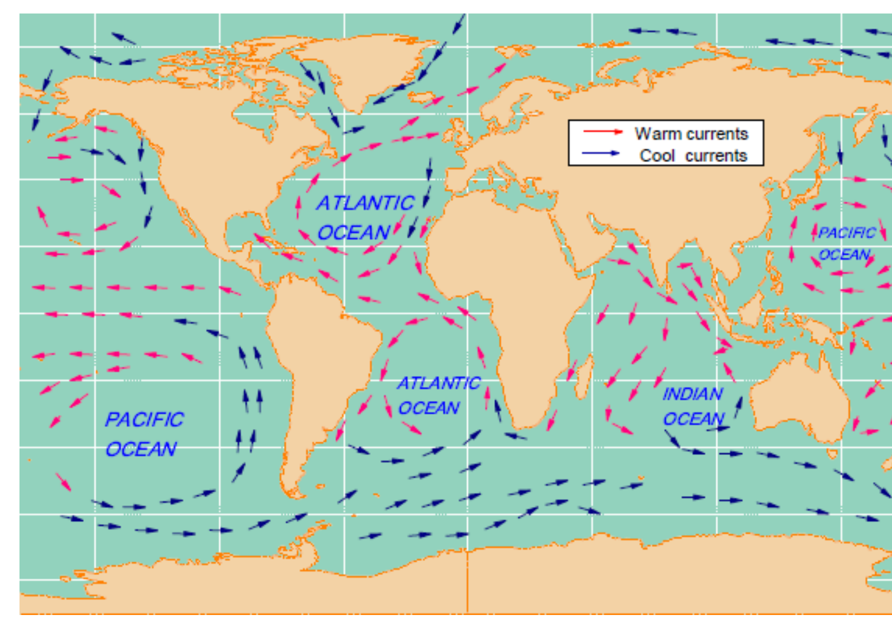
**Scalar field:**
We can associate a number at each and every point in space.

Example: We can associate a temperature (a number) at every point in this room. The field is the temperature field and it is a scalar field because the field quantity “temperature” is a scalar. Similarly density, potential energy etc…



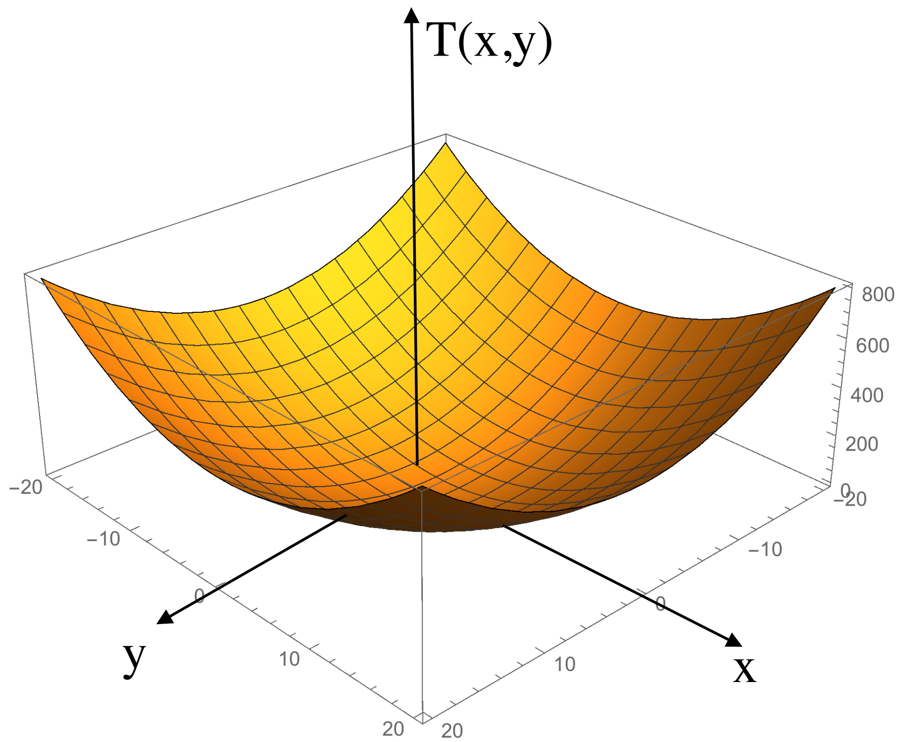

**Vector field:**
We can associate a vector at each and every point in space.

Example: Gravitational field, electric & magnetic field, ocean current etc…



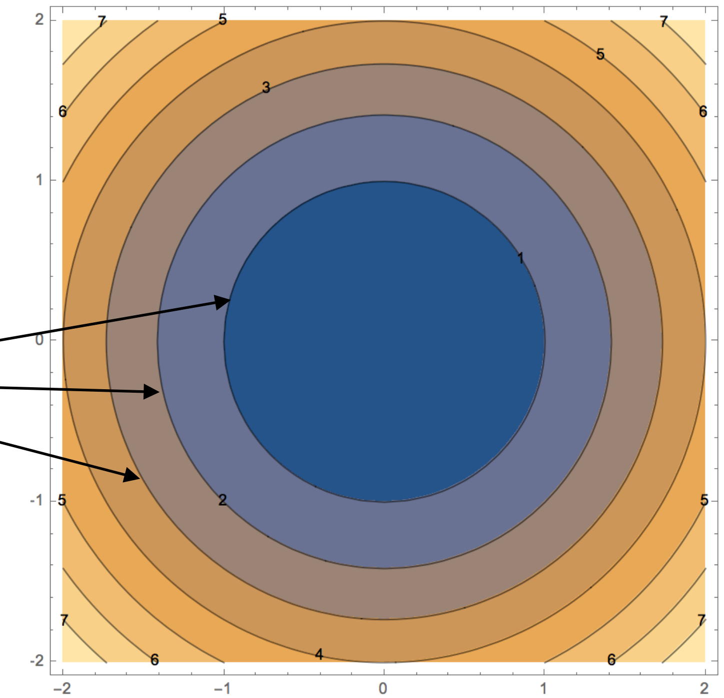
# Scalar Fields: How to "see" them?

Imagine "contours" which are imaginary surfaces drawn through all points for which the field has the same value: level surfaces (in 3 D)/level curves (in 2 D). For a temperature field these are called "isothermal surfaces" or isotherms.

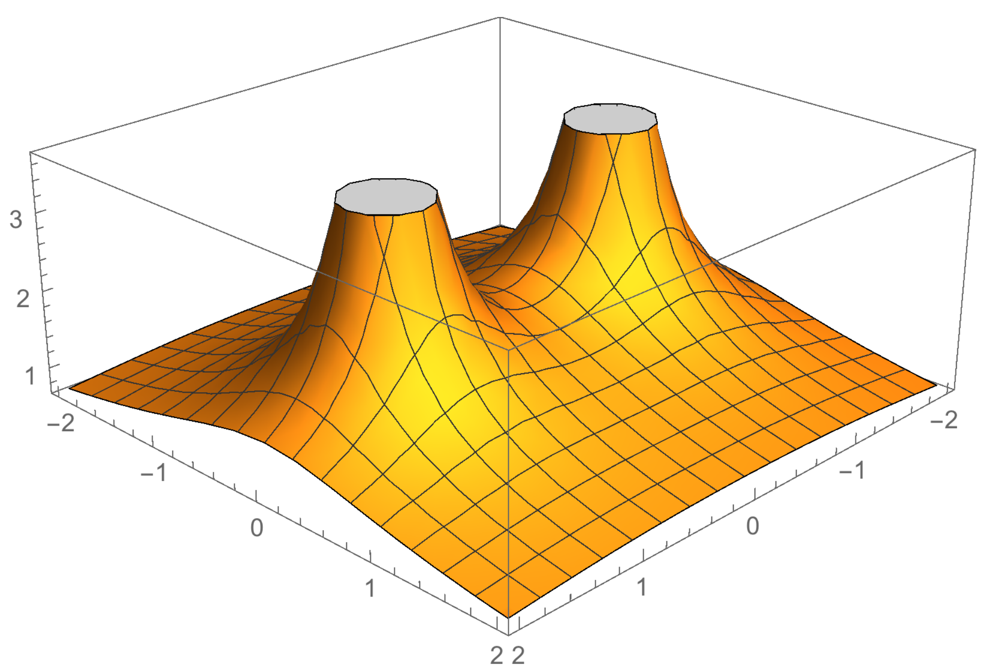



$$T(x, y) = x^2 + y^2$$

Isotherms

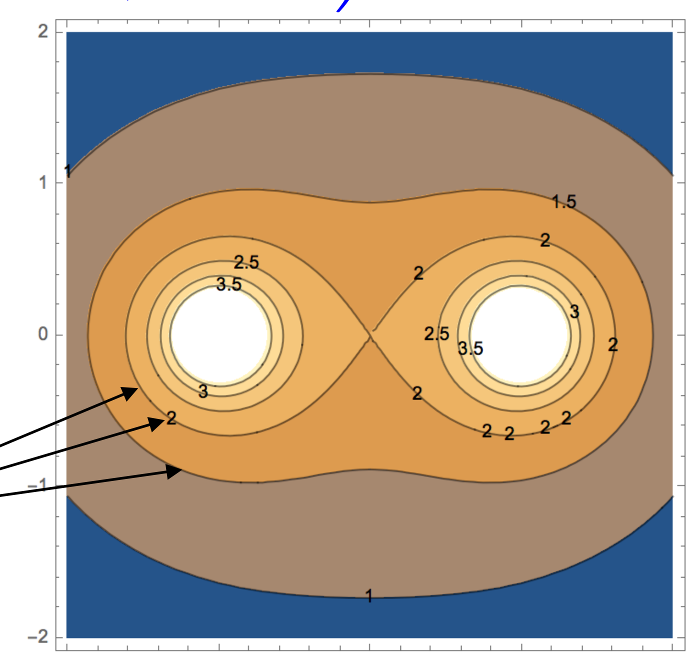


$$V(x, y) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{(x-1)^2+y^2}} + \frac{1}{\sqrt{(x+1)^2+y^2}}\right)$$



Potential due to 2 identical point charges at (1,0,0) and (-1,0,0)

Equipotentials

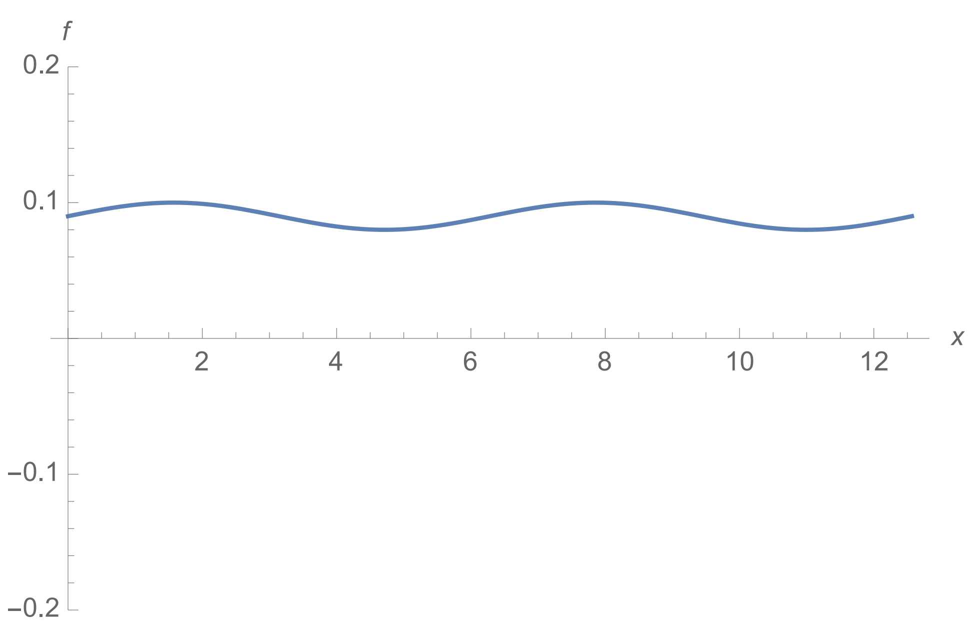


# Differential Calculus of the scalar field

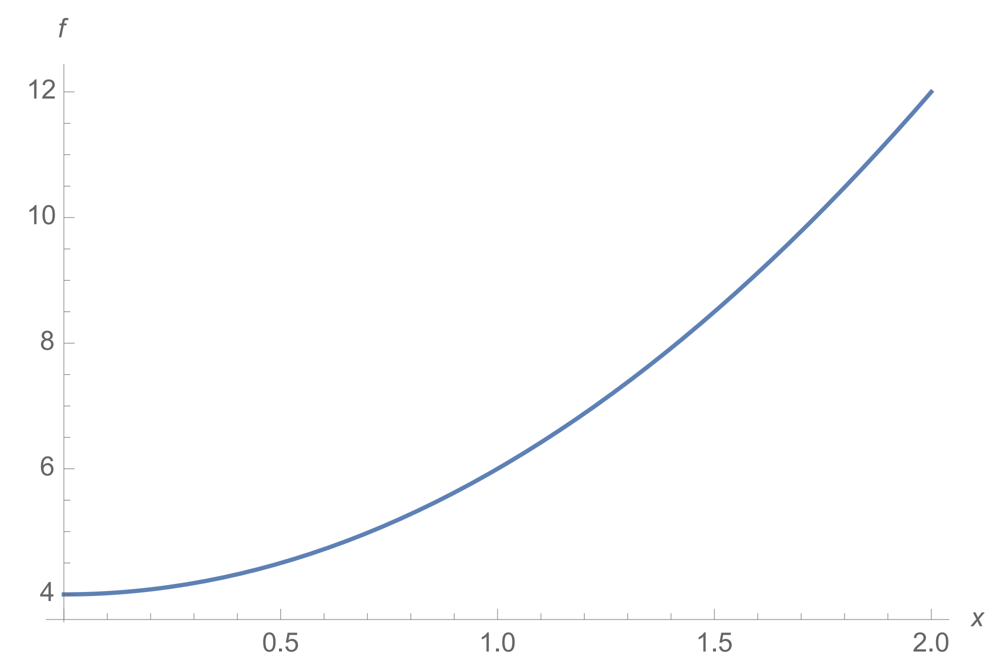
We already know the definition of ordinary derivative of a function $f(x)$ of a single variable $x$

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Tells us how rapidly the function $f(x)$ varies when we change the argument $x$ by a tiny amount $\Delta x$.




The function varies slowly with x
The derivative is small




The function varies rapidly with x
The derivative is large

Derivative $df/dx$ is the slope of the function.

# Differential Calculus of the scalar field

What happens in higher dimensions? i.e. if the scalar field is function of more variables?

Suppose we have a scalar field $T(x,y,z)$ describing a temperature field and we want to know how the temperature at one place is related to the temperature at a nearby place.

How does $T$ vary with position? Caution: T now depends on 3 variables x, y and z!

$\longrightarrow$ Do we differentiate T with respect to x? Or with respect to y, or z?

Remember: Useful physical laws do not depend upon the orientation of the coordinate system. Therefore the laws should be written in a form in which either both sides are scalars or both are vectors!

What is the derivative of a scalar field, say $\partial T/\partial x$? Is it a scalar, or a vector?

$\longrightarrow$ It is neither. Because if you took a different $x$-axis, $\partial T/\partial x$ would certainly be different.

# The Gradient

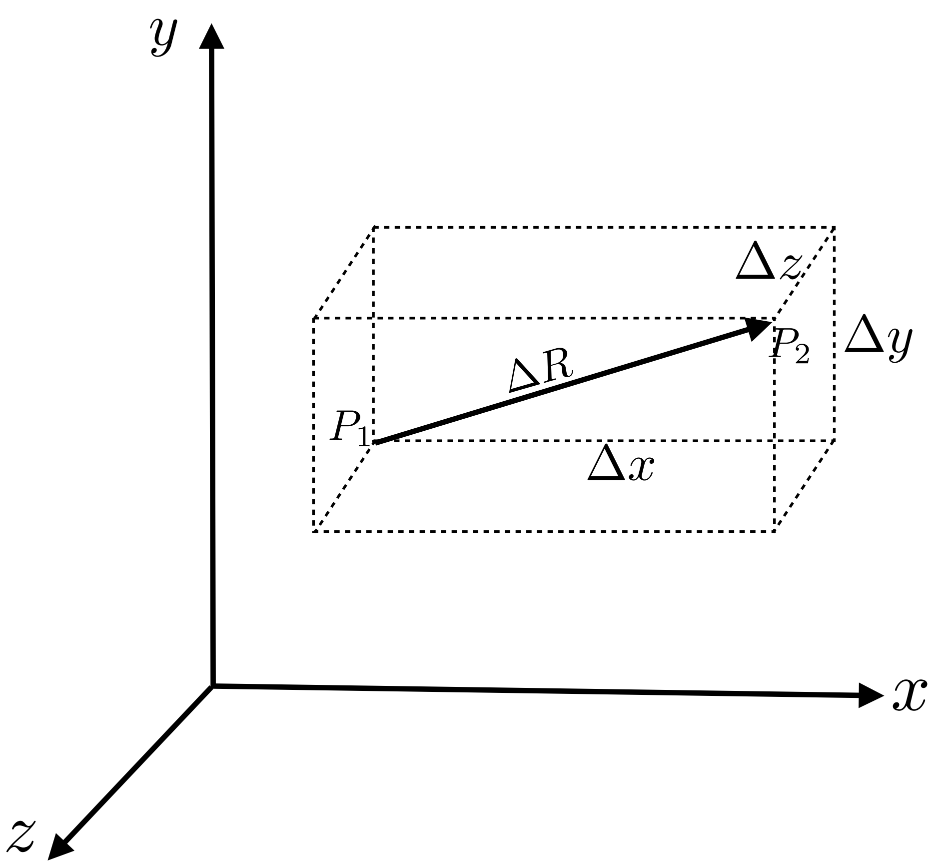
Notice: since $T = T(x, y, z)$, we have three possible derivatives: $\partial T/\partial x$, $\partial T/\partial y$ , $\partial T/\partial z$.
And recall: it takes three numbers to form a vector.

Perhaps, these three derivatives are the components of a vector:

$$\left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z}\right) = \text{A vector?}$$

But, any 3 numbers do not form a vector. The components has to transform among themselves correctly under a rotation of the coordinate system.

Think about the temperature field at $P_1$ and $P_2$ separated by $\Delta\vec{R}$




$$\Delta T = T_2 - T_1 \longrightarrow \text{Scalar}$$

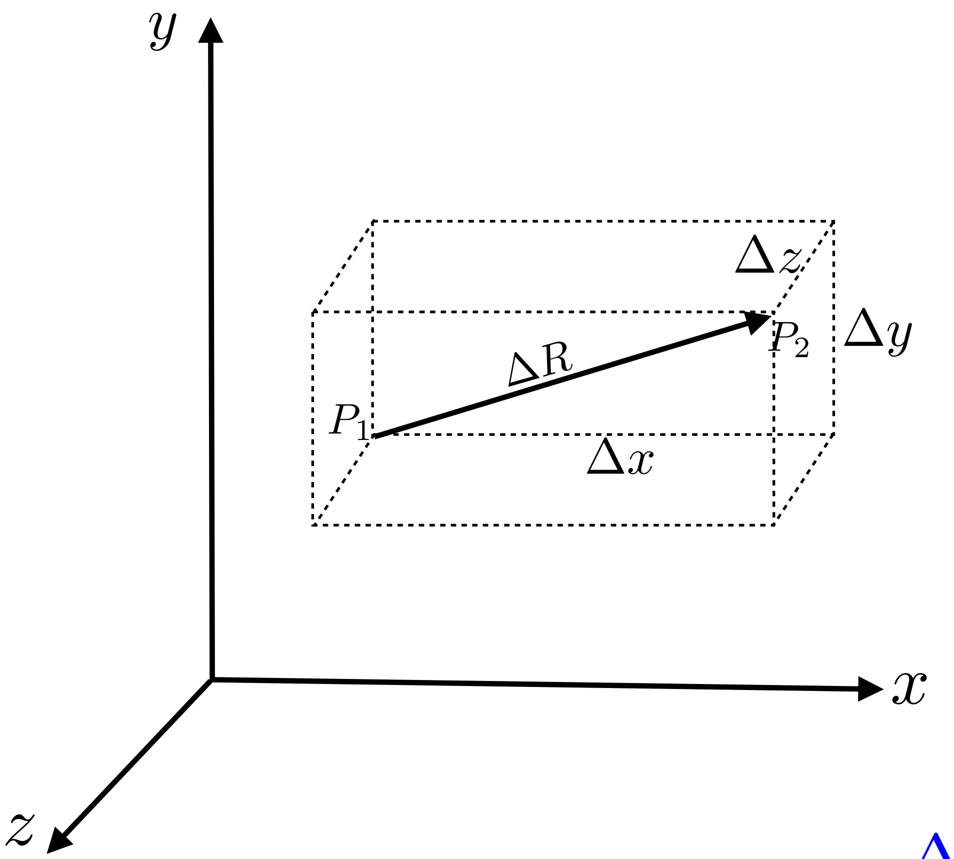
In certain coord. system $T_1 = T(x, y, z)$ and $T_2 = T(x + \Delta x, y + \Delta y, z + \Delta z)$.

$\Delta x, \Delta y,\ \&\ \Delta z$ are components of the vector $\Delta\vec{R}$.

$$\Delta T = \frac{\partial T}{\partial x}\Delta x + \frac{\partial T}{\partial y}\Delta y + \frac{\partial T}{\partial z}\Delta z$$

(Theorem on "partial derivatives": true in the limit $\Delta x, \Delta y, \Delta z$ tends to zero)

# The Gradient




$$\Delta T = \frac{\partial T}{\partial x}\Delta x + \frac{\partial T}{\partial y}\Delta y + \frac{\partial T}{\partial z}\Delta z$$

Scalar

Components of vector $\Delta \vec{R}$

Must be x, y, z component
of another vector

$$\Delta T = \left(\hat{x}\frac{\partial T}{\partial x} + \hat{y}\frac{\partial T}{\partial y} + \hat{z}\frac{\partial T}{\partial z}\right).(\Delta x\hat{x} + \Delta y\hat{y} + \Delta z\hat{z})$$

$$\Delta T = \vec{\nabla} T . \Delta \vec{R}$$

We call this new vector Gradient of $T$ (or "del-$T$", or "grad-$T$"). The symbol $\vec{\nabla}$ (called "del") is supposed to remind us of differentiation!

Important: The quantity $\vec{\nabla} T$ is a vector.

$\Delta T = \vec{\nabla} T . \Delta \vec{R} \longrightarrow$ says that the difference in temperature between two nearby points is the dot product of the gradient of $T$ and the vector displacement between the points.

# The Gradient: Geometrical meaning

$\Delta T = \vec{\nabla} T . \Delta \vec{R} = |\vec{\nabla} T||\Delta \vec{R}| \cos\theta$ $\quad$ $\theta$ is the angle between $\vec{\nabla} T$ and $\Delta \vec{R}$.

If we fix magnitude $|\Delta \vec{R}|$ and search around in various directions (i.e. vary $\theta$), the maximum change in $T$ occurs when $\theta = 0$ (i.e. $\cos\theta = 1$).

i.e. for a fixed distance $|\Delta \vec{R}|$, $\Delta T$ is greatest when you move in the same direction as $\vec{\nabla} T$.

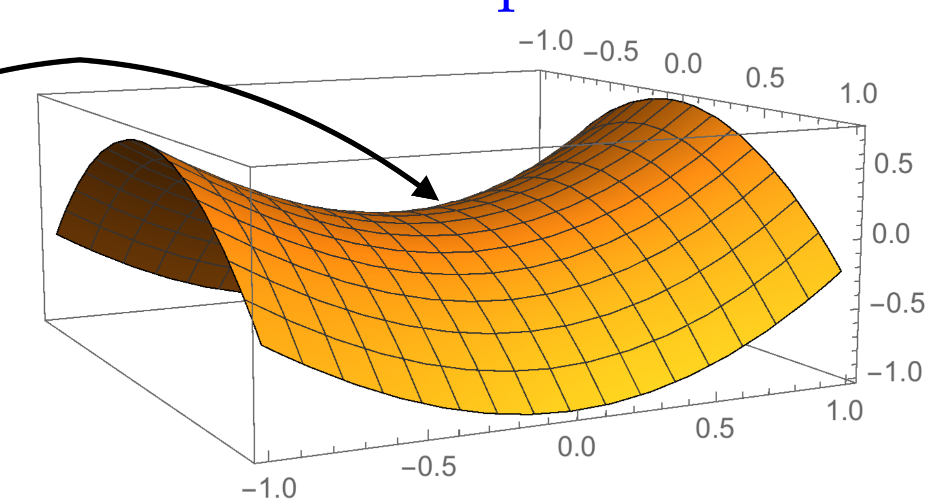
The gradient $\vec{\nabla} T$ points in the direction of maximum increase of the scalar function $T$.

The magnitude $|\vec{\nabla} T|$ gives the slope (rate of change of $T(x, y, z)$) along this maximal direction.

$\vec{\nabla} T = 0$ at a point $(x, y, z)$ $\implies$ maximum or minimum or saddle point.

as $\Delta T = 0$

Saddle point




Analogous to the situation for functions of one variable!

# The Gradient: Geometrical meaning

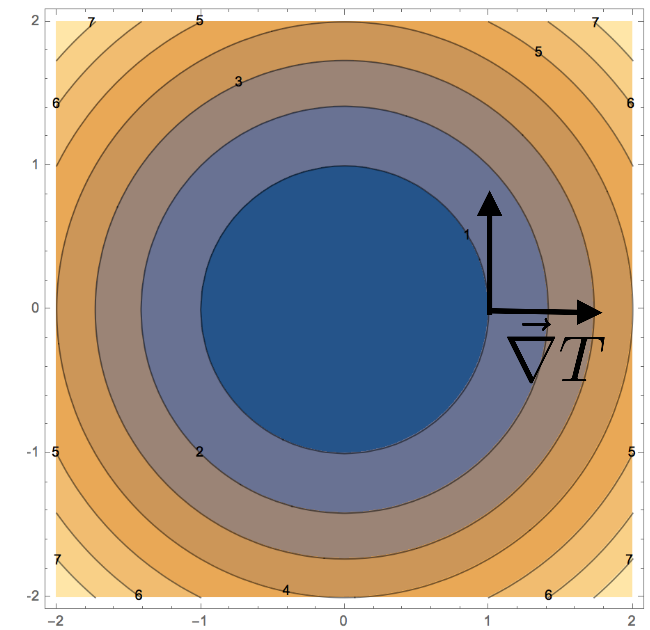
Gradient of a scalar function $T$ is always normal to the level surfaces or level curves.

Level curve : parametrised by a variable $t$, which varies from point to point on the curve. Example of such a parameter for the circle is angle $\theta$, so that $x = R\cos\theta, y = R\sin\theta$, where $R$ (fixed) is the radius and $0 < \theta < 2\pi$ is the polar angle.

$$\text{Position vector on the level curve } \vec{r}(t) = x(t)\hat{x} + y(t)\hat{y} + 0\hat{z}$$

$$\text{Equation of the level curve } T(x(t), y(t), z(t) = 0) = \text{constant} \implies \frac{\mathrm{dT}}{\mathrm{dt}} = 0$$

$$\text{Tangent to the curve } \vec{r}\,' = \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + 0\hat{z}$$




$$\frac{dT}{dt} = \frac{\partial T}{\partial x}\frac{dx}{dt} + \frac{\partial T}{\partial y}\frac{dy}{dt} = \vec{\nabla}T.\vec{r}\,' = 0$$

Which shows that the gradient is normal to the level curve.

# Directional Derivative

Recall: $\overrightarrow{A} \cdot \overrightarrow{B} = AB\cos\theta$

Projection of $\overrightarrow{A}$ along $\overrightarrow{B}$ OR vice versa.

$dT = d\overrightarrow{R} \cdot \overrightarrow{\nabla} T$

Variation of T along $d\overrightarrow{R}$.

The change of any quantity (Q) along a vector, say $\overrightarrow{N}$ :

$$\overrightarrow{N} \cdot \overrightarrow{\nabla} Q = N_i \frac{\partial}{\partial x_i} Q$$ : Directional derivative

Q can be scalar or vector.

# The operator $\vec{\nabla}$

$$\vec{\nabla} \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \quad \text{which means of course} \quad \nabla_x \equiv \frac{\partial}{\partial x}, \quad \nabla_y \equiv \frac{\partial}{\partial y}, \quad \nabla_z \equiv \frac{\partial}{\partial z}$$

Important: $\vec{\nabla}$ is a vector operator, alone it does not have a meaning.

In PH101, you have already encountered the derivative as an operator (recall momentum operator $-i\hbar\partial/\partial x$ in quantum mechanics).

$\vec{\nabla}T$ means that $\vec{\nabla}$ is a vector operator that acts upon a scalar field $T$ to give a vector field. (<u>$\vec{\nabla}T$ does not mean $\vec{\nabla}$ is a vector that multiplies a scalar $T$</u>).

Since $\vec{\nabla}$ is an operator, $T\vec{\nabla} \neq \vec{\nabla}T$ (unlike ordinary algebra)

Still an operator, "hungry for something to differentiate"

The operator $\vec{\nabla}$ has already acted to give a Vector field

**Examples on gradient operation:**

1. Find $\overrightarrow{\nabla} f(r)$ where $f(r) = \ln r, \vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$.

$$r = \sqrt{x^2 + y^2 + z^2} \Rightarrow f(r) = \ln r = \frac{1}{2}\ln(x^2 + y^2 + z^2).$$

Now $\overrightarrow{\nabla} f(r) = \hat{x}\dfrac{\partial f}{\partial x} + \hat{y}\dfrac{\partial f}{\partial y} + \hat{z}\dfrac{\partial f}{\partial z}$

Here $\dfrac{\partial f}{\partial x} = \dfrac{2x}{2(x^2 + y^2 + z^2)} = \dfrac{x}{r^2}$

Similarly, $\dfrac{\partial f}{\partial y} = \dfrac{y}{r^2}$ and $\dfrac{\partial f}{\partial z} = \dfrac{z}{r^2}$ .

Therefore, $\overrightarrow{\nabla} f(r) = \dfrac{x\hat{x} + y\hat{y} + z\hat{z}}{r^2} = \dfrac{\vec{r}}{r^2}$.

More examples to work out will be given in Tutorial 1

2. Find the directional derivative of $\phi = x^2yz + 4xz^2$ at $(1,-2,-1)$ in the direction $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

$$\begin{aligned}\nabla\phi &= \nabla(x^2yz + 4xz^2) = (2xyz + 4z^2)\mathbf{i} + x^2z\,\mathbf{j} + (x^2y + 8xz)\mathbf{k} \\ &= 8\mathbf{i} - \mathbf{j} - 10\mathbf{k} \quad \text{at} \quad (1,-2,-1).\end{aligned}$$

The unit vector in the direction of $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ is

$$\mathbf{a} = \frac{2\mathbf{i} - \mathbf{j} - 2\mathbf{k}}{\sqrt{(2)^2 + (-1)^2 + (-2)^2}} = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$$

Then the required directional derivative is

$$\nabla\phi \cdot \mathbf{a} = (8\mathbf{i} - \mathbf{j} - 10\mathbf{k}) \cdot (\tfrac{2}{3}\mathbf{i} - \tfrac{1}{3}\mathbf{j} - \tfrac{2}{3}\mathbf{k}) = \frac{16}{3} + \frac{1}{3} + \frac{20}{3} = \frac{37}{3}$$

Since this is positive, $\phi$ is increasing in this direction.

3. Find the angle between the surfaces $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at the point $(2,-1,2)$.

The angle between the surfaces at the point is the angle between the normals to the surfaces at the point.

A normal to $x^2+y^2+z^2=9$ at $(2,-1,2)$ is

$$\nabla\phi_1 = \nabla(x^2+y^2+z^2) = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k} = 4\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$$

A normal to $z=x^2+y^2-3$ or $x^2+y^2-z=3$ at $(2,-1,2)$ is

$$\nabla\phi_2 = \nabla(x^2+y^2-z) = 2x\,\mathbf{i} + 2y\,\mathbf{j} - \mathbf{k} = 4\mathbf{i} - 2\mathbf{j} - \mathbf{k}$$

$(\nabla\phi_1)\cdot(\nabla\phi_2) = |\nabla\phi_1|\,|\nabla\phi_2|\cos\theta$, where $\theta$ is the required angle. Then

$$(4\mathbf{i}-2\mathbf{j}+4\mathbf{k})\cdot(4\mathbf{i}-2\mathbf{j}-\mathbf{k}) = |4\mathbf{i}-2\mathbf{j}+4\mathbf{k}|\,|4\mathbf{i}-2\mathbf{j}-\mathbf{k}|\cos\theta$$

$$16+4-4 = \sqrt{(4)^2+(-2)^2+(4)^2}\,\sqrt{(4)^2+(-2)^2+(-1)^2}\cos\theta$$

and $\cos\theta = \dfrac{16}{6\sqrt{21}} = \dfrac{8\sqrt{21}}{63} = 0.5819$; thus the acute angle is $\theta = \arccos 0.5819 = 54^\circ 25'$.

# Differential Calculus of the vector field

What else can we do with the del or grad operator? $\longrightarrow$ Try combining it with a vector field

$(\text{a vector}).\vec{\nabla}$ (Still a hungry operator)

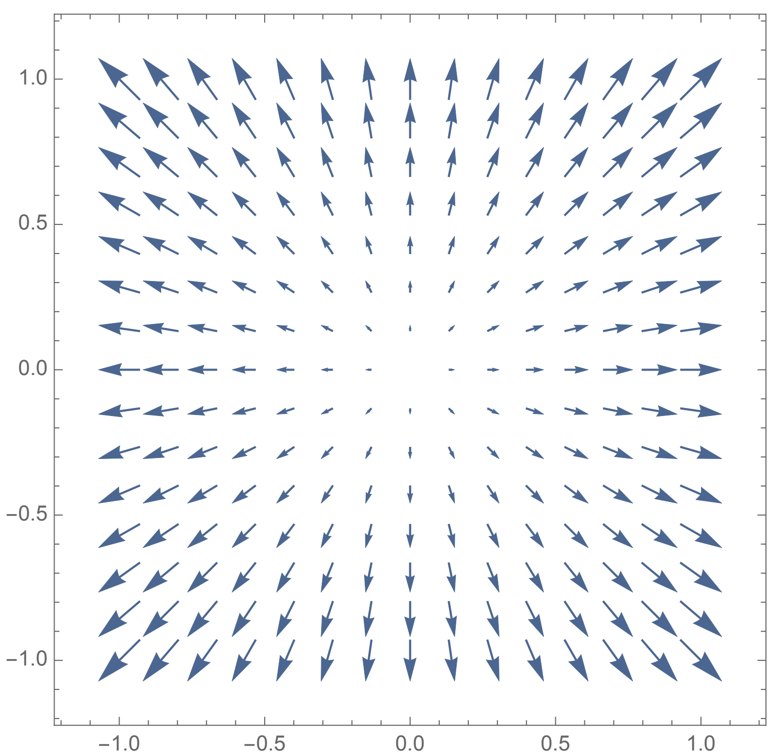
$\vec{\nabla}.(\text{a vector})$ (is a Scalar)

**The Divergence:**

$$\vec{\nabla}.\vec{v} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right).(v_x\hat{x} + v_y\hat{y} + v_z\hat{z})$$

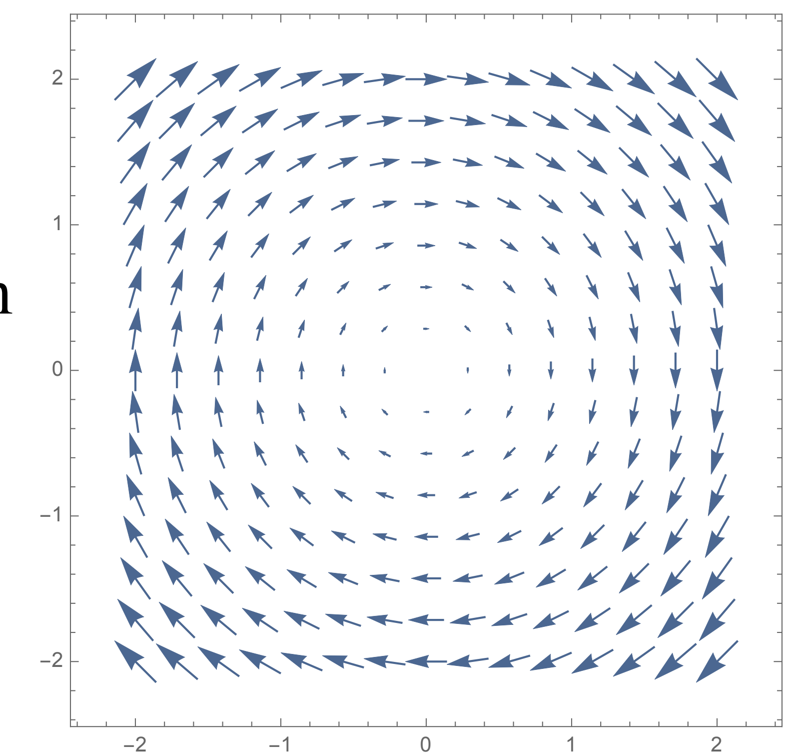
Must be a vector

$$= \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$



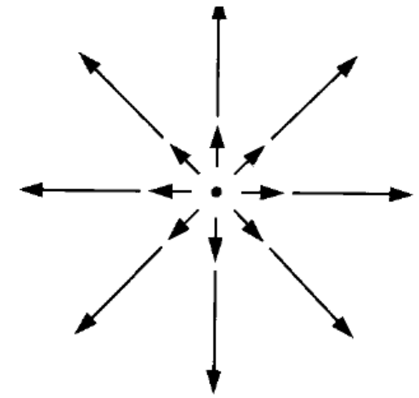

non-zero divergence

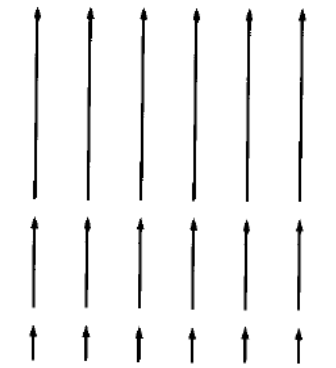
Divergence is a measure of how much the vector is spread out (diverges) from the point in question.



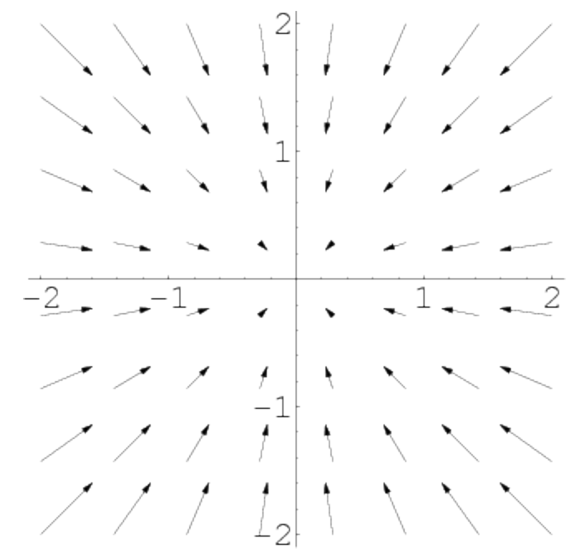

zero divergence: `solenoidal field'

- Source: $\overrightarrow{\nabla} . \overrightarrow{v} > 0$

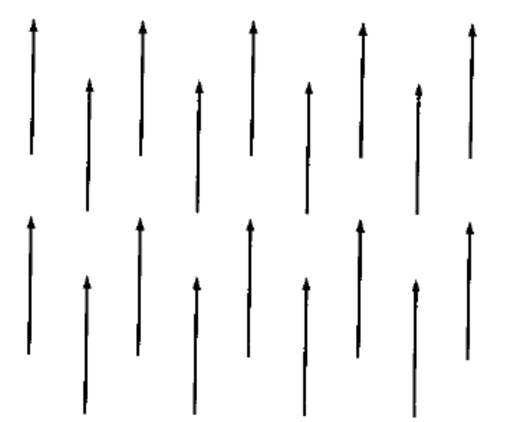




- Sink: $\overrightarrow{\nabla} . \overrightarrow{v} < 0$




- Solinoidal: $\overrightarrow{\nabla} . \overrightarrow{v} = 0$



1. Given $\phi = 2x^3y^2z^4$. (*a*) Find $\nabla\cdot\nabla\phi$ (or div grad $\phi$).

(*b*) Show that $\nabla\cdot\nabla\phi = \nabla^2\phi$, where $\nabla^2 \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ denotes the Laplacian operator

(*a*)
$$\nabla\phi = \mathbf{i}\frac{\partial}{\partial x}(2x^3y^2z^4) + \mathbf{j}\frac{\partial}{\partial y}(2x^3y^2z^4) + \mathbf{k}\frac{\partial}{\partial z}(2x^3y^2z^4)$$
$$= 6x^2y^2z^4\,\mathbf{i} + 4x^3yz^4\,\mathbf{j} + 8x^3y^2z^3\,\mathbf{k}$$

Then
$$\nabla\cdot\nabla\phi = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot(6x^2y^2z^4\,\mathbf{i} + 4x^3yz^4\,\mathbf{j} + 8x^3y^2z^3\,\mathbf{k})$$
$$= \frac{\partial}{\partial x}(6x^2y^2z^4) + \frac{\partial}{\partial y}(4x^3yz^4) + \frac{\partial}{\partial z}(8x^3y^2z^3)$$
$$= 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2$$

(*b*)
$$\nabla\cdot\nabla\phi = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\cdot\left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right)$$
$$= \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$
$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = \nabla^2\phi$$

2. Determine the constant $a$ so that the vector $\mathbf{V} = (x+3y)\mathbf{i} + (y-2z)\mathbf{j} + (x+az)\mathbf{k}$ is solenoidal.

A vector $\mathbf{V}$ is solenoidal if its divergence is zero (Problem 21).

$$\nabla \cdot \mathbf{V} = \frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+az) = 1 + 1 + a$$

Then $\nabla \cdot \mathbf{V} = a + 2 = 0$ when $a = -2$.

# Differential Calculus of the vector field

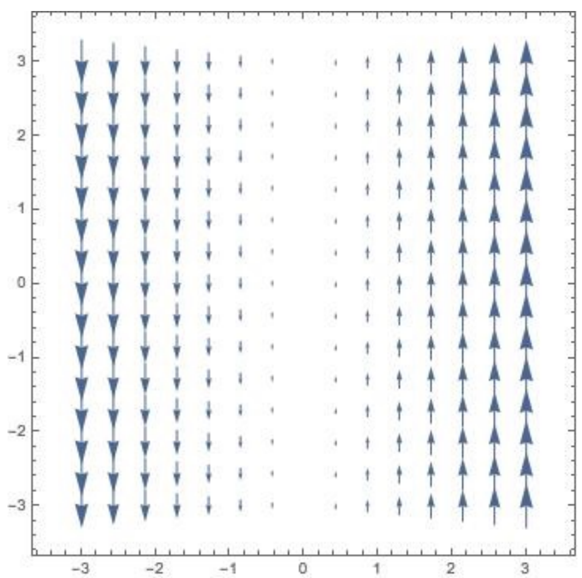
**The Curl:** Another operation with the gradient operator

$$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

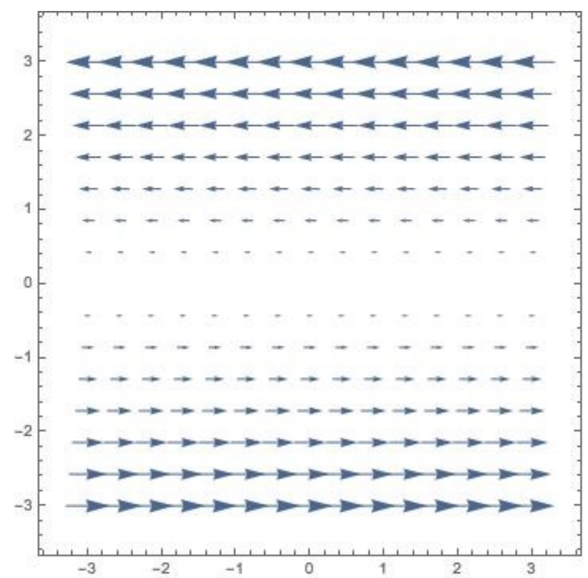
Curl means how much a vector swirls around the point in question. If vector field is a velocity field of a fluid then non-zero curl indicates rotational flow.

Or, if you like:

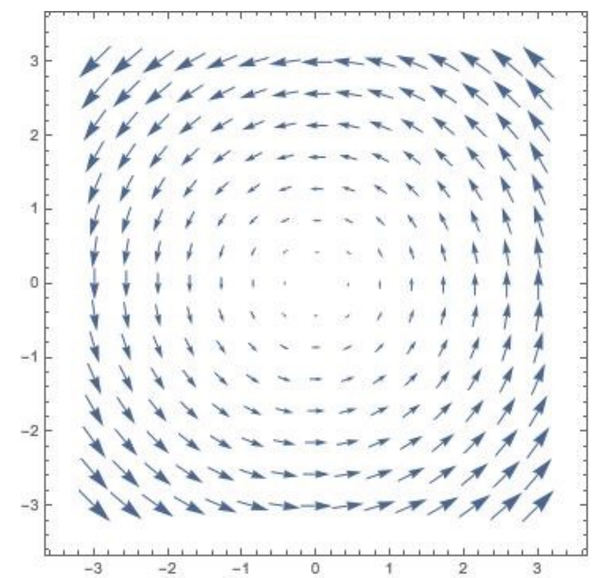
$$(\vec{\nabla} \times \vec{v})_i = \sum_{j,k} \varepsilon_{ijk} \nabla_j v_k$$



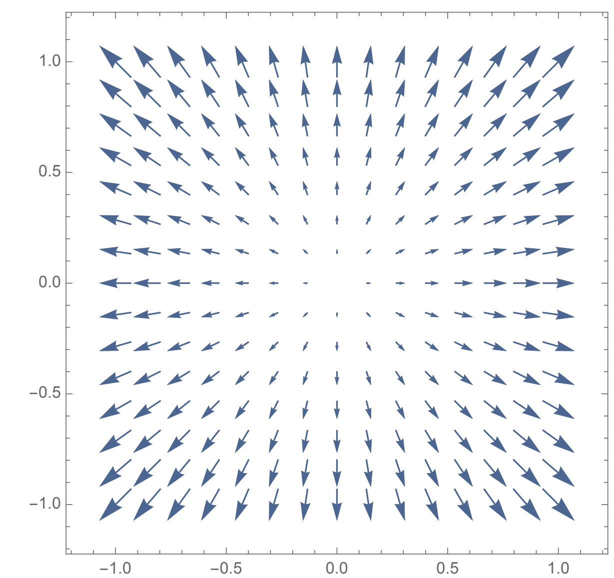

$$\vec{V}_1 = x\hat{y}$$
$$\vec{\nabla} \times \vec{V}_1 = \hat{z}$$




$$\vec{V}_2 = -y\hat{x}$$
$$\vec{\nabla} \times \vec{V}_2 = \hat{z}$$




$$\vec{V}_3 = -y\ \hat{x} + x\ \hat{y}$$
$$\vec{\nabla} \times \vec{V}_3 = 2\hat{z}$$

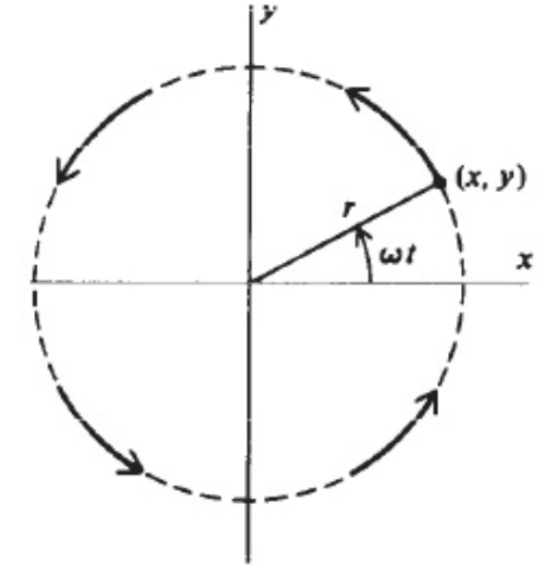



$$\vec{V}_4 = x\hat{x} + y\hat{y}$$
$$\vec{\nabla} \times \vec{V}_4 = 0$$

Vector fields with zero curl is called "irrotational" for obvious reasons!

# How does the curl represent swirling of a vector field

Consider water is flowing on a circular path shown in figure. A small volume of water at point (x,y)at time t has coordinates

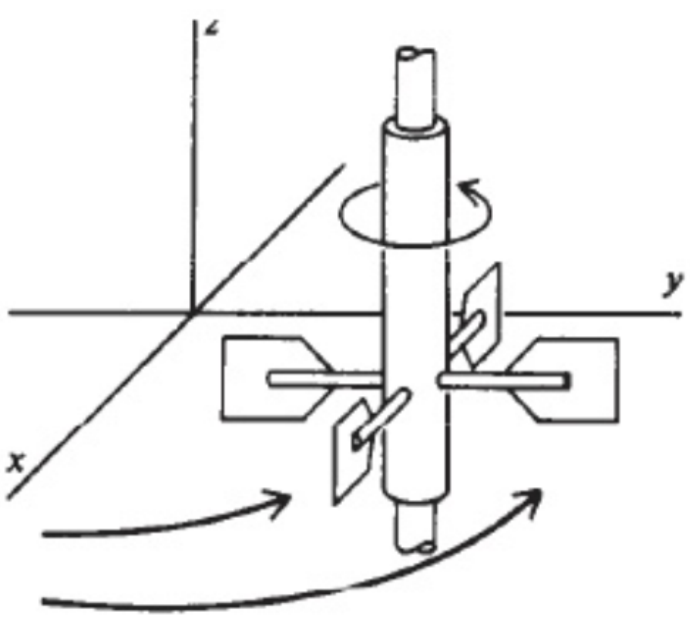



$$x = r\cos\omega t, y = r\sin\omega t$$

Velocity vector field at point (x,y)

$$\begin{aligned}\vec{v} &= \hat{x}(dx/dt) + \hat{y}(dy/dt) = r\omega[-\hat{x}\sin\omega t + \hat{y}\cos\omega t] \\ &= \omega(-\hat{x}y + \hat{y}x)\end{aligned}$$

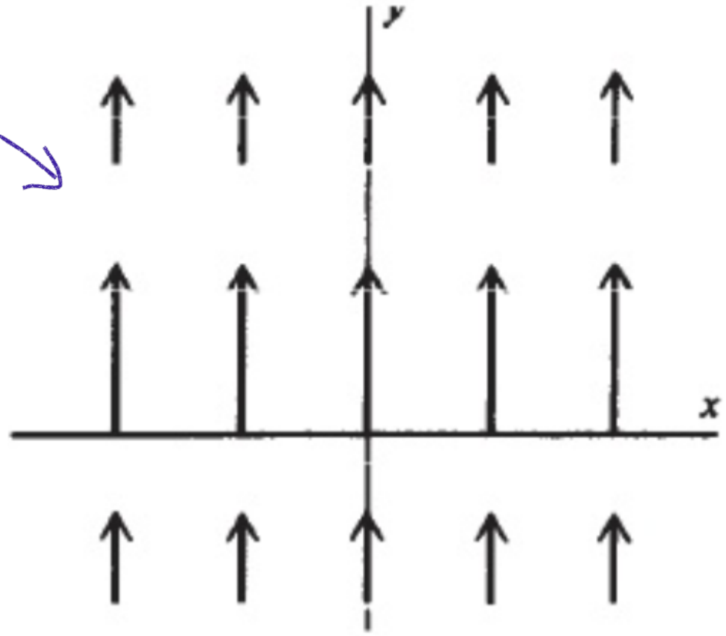
The curl of the velocity vector field: $\vec{\nabla} \times \vec{v} = 2\hat{z}\omega$ **(Non-zero)**




Curl of a vector field can be ideated by placing a paddle wheel in the flow. If the wheel rotates, the curl is non-zero. The wheel will rotate with its axis pointing in the direction of the curl.

# More interesting example

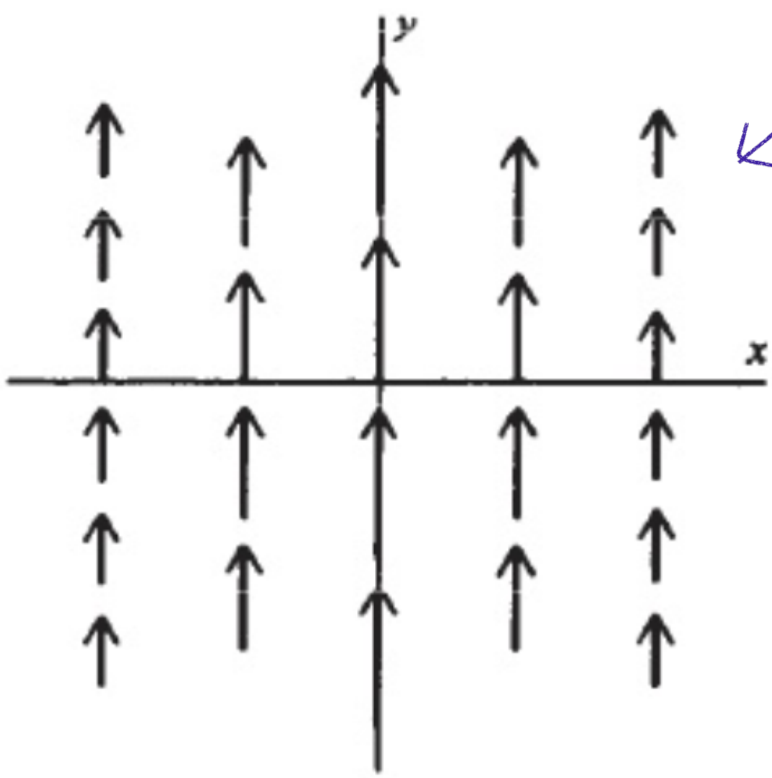
Imagine a vector field $\vec{v} = \hat{y} v_0 e^{-y^2/\lambda^2}$




Field lines point along *y* and the magnitude varies with *y*!

As expected the curl will be zero: $\vec{\nabla} \times \vec{v} = 0$

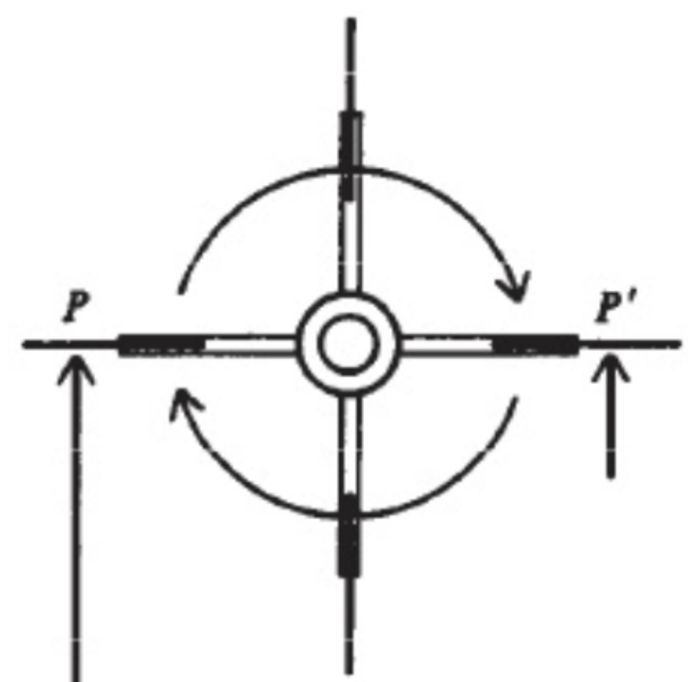
However, consider $\vec{v} = \hat{y} v_0 e^{-x^2/\lambda^2}$




Field lines still point along *y* but the magnitude varies with *x*!

$$\vec{\nabla} \times \vec{v} = -\hat{z} v_0 \frac{2x}{\lambda^2} e^{-x^2/\lambda^2}$$

**Non-zero curl!**




This can be understood by the non-zero torque on the paddle wheel in the field, although, there is no visible swirling of the field.

# Second derivatives of vector fields

So far we had only first derivatives. Why not second derivatives?

Combinations which are possible

$$\vec{\nabla}.(\vec{\nabla}T) = \nabla_x(\nabla_x T) + \nabla_y(\nabla_y T) + \nabla_z(\nabla_z T) = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

$$\vec{\nabla}.(\vec{\nabla}T) = \vec{\nabla}.\vec{\nabla}T = \nabla^2 T$$

$\nabla^2$ : new operator (scalar) - "Laplacian"

Scalar field

$\vec{\nabla} \times (\vec{\nabla}T) = 0$ (why?) Theorem: If $\vec{\nabla} \times \vec{v} = 0$, there is a $\phi$ such that $\vec{v} = \vec{\nabla}\phi$

$\vec{\nabla}.(\vec{\nabla} \times \vec{v}) = 0$ (why?) Theorem: If $\vec{\nabla}.\vec{D} = 0$, there is a $\vec{C}$ such that $\vec{D} = \vec{\nabla} \times \vec{C}$

$\vec{\nabla}(\vec{\nabla}.\vec{v})$ = a vector field.

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{v}) = \vec{\nabla}(\vec{\nabla}.\vec{v}) - \nabla^2 \vec{v}$$

1. Evaluate $\vec{\nabla}\cdot(\vec{A}\times\vec{r})$ if $\vec{\nabla}\times\vec{A} = \vec{0}$.

$$\vec{\nabla}\cdot(\vec{A}\times\vec{r}) = \frac{\partial}{\partial x_i}(\vec{A}\times\vec{r})_i$$

$$= \frac{\partial}{\partial x_i}\left(\epsilon_{ijk} A_j x_k\right) \quad \left[\because \vec{r} = x\hat{x} + y\hat{y} + z\hat{z} = x_k\hat{x}_k\right]$$

$$= \epsilon_{ijk}\frac{\partial A_j}{\partial x_i} x_k + \epsilon_{ijk} A_j \frac{\partial x_k}{\partial x_i}$$

Now, $\vec{\nabla}\times\vec{A} = \vec{0} \Rightarrow (\vec{\nabla}\times\vec{A})_i = 0$

$$\Rightarrow \epsilon_{ijk}\frac{\partial A_k}{\partial x_j} = 0$$

Concentrate on

$$\epsilon_{ijk} \frac{\partial A_j}{\partial x_i} = \epsilon_{kij} \frac{\partial A_j}{\partial x_i}$$

$$= (\vec{\nabla} \times \vec{A})_k = 0$$

$$\therefore \quad \vec{\nabla} \cdot (\vec{A} \times \vec{r}) = \epsilon_{ijk} A_j \underbrace{\frac{\partial x_k}{\partial x_i}}_{\delta_{ik}}$$

$$= \epsilon_{iji} A_j$$

$$= 0$$

Home work:

1. If $\vec{\nabla} \cdot \vec{E} = 0$, $\vec{\nabla} \cdot \vec{H} = 0$, $\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{H}}{\partial t}$

and $\vec{\nabla} \times \vec{H} = \dfrac{\partial \vec{E}}{\partial t}$,

then show that $\vec{E}$ and $\vec{H}$ satisfy

$$\nabla^2 u = \frac{\partial^2 u}{\partial t^2}$$

with $u \in \{\vec{E}, \vec{H}\}$.

2. If $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, prove $\boldsymbol{\omega} = \frac{1}{2}\,\text{curl}\,\mathbf{v}$ where $\boldsymbol{\omega}$ is a constant vector.

# Some identities involving Gradient , Divergence and Curl

$$\vec{\nabla}.(\vec{A}+\vec{B}) = \vec{\nabla}.\vec{A}+\vec{\nabla}.\vec{B} \qquad \vec{\nabla}\times(\vec{A}+\vec{B}) = \vec{\nabla}\times\vec{A}+\vec{\nabla}\times\vec{B}$$

$$\vec{\nabla}(f+g) = \vec{\nabla}f+\vec{\nabla}g$$

$$\vec{\nabla}(fg) = f\vec{\nabla}g+g\vec{\nabla}f$$

$$\vec{\nabla}.(f\vec{A}) = f\vec{\nabla}.\vec{A}+\vec{A}.\vec{\nabla}f$$

$$\vec{\nabla}(\vec{A}.\vec{B}) = \vec{A}\times(\vec{\nabla}\times\vec{B})+\vec{B}\times(\vec{\nabla}\times\vec{A})+(\vec{A}.\vec{\nabla})\vec{B}+(\vec{B}.\vec{\nabla})\vec{A}$$

$$\vec{\nabla}.(\vec{A}\times\vec{B}) = \vec{B}.(\vec{\nabla}\times\vec{A})-\vec{A}.(\vec{\nabla}\times\vec{B})$$

$$\vec{\nabla}\times(\vec{A}\times\vec{B}) = (\vec{B}.\vec{\nabla})\vec{A}-(\vec{A}.\vec{\nabla})\vec{B}+\vec{A}(\vec{\nabla}.\vec{B})-\vec{B}(\vec{\nabla}.\vec{A})$$

$$\vec{\nabla}\times(f\vec{A}) = f(\vec{\nabla}\times\vec{A})-\vec{A}\times\vec{\nabla}f$$

# What did we learn today:

- The gradient of a scalar field is defined as $\vec{\nabla}T = (\hat{x}\frac{\partial T}{\partial x} + \hat{y}\frac{\partial T}{\partial y} + \hat{z}\frac{\partial T}{\partial z})$.

- $\vec{\nabla}T$ is a vector field and it points in the direction of maximum increase of $T(x, y, z)$.

- $\vec{\nabla}T$ is perpendicular to level curves/surfaces.

- The scalar field $T$ has extremum/saddle point if $\vec{\nabla}T = 0$.

- Divergence of a vector field is denoted by $\vec{\nabla}.\vec{v}$ and is a scalar. It gives the outward flux of the field around a point.

- Curl of a vector field is a vector field and denoted by $\vec{\nabla} \times \vec{v}$ and non-zero curl implies rotational flow in case of a velocity field of a fluid.